

Math 8: Homework #5 Solution

2.5 - 7

Let $A = \{a \in \mathbb{N} : \exists b \in \mathbb{N} \text{ such that } a^2 = 2b^2\}$. Then by the well ordering principle, A has a smallest element. Call this smallest element a . Then there is some b such that $a^2 = 2b^2$. Then $b < a$, or else $a = 2b^2 < b^2$. Since $a^2 = 2b^2$, then a^2 is even. So a is even also. So $a = 2c$ for some $c \in \mathbb{N}$. Therefore $(2c)^2 = 2b^2$. So $b^2 = 2c^2$. So $b \in A$, and $b < a$ which is a contradiction of a being the smallest element. So $A = \emptyset$. Therefore $\sqrt{2}$ is irrational.

2.5 - 8

- a) $7 = 2 * 3 + 1$, $-6 = -2 * 3 + 0$, $-5 = 3 * (-2) + 1$, $-5 = -3 * 2 + 1$,
 $1 = -4 * (0) + 1$, $0 = 3 * 0 + 0$.
- b) Let a and b be integers with $b > 0$. Let $T = \{a - bx : x \in \mathbb{Z} \text{ and } a - bx \geq 0\}$. If $a \geq 0$, then choose $x = 0$, so $a - bx = a \geq 0$, so $a \in T$. If $a < 0$, choose $x = a$, then $a - bx = a - ba = a(1 - b) \geq 0$. So $T \neq \emptyset$. Therefore by the well ordering principle, T has a smallest element. Call this smallest element r . So by definition of T , $r \geq 0$. And there is some integer q such that $a - bq = r$. Suppose that $r \geq b$. Then $r = b + s$ for some $s \geq 0$. So $a - bq = r = b + s$ implies that $s = a - bq - b = a - b(q + 1) \geq 0$. So $s \in T$. But $s = r - b < b$ so s is less than r which contradicts the minimality of r . So therefore $r < b$. So now we have proven the division algorithm for integers with $b > 0$. To prove the division algorithm for any $a, b \in \mathbb{Z}$, we must prove it for $b < 0$. So assume that $b < 0$. Then $-b > 0$ so by division algorithm proved above, there are integers n and r such that $a = (-b)n + r$. Where $0 \leq r < (-b)$. Now since $(-b) = |b|$ and $a = (-b)n + r = b(-n) + r$, let $q = -n$. So given any integers a and b with $b \neq 0$, there exists integers q and r with $0 \leq r < |b|$ such that $a = qb + r$.

2.5 - 10

If $a = 1$, choose s to be $a + 1$, then $a < a + 1 \leq (a + 1)b = sb$. Suppose the statement is true when $a = n$ for some natural number n . Then there is some natural number s such that $n < sb$. Then $n + 1 < sb + 1 \leq sb + b = (s + 1)b$. So the statement is true when $a = n + 1$. So by the PMI, the statement is true for all natural numbers a and b .