Math 8: Homework #2 Solution

1.3 - 1 b, c, d, j, m

b) $(\forall x)(x \text{ is precious} \Rightarrow x \text{ is not beautiful})$

c) $(\exists x)(x \text{ is an isosceles triangle} \wedge x \text{ is a right triangle})$

d) $\sim (\exists x)(x \text{ is a right triangle} \wedge x \text{ is an isosceles triangle})$

j) $(\forall x)(\exists y)(x \text{ loves } y)$

m) $(\forall x)(\forall y)(\exists a)(\exists b)(x + iy \neq 0 \Rightarrow (c + di)(x + iy) = \pi)$

1.3 - 4 a, d

a) This statement is false since $-1 + -1 = -2 < -1$ and -1 is a real number.

d) This statement is true by the intermediate value theorem applied to $f(x) = 3^x - x^2$. $f(0) = 1$ and $f(-1) = -2/3$, so there exists some real number $c \in [-1, 0]$ such that $f(c) = 0 \in [-2/3, 1]$. So $3^c = c^2$.

1.3 - 5 e, f

e) There does not exists a real number whose square is negative.

f) There is a unique real number whose square is zero.

1.3 - 11

The phrase can be translated into quantifiers as follows: (let the universe be all people)
$(\exists x)(x \text{ can be fooled all of the time}) \wedge (\forall x)(x \text{ can be fooled some of the time}) \wedge \sim (\forall x)(x \text{ can be fooled all of the time})$.

The denial of this phrase is therefore:

$(\forall x)(x \text{ can not be fooled all of the time}) \vee (\exists x)(x \text{ can not be fooled some of the time}) \vee (\forall x)(x \text{ can be fooled all of the time})$.

Which can be translated back into english as:

You can not fool all of the people all of the time or you can not fool some of the people some of the time or you can fool all of the people all of the time.

1.4 - 2 d, g

d) Suppose that the maximum value of a differentiable function $f(x)$ on the closed interval $[a, b]$ occurs at x_0

$\vdots$

Thus, either $x_0 = a$ or $x_0 = b$ of $f'(x_0) = 0$. Therefore, if $f(x)$ is a differentiable function on the closed interval $[a, b]$ with maximum value at x_0, then either $x_0 = a$ or $x_0 = b$ or $f'(x_0) = 0$.

g) There is no g)

1.4 - 4 a, b

Let x and y be integers.

a) Suppose that x and y are even. Then $\exists m \in \mathbb{Z}$ such that $x = 2m$ and $\exists n \in \mathbb{Z}$ such that $y = 2n$ (Recall that $\mathbb{Z}$ is the set of all integers). So $x + y = 2m + 2n = 2(m + n)$. And since $x + y$ is two times the integer $m + n$, then $x + y$ is even. Therefore, if x and y are even, then $x + y$ is even.

b) Suppose that x and y are even. Then $\exists m \in \mathbb{Z}$ such that $x = 2m$ and $\exists n \in \mathbb{Z}$ such that $y = 2n$. So $xy = (2m)(2n) = 4(mn)$. So xy is 4 times the integer mn, so xy is divisible by 4. Therefore, if x and y are even, then xy is divisible by 4.

1.4 - 7

i) Suppose n is an odd number. Then $n = 2k + 1$ for some natural number k. So $n^2 + n + 3 = (2k + 1)^2 + (2k + 1) + 3 = 4k^2 + 4k + 1 + 2k + 1 + 3 = 4k^2 + 6k + 5 = (4k^2 + 6k + 4) + 1 = 2(k^2 + 3k + 2) + 1$. So $n^2 + n + 3 = 2n + 1$ where $n = (k^2 + 3k + 2)$. So $n^2 + n + 3$ is odd.

ii) Suppose n is an even number. Then $n = 2m$ for some natural number m. So $n^2 + n + 3 = (2m)^2 + 2m + 3 = (4m^2 + 2m + 2) + 1 = 2(m^2 + m + 1) + 1$. So $n^2 + n + 3 = 2n + 1$ where $n = m^2 + m + 1$. So $n^2 + n + 3$ is odd.

Therefore since any natural number is either even or odd, $n^2 + 2n + 3$ is odd.

1.4 - 10b

b) This argument doesn't quite work. The person assumes that $\exists q \in \mathbb{Z}$ such that $b = aq$ and $\exists q \in \mathbb{Z}$ such that $c = aq$ (Since $\exists$ only extends to the following statement, the 2 q's are distinct). However, the person then writes $b + c = aq + aq = 2aq$ but since the q's are distinct, this is not true. $b + c = a(q + q)$. This person became confused with notation. He or she should have used 2 different variables representing the two integers, say p and q respectively. Then $b + c = aq + ap = a(p + q)$, and the result follows. I would assign this a C for a proof which is largely correct.

1.5 - 3 d, f Let x and y be integers.

d) The contrapositive of this statement would be: if x and y are odd, then xy is odd. So assume that x and y are odd, so $\exists m \in \mathbb{Z}$ such that $x = 2m+1$ and $\exists n \in \mathbb{Z}$ such that $y = 2n+1$. So $xy = (2m+1)(2n+1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$. So since $xy = 2p + 1$ where $p = 2mn + m + n$, then xy is odd. So if x and y are odd, then xy is odd. Therefore by contraposition, if xy is even, then either x or y is even.

f) The contrapositive of this statement is: if either x or y is even, then xy is even. So assume that either x or y is even, say x is even. So $\exists m \in \mathbb{Z}$ such that $x = 2m$. Then $xy = 2my = 2(my)$ which is even. Therefore if either x or y is even, then xy is even. So by contraposition, if xy is odd, then both x and y are odd.

1.5 -5 A circle has center $(2, 4)$.

a) Prove this by contradiction. So assume that $(-1, 5)$ and $(5, 1)$ are both on the circle. The equation of a circle with center $(2, 4)$ and radius R is: $(x-2)^2 + (y-4)^2 = R^2$. By plugging in $(-1, 5)$ into this equation, we find that $R^2 = 10$. But by plugging in $(5, 1)$, we would find that $R^2 = 18$ which is a contradiction. Therefore $(-1, 5)$ and $(5, 1)$ cannot both lie on the same circle with center $(2, 4)$.

b) Prove this by contraposition. So assume that circle intersects the line $y = x - 6$. The circle can either intersect the line at one point or at two points. The minimum radius that will lead to an intersection, is clearly the case of intersection at one point. So assume that the circle intersects the line at one point. Then we must show that the radius of this circle is greater than or equal to 5. As shown in class, the line through the center of the circle and the point of intersection will be $y = -x + 6$. So now by setting $-x + 6 = x + 6$, we see that the two lines will intersect at the point $(6, 0)$. And if $(6, 0)$ lies on the circle, the circle would have radius $R = \sqrt{32} > 5$. So if the circle intersects the line, it will have radius greater than or equal to 5. Thus, by contraposition, if the radius of the circle is less than 5, then the circle does not intersect the line.

1.5 - 12 c

c) This argument also doesn't quite work. Everything is good up to the point where the person writes "the left side of the equation is even because it is the sum of even numbers." So the person is assuming the claim he or she is trying to prove. What he or she would need to do before writing the above incorrect idea, would be to say $x = 2m$ for some m and $y = 2n$ for some n, then add everything to get $2m + 2n - 2k = 2(m + n - k)$ which is clearly divisible by 2, but the right side is not divisible by 2. Then he or she would get the desired result.