

Math 8: Homework #9 Solution

5.1 - 1

Take any set A , then define $f : A \rightarrow A$ by $f(x) = x$. This is a bijection, so $A \approx A$. Now suppose $A \approx B$. Then there is a bijection $f : A \rightarrow B$. So $f^{-1} : B \rightarrow A$ is a bijection. So $B \approx A$. Now suppose $A \approx B$ and $B \approx C$. Then there exist bijections $f : A \rightarrow B$ and $g : B \rightarrow C$. So $g \circ f : A \rightarrow C$ is a bijection. So $A \approx C$.

5.1 - 5 i, j, k, l, m

- i) infinite
- j) infinite
- k) finite (since $x^2 + x = x(x + 1)$ isn't prime for $x > 1$)
- l) finite (only 8 solutions)
- m) infinite

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- a) $A \cap B \subseteq A$. And any subset of a finite set is finite. So $A \cap B$ is finite.
- b) Prove by contraposition. If B is finite, and $A \subseteq B$, then A is finite. Any subset of a finite set is finite.

9

A function is a relation. A relation is a subset of $A \times B$. The number of subsets of $A \times B$ is the number of elements in the power set of $A \times B$. Suppose that $\#A = m$ and $\#B = n$, then the number of elements in the power set of $A \times B$ is 2^{mn} . So there are 2^{mn} relations from A to B . This is a finite number. So the number of functions from A to B must be finite.

5.2 - 5 b, d, f, g

- b) c (define $f : (0, 1) \rightarrow (5, \infty)$ by $f(x) = 5/x$ which is a bijection)
- d) N_0 (bijection $f : \mathbb{N} \rightarrow \{2^x : x \in \mathbb{N}\}$ given by $f(x) = 2^x$)
- f) c
- g) N_0

5.3 - 6

Let Γ be the finite collection of countable sets. Suppose $\Gamma = \emptyset$. Then everything is true. Suppose Γ has one countable set. Then the union in question is countable (as it is one countable set). So suppose this is true for Γ having k sets. Add another countable set to the mix, A_{k+1} . Then by Theorem 5.25, $A_{k+1} \cup \bigcup_{1 \leq i \leq k} A_i$ is countable. So by PMI, this theorem is proved.

5.3 - 7

If we only have 1 denumerable set A_1 , then $B_1 = A_1$ is denumerable and pairwise disjoint from the other B_i 's (which are all $\emptyset$) and that union is satisfied by $A_1 = B_1$. Suppose our theorem is true for k . Add another denumerable set A_{k+1} . Define $B_{k+1} = A_{k+1} - \bigcup_{1 \leq i \leq k} A_i$. Then $B_{k+1} \cap B_i = \emptyset$ for $1 \leq i \leq k$ by definition (B_{k+1} has no elements from $A_1, A_2, \dots, A_k$). Also B_{k+1} is denumerable. So $\{B_i : 1 \leq i \leq k+1\}$ is a denumerable family of pairwise disjoint sets. Clearly $\bigcup_{1 \leq i \leq k+1} B_i \subseteq \bigcup_{1 \leq i \leq k+1} A_i$. To show the other containment, take any $a \in \bigcup_{1 \leq i \leq k+1} A_i$ then $a \in A_i$ for some i . Choose the smallest i such that this is true. Can do this by WOP. Then $a \in B_i$ for this smallest i .

5.3 - 15

If A and B are finite, then AXB is finite, so countable. So suppose A and B are denumerable. Need to show AXB is denumerable. $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$ by considering $f(m, n) = 2^{m-1}(2n-1)$ is a bijection. Since A is denumerable then there is a bijection $g : A \rightarrow \mathbb{N}$ and since B is denumerable then there is a bijection $h : B \rightarrow \mathbb{N}$. Define $j : AXB \rightarrow \mathbb{N} \times \mathbb{N}$ by $j(a, b) = (g(a), h(b))$ then this is a bijection. So AXB is denumerable. If A is denumerable and B is finite with k elements. Define $f : \mathbb{N} \times \mathbb{N} \rightarrow AXB$ by $f(m, n) = (g^{-1}(m), n \bmod k)$ where g is defined above is a bijection. So AXB is denumerable.