Summary of Integral Theorems

Line Integrals:

Definition 1. *A parametrized curve is a vector-valued function* $\mathbf{c}(t) : I \subseteq \mathbb{R} \to \mathbb{R}^n$.

-its image should be the curve that you want to integrate over

Common Parametrizations

1. A circle of radius r in the xy-plane centered around the origin, $\mathbf{c}(t) = (r\cos t, r\sin t, 0)$, $t \in [0, 2\pi]$
2. A line segment joining the points (a, b, c) to (p, q, r), $\mathbf{c}(t) = (a, b, c) + t(p - a, q - b, r - c)$, $t \in [0, 1]$

Types of line integrals

1. Vector line integrals: $\int \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{F}$ is a vector field and $d\mathbf{s} = \mathbf{c}'(t)dt$
2. Scalar line integrals: $\int f ds$ where f is a scalar function and $ds = ||d\mathbf{s}|| = ||\mathbf{c}'(t)||dt$

How to set up a line integral

1. Find a parametrization for your curve: $\mathbf{c}(t)$.
2. Calculate $\mathbf{c}'(t)$.
3. Are you doing a vector line integral or a scalar line integral?
   (a) If vector line integral, plug in $\mathbf{c}(t)$ into $\mathbf{F}$ and integrate $\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt$.
   (b) If scalar line integral, plug in $\mathbf{c}(t)$ into f and integrate $f(\mathbf{c}(t))||\mathbf{c}'(t)||dt$.

Surface Integrals:

Definition 2. *A parametrized surface is a vector-valued function* $\mathbf{r}(u,v) : D \subseteq \mathbf{R}^2 \to \mathbf{R}^n$.

-its image should be the surface that you want to integrate over

Common Parametrizations

1. A sphere of radius r centered around the origin: $\mathbf{r}(\phi,\theta) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$, $\phi \in [0,\pi]$, $\theta \in [0,2\pi]$.
2. A cylinder of radius r centered around the z-axis: $\mathbf{r}(\theta,z) = (r\cos\theta, r\sin\theta, z)$, $\theta \in [0,2\pi]$, $z \in [0,h]$ where h is the height of the cylinder.
3. A plane with normal vector (a,b,c), $\mathbf{r}(u,v) = (u, v, \frac{-1}{c}(au+bv+d))$ $u \in (-\infty,\infty)$, $v \in (-\infty,\infty)$.

Types of surface integrals

1. Vector surface integrals: $\int \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}$ is a vector field and $d\mathbf{S} = \mathbf{N}(u,v)dudv = \hat{\mathbf{n}}(u,v)dS$ where $\hat{\mathbf{n}} = \mathbf{N}/||\mathbf{N}||$ and $dS = ||\mathbf{N}(u,v)||dudv$. You can use either definition of $d\mathbf{S}$, as they are equal.
2. Scalar surface integral: $\int f dS$ where f is a scalar function and $dS = ||d\mathbf{S}|| = ||\mathbf{N}(u,v)||dudv$.

How to set up a surface integral

1. Find a parametrization for your surface: $\mathbf{r}(u,v)$
2. Calculate the tangent vectors, $\mathbf{T}_u = \frac{\partial \mathbf{r}}{\partial u}$ and $\mathbf{T}_v = \frac{\partial \mathbf{r}}{\partial v}$
3. Calculate $\mathbf{N}(u,v) = \mathbf{T}_u \times \mathbf{T}_v$
4. Are you doing a vector surface integral or a scalar surface integral?
   (a) If vector surface integral, plug in $\mathbf{r}(u,v)$ into $\mathbf{F}$ and integrate $\mathbf{F}(\mathbf{r}(u,v))\cdot \mathbf{N}(u,v)dudv$
   (b) If scalar surface integral, plug in $\mathbf{r}(u,v)$ into f and integrate $f(\mathbf{r}(u,v))||\mathbf{N}(u,v)||dudv$

Triple integrals

-In rectangular coordinates (x,y,z), $dV = dxdydz$.
-In cylindrical coordinates (r,θ,z), $dV = rdrd\theta dz$.
-To switch from rectangular coordinates to cylindrical coordinates, $x = r\cos\theta, y = r\sin\theta, z = z$.
-In spherical coordinates, (r,θ,ϕ), $dV = r^2\sin\phi drd\theta d\phi$.
-To switch from rectangular coordinates to spherical coordinates, $x = r\cos\theta\sin\phi, y = r\sin\theta\sin\phi, z = r\cos\phi$.

Stoke's Theorem:

Theorem 1. *If* $\mathbf{F}$ *is a vector field defined on a surface* S*, then* $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{c=\partial S} \mathbf{F} \cdot d\mathbf{s}$ *if* S *and* c *are oriented positively.*

-Look at what this is saying: The vector surface integral of the curl of a vector field $\mathbf{F}$ is equal to the vector line integral of $\mathbf{F}$ around the boundary curve of the surface.
-You can only apply this theorem when you have a curl involved.
-This theorem is generally not applied when the surface is closed (when it has no boundary curve, like a sphere for example).

Divergence Theorem:

Theorem 2. *If* $\mathbf{F}$ *is a vector field defined on a* 3*-dimensional region* W *which is bounded by a closed surface* S*, then* $\int\int_{S=\partial W} \mathbf{F} \cdot d\mathbf{S} = \int\int\int_W \nabla \cdot \mathbf{F} dV$ *assuming that the normal vector for* S *is pointing outwards.*

-This theorem is saying: The vector surface integral of $\mathbf{F}$ on the boundary of W is equal to the triple integral of the scalar function $\nabla \cdot \mathbf{F}$ over W.
-This theorem can be applied to any vector surface integral over a closed surface.
- This theorem can only be applied when the surface is closed.

Green's Theorem:

Theorem 3. *If* $P(x, y)$ *and* $Q(x, y)$ *are differentiable, then* $\int_{c=\partial D} Pdx + Qdy = \int\int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dxdy$ *if* c *is oriented positively.*

- This theorem says: The line integral around the curve c is equal to an integral over the interior of c as long as c and the region D lie in $\mathbb{R}^2$.
- This theorem can only be applied to regions that lie in the xy-plane.
- If $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$, then the left hand side of the theorem is just $\int_{c=\partial D} \mathbf{F} \cdot d\mathbf{s}$.

Which integral theorem can I apply? Ask yourself these 3 questions:

1. Are the regions involved lying in the xy-plane? If yes, apply Green's Theorem. If no, go to next question.

2. Am I integrating over a closed surface (this means that the surface has a definite inside and outside, like a sphere, or a cylinder with top and bottom attached, or a cube)? If yes, apply Divergence theorem. If no, go to the next question.

3. Am I integrating a curl over a surface? If yes, apply Stoke's theorem. If you answered no to this question and the previous 2 questions, you can not apply any of the classical integration theorems.