

Math 8: Homework #8 Solution

4.1 - 2 b, e, h

- b) Domain = $\mathbb{R}$, Possible Codomain - $\mathbb{R}$ or $[-2, \infty]$, etc. Range = $[5, \infty)$
- e) Domain = $\mathbb{R} - \{\pi/2 + n\pi | n \in \mathbb{Z}\}$, Possible Codomain - $\mathbb{R}$, Range = $\mathbb{R}$
- h) Domain = $\mathbb{R}$, Possible Codomain - $\mathbb{R}$ or $[0, \infty)$, etc. Range = $[1, \infty)$

4

- a) 24
- b) 4 or -4
- c) 5 and -5
- d) $\sqrt{21}$ or $-\sqrt{21}$
- e) 0
- f) There is no such preimage, as $-10 \notin \text{Rng}(f)$

16

- a) The absolute value is always non-negative, so $d(x, y) \geq 0 \forall x, y \in \mathbb{N}$. Take any $x, y, z \in \mathbb{N}$. $|x - y| = 0 \Leftrightarrow x = y$. So $d(x, y) = 0 \Leftrightarrow x = y$. Since $|x - y| = |y - x|$, then $d(x, y) = d(y, x)$. Now $|x - z| = |x - y + y - z| \leq |x - y| + |y - z|$ by the triangle inequality property of the absolute value. So $d(x, z) \leq d(x, y) + d(y, z)$. Therefore d is a metric.
- b) Take any $x, y, z \in \mathbb{R}$. By definition, $d(x, y) \geq 0$. By definition, $d(x, y) = 0 \Leftrightarrow x = y$. Suppose $d(x, y) = 0$ then $x = y$ so $y = x$, therefore $d(y, x) = 0$. Suppose that $d(x, y) = 1$ then $x \neq y$, so $y \neq x$, so $d(y, x) = 1$. Therefore $d(x, y) = d(y, x)$. Now suppose that $x = z$, then $d(x, z) = 0$. And $0 \leq 0$ and $0 \leq 1$ and $0 \leq 2$. And $d(y, z) + d(x, y)$ can only be 0, 1, 2 by definition of d . Now suppose that $x \neq z$ then $d(x, z) = 1$. Since $x \neq z$, either $x = y \wedge y \neq z$ or $x \neq y \wedge y = z$ or $x \neq y \wedge y \neq z$. In any event, $d(x, z) \leq d(x, y) + d(y, z)$. So since these are the only 2 cases ($x = z$ or $x \neq z$), the triangle inequality holds for d .

- c) Take any $(x, y), (z, w), (t, s) \in \mathbb{R} \times \mathbb{R}$. The square root function is always nonnegative, so $d((x, y), (z, w)) \geq 0$. Now underneath the square root, we are adding 2 squared real numbers. And a real number squared is non-negative. So the only way for the sum underneath the square root to be zero is if each term is zero. So that implies that $x = z$ and $y = w$, so $(x, y) = (z, w)$, so property ii is satisfied. It can be easily seen that $(x - z)^2 = (z - x)^2 \forall x, z \in \mathbb{R}$. This implies that $d((x, y), (z, w)) = d((z, w), (x, y))$.
- d) Take any $(x, y), (z, w), (t, s) \in \mathbb{R} \times \mathbb{R}$. Absolute values are always non-negative, so are the sums of two absolute values. Therefore $d((x, y), (z, w)) \geq 0$. The only way to add 2 nonnegative numbers to get zero is if each one is zero. Therefore, $x = z$ and $y = w$, so $(x, y) = (z, w) \Leftrightarrow d((x, y), (z, w)) = 0$. The absolute value is symmetric from part a), so this implies that d is symmetric (satisfies property iii). Well $|x - t + t - z| + |y - s + s - w| \leq |x - t| + |t - z| + |y - s| + |s - w|$. This implies the triangle inequality holds for d .

4.2 - 1 g, h

- g) $(f \circ g)(x) = \frac{x^2+2}{x^2+3}, (g \circ f)(x) = (\frac{x+1}{x+2})^2 + 1$
- h) $(f \circ g)(x) = 3|x| + 2, (g \circ f)(x) = |3x + 2|$

4.2 - 8

If $f \cap g = \emptyset$, then $A = \emptyset$, so $g|_A = \emptyset$. So suppose $f \cap g \neq \emptyset$. So $\exists x$ such that $f(x) = g(x)$. So $f \cap g = \{(x, f(x)) : f(x) = g(x)\}$ And $g|_A = \{(x, g(x)) : x \in A\}$. And $A = \{x : g(x) = f(x)\}$. So these two sets are equal.

4.2 - 15 b, c, e

- b) Let $fx = -x$ and $g(x) = -x$ then $f \circ g = x$.
- c) Suppose f and g are decreasing. Take any $x, y \in \text{Dom}(g)$, suppose $x < y$. Then $g(x) > g(y)$. So $f(g(y)) > f(g(x))$. So $f \circ g$ is increasing.

4.3 - 4

Take any $c \in C$ then since g is onto, there is a $b \in B$ such that $g(b) = c$. Since f is onto, there is a $a \in A$ such that $f(a) = b$. So now $g \circ f(a) = c$. So $g \circ f$ is onto.

4.3 - 6

Suppose f is not injective. So there are $a, b \in A$ where $a \neq b$ such that $f(a) = f(b)$. So $g \circ f(a) = g \circ f(b)$. So $g \circ f$ is not injective. This is a proof by contraposition.

4.3 - 8 b, c, d, f

- b) Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be given by $g(x) = x$. This is onto. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = 2$. Then $g \circ f(x) = 2$ is not onto.

- c) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = 2$. Then f is not onto. And let $g : \mathbb{R} \rightarrow 2$ be given by $g(x) = 2$. Then $g \circ f(x) = 2$ is onto.
- d) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x$. So f is injective. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be given by $g(x) = x^2$. Then $g \circ f(x) = x^2$. So $g \circ f$ is not injective.
- f) Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be given by $g(x) = x^2$. So g is not injective. Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be given by $f(x) = x$. Then $g \circ f$ is injective.

13 Suppose f is not injective. So there are distinct points $a, b \in \text{Dom} f$ such that $f(a) = f(b)$ and $a < b$. This means that f is neither strictly increasing or strictly decreasing. This is a proof by contraposition.