

Math 8: Homework #4 Solution

2.3 - 6b

- b) Take any $b \in B \cup (\bigcap_{\alpha \in \Delta} A_\alpha)$. Then $b \in B$ or $b \in \bigcap_{\alpha \in \Delta} A_\alpha$. If $b \in B$, then $b \in B \cup A_\alpha, \forall \alpha \in \Delta$. So $b \in \bigcap_{\alpha \in \Delta} (B \cup A_\alpha)$. If $b \in \bigcap_{\alpha \in \Delta} A_\alpha$, then $b \in A_\alpha, \forall \alpha \in \Delta$. So $b \in \bigcap_{\alpha \in \Delta} (B \cup A_\alpha)$. Therefore $B \cup (\bigcap_{\alpha \in \Delta} A_\alpha) \subseteq \bigcap_{\alpha \in \Delta} (B \cup A_\alpha)$. Now take any $a \in \bigcap_{\alpha \in \Delta} (B \cup A_\alpha)$ then $a \in B \cup A_\alpha, \forall \alpha \in \Delta$. So $a \in B$ or $a \in A_\alpha, \forall \alpha \in \Delta$. If $a \in B$, then $a \in B \cup (\bigcap_{\alpha \in \Delta} A_\alpha)$. If $a \in A_\alpha, \forall \alpha \in \Delta$, then $a \in \bigcap_{\alpha \in \Delta} A_\alpha$. So $\bigcap_{\alpha \in \Delta} (B \cup A_\alpha) \subseteq B \cup (\bigcap_{\alpha \in \Delta} A_\alpha)$. Therefore, $\bigcap_{\alpha \in \Delta} (B \cup A_\alpha) = B \cup (\bigcap_{\alpha \in \Delta} A_\alpha)$.

2.3 - 7b

- b) $(\bigcap_{\alpha \in \Delta} A_\alpha) \cup (\bigcap_{\beta \in \Gamma} B_\beta) = \bigcap_{\beta \in \Gamma} ((\bigcap_{\alpha \in \Delta} A_\alpha) \cup B_\beta) = \bigcap_{\beta \in \Gamma} (\bigcap_{\alpha \in \Delta} (A_\alpha \cup B_\beta))$.

2.3 - 8a, d

- a) false. Choose your sets such that each $A_\alpha \subseteq B$, and all the A_α 's are disjoint. For instance $B = \{1, 2, 3, 4\}$ and $A_1 = \{1\}, A_2 = \{2\}$.
- d) true. Writing out the definition of the left hand side, $\{x : x \in A_\alpha \text{ for some } \alpha \in \Delta \text{ and } x \notin B\}$. Definition of the right hand side, $\{x : x \in (A_\alpha - B) \text{ for some } \alpha \in \Delta\}$. So $x \in A_\alpha$ for some $\alpha \in \Delta$ and $x \notin B$. Both definitions are the same.

2.4 - 8a, c, e

- a) $n = 1$ case: $1 = \frac{1}{2}(1)(3(1) - 1) = \frac{1}{2}(2) = 1$. Assume that $\sum_{i=1}^n (3i - 2) = \frac{1}{2}n(3n - 1)$ for some $n \geq 1$. And $\sum_{i=1}^{n+1} (3i - 2) = 3(n + 1) - 2 + \sum_{i=1}^n (3i - 2) = 3(n + 1) - 2 + \frac{1}{2}n(3n - 1) = 3n + 1 + \frac{1}{2}n(3n - 1) = \frac{1}{2}(6n + 2 + 3n^2 - n) = \frac{1}{2}(3n^2 + 5n + 2) = \frac{1}{2}(n + 1)(3(n + 1) - 1)$.
- c) $n = 1$ case: $2^1 = 2^2 - 2$. Assume that $\sum_{i=1}^n 2^i = 2^{n+1} - 2$ for some $n \geq 1$. Then $\sum_{i=1}^{n+1} 2^i = 2^{n+1} + \sum_{i=1}^n 2^i = 2^{n+1} + 2^{n+1} - 2 = 2(2^{n+1}) - 2 = 2^{n+2} - 2$.
- e) First get a common denominator: $\frac{3n^5 + 5n^3 + 7n}{15}$. $n = 1$ case: $\frac{3+5+7}{15} = 1$ which is an integer. Now assume that the claim is true up to some $n \geq 1$. Then $\frac{3(n+1)^5 + 5(n+1)^3 + 7(n+1)}{15} = \frac{3n^5 + 5n^3 + 7n + 15n^4 + 30n^3 + 45n^2 + 30n + 15}{15}$. Now by breaking up the fraction into: $\frac{3n^5 + 5n^3 + 7n}{15} + \frac{15n^4 + 30n^3 + 45n^2 + 30n + 15}{15}$ we see that the first term is the integer we got by induction, and the second term is clearly an integer since each term is divisible by 15.

2.4 - 9b, d

- b) $n = 5$ case: $(5 + 1)! = 720 > 256$. Assume that $(n + 1)! > 2^{n+3}$ for some $n \geq 5$. Then $(n+1+1)! = (n+2)(n+1)! > (n+2)2^{n+3} > 2 \cdot 2^{n+3} = 2^{n+1+3}$
- d) $n = 2$ case: $\sqrt{2} < 1 + \frac{1}{\sqrt{2}}$. Assume that $\sqrt{n} < \sum_{i=1}^n \frac{1}{\sqrt{i}}$ for some $n \geq 2$. Then $\sqrt{n} + \frac{1}{\sqrt{n+1}} < \sum_{i=1}^n \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}}$. And since $n < \sqrt{n(n+1)}$, then $n + 1 < \sqrt{n(n+1)} + 1$, so $\sqrt{n+1} < \sqrt{n} + \frac{1}{\sqrt{n+1}} < \sum_{i=1}^{n+1} \frac{1}{\sqrt{i}}$.

2.4 - 11 $n = 1$ case: with only one point, there should be no lines, and $\frac{1^2-1}{2} = 0$. Now assume for some $n \geq 1$ that there are n points, and $\frac{n^2-n}{2}$ lines. Now add another point. So you will have to connect the new point to the n other points. So there must be $\frac{n^2-n}{2} + n$ lines. And $\frac{n^2-n}{2} + n = \frac{(n+1)^2-n}{2}$.

2.4 - 13 Moving 1 disk takes 1 move, and $1 = 2^1 - 1$. Now assume that n disks can be moved in $2^n - 1$ moves for $n \geq 1$. Now consider the case of moving $n + 1$ disks. Since n disks can be moved in $2^n - 1$ moves, the top n disks can be moved to the second peg in $2^n - 1$ moves. The $n + 1^{th}$ disk can be moved to the third peg in 1 move. And the n disks on the second peg can be moved onto the third peg in $2^n - 1$ moves. So to move $n + 1$ disks, it takes $2^n - 1 + 1 + 2^n - 1 = 2(2^n) - 1 = 2^{n+1} - 1$ moves.

2.4 - 15a

- a) This proof is incorrect. The writer assumes that for any integer n , the set of n horses all have the same color. This argument would require a different proof for going from $n = 1$ to $n = 2$. Given 2 horses, removing 1 horse will certainly give you a set of horses with all the same color, but when you remove the other horse, the 2 sets may not have horses of the same color.