THE LOWER ALGEBRAIC K-GROUPS

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1. Introduction

Algebraic K-theory is the study and application of certain functors K_n from the category of rings with $1 \neq 0$ to abelian groups. The functors K_0, K_1 (called the "lower" or classical K-groups) are easier to define than the others and have the most immediate applications.

My primary two resources are Milnor's classic text ([3]) and Rosenberg's more modern treatment ([4]), which books I use throughout. I also like Charles Weibel's online work in progress ([5]) which includes errata for [3] and [4].

I have omitted some proofs of technical lemmas which can be found in these standard resources, but have intentionally included details which are left out or glossed over (for instance, see 2.11, 2.14, 3.6, 3.7, 3.11, 3.18) as well as a few exercises and additional examples (for instance, see 2.16, 2.23, 3.15, 3.16).

2. K_0

2.1. Definitions and Basic Properties.

Definition 2.1. Denote the category of rings with identity where morphisms preserve $1 \neq 0$ by $\underline{\text{Ring}}$, and for $R \in |\underline{\text{Ring}}|$ (= objects of $\underline{\text{Ring}}$) define $\text{Proj}(R)$ to be the set of isomorphism classes of finitely generated projective left R-modules. For a finitely generated projective left R-module M let $\overline{M} \in \text{Proj}(R)$ denote the isomorphism class of M.

Remark 2.2. We consider arbitrary rings only in this subsection and the first subsection in the K_1-section for the sake of more generality in the construction of these functors, but we almost exclusively work with commutative rings in other subsections.

Definition 2.3. For $R \in |\underline{\text{Ring}}|$ define $K_0(R)$ to be the quotient of the free abelian group on $\text{Proj}(R)$ modulo the subgroup generated by elements of the form $\overline{M \oplus N} - \overline{M} - \overline{N}$. Denote the coset of $\overline{M} \in \text{Proj}(R)$ in $K_0(R)$ by $[M]$.

Lemma 2.4. *Let $R \in |\underline{\text{Ring}}|$ and M, N be finitely generated projective left R-modules. Then $[M] = [N]$ if and only if M is stably isomorphic to N (i.e., $R^n \oplus M \cong R^n \oplus N$ as left R-modules for some $n \in \mathbb{N}$).*

Proof. ($\Leftarrow$) If $R^n \oplus M \cong R^n \oplus N$, then $n[R] + [M] = [R^n \oplus M] = [R^n \oplus N] = n[R] + [N]$, so $[M] = [N]$. ($\Rightarrow$) Suppose $[M] = [N]$. Then there are finitely generated projective R-modules P_i, Q_i, P'_j, Q'_j with

$$\overline{M} - \overline{N} = \sum_i (\overline{P_i \oplus Q_i} - \overline{P_i} - \overline{Q_i}) - \sum_j (\overline{P'_j \oplus Q'_j} - \overline{P'_j} - \overline{Q'_j}),$$

so

$$\overline{M} + \sum_j \overline{P'_j \oplus Q'_j} + \sum_i (\overline{P_i} + \overline{Q_i}) = \overline{N} + \sum_i \overline{P_i \oplus Q_i} + \sum_j (\overline{P'_j} + \overline{Q'_j}),$$

giving

$$M \oplus X \cong N \oplus X$$

where

$$X \cong \sum_j (P'_j \oplus Q'_j) \oplus \sum_i P_i \oplus \sum_i Q_i \cong \sum_i (P_i \oplus Q_i) \oplus \sum_j P'_j \oplus \sum_j Q'_j.$$

Since X is finitely generated and projective, there is an R-module Y such that $X \oplus Y \cong R^n$ for some $n \in \mathbb{N}$, so

$$M \oplus R^n \cong (M \oplus X) \oplus Y \cong (N \oplus X) \oplus Y \cong N \oplus R^n,$$

whence M, N are stably isomorphic. □

Example 2.5. Let R be a division ring. Then every finitely generated R-module is a finite dimensional vector space. Hence for each generator $[M] \in K_0(R)$ we have

$$[M] = [R^d] = d[R]$$

where $d = \dim_R(M)$. Thus $K_0(R)$ is generated by $[R]$. Now suppose

$$[0] = m[R] = [R^m]$$

for some $m \in \mathbb{N}_0$. Then by lemma 2.4 there is an $n \in \mathbb{N}$ with

$$R^n \oplus 0 \cong R^n \oplus R^m,$$

so

$$n = \dim_R(R^n \oplus 0) = \dim_R(R^n \oplus R^m) = n + m,$$

whence $m = 0$. Therefore $[R]$ does not have finite order in $K_0(R) = \langle [R] \rangle$, so

$$K_0(R) \cong \mathbb{Z}.$$

Example 2.6. Every finitely generated abelian group is the direct sum of a torsion free part consisting of finitely many copies of $\mathbb{Z}$ along with a torsion part consisting of finitely many primary cyclic groups. Hence every finitely generated projective $\mathbb{Z}$-module is free since projective implies torsion-free. Therefore we again have that

$$K_0(\mathbb{Z}) = \langle [\mathbb{Z}] \rangle$$

is cyclic. In fact, using an argument similar to that in the last example it's easy to see $[\mathbb{Z}]$ has infinite order by 2.4 since the rank of a free $\mathbb{Z}$-module is well-defined. Thus we again have

$$K_0(\mathbb{Z}) \cong \mathbb{Z}.$$

Remark 2.7. Let $f : R \to S$ be a morphism in $\underline{\text{Ring}}$ and M be a finitely generated projective left R-module. Then $R^n \cong M \oplus N$ as left R-modules for some $n \in \mathbb{N}$, so viewing S as an S-R-bimodule, we have

$$S^n \cong (S \otimes_R R)^n \cong S \otimes_R R^n \cong (S \otimes_R M) \oplus (S \otimes_R N)$$

as S-modules, whence $S \otimes_R M$ is a finitely generated projective left S-module. Moreover, if $M \cong N$ as left R-modules, then $S \otimes_R M \cong S \otimes_R N$ as left S-modules. Therefore we have a well-defined map

$$\text{Proj}(R) \to \text{Proj}(S)$$

given by

$$\overline{M} \mapsto \overline{S \otimes_R M}.$$

In fact, since tensor product distributes over direct sums, the above mapping induces a group homomorphism

$$f_* : K_0(R) \to K_0(S)$$

given by

$$[M] \mapsto [S \otimes_R M].$$

Theorem 2.8. *K_0 is a functor from $\underline{\text{Ring}}$ to the category $\underline{Ab}$ of abelian groups where K_0 sends morphisms f to f_*.*

Proof. Let $f : R \to S$ and $g : S \to T$ be morphisms in $\underline{\text{Ring}}$. Then for a generator $[M] \in K_0(R)$ we have

$$\begin{aligned} (g \circ f)_*([M]) &= [T \otimes_R M] = [(T \otimes_S S) \otimes_R M] \\ &= [T \otimes_S (S \otimes_R M)] = g_*(f_*([M])). \end{aligned}$$

Also, if $i : R \to R$ is an identity morphism in $\underline{\text{Ring}}$, then

$$i_*([M]) = [R \otimes_R M] = [M].$$

□

Remark 2.9. There is a more concrete interpretation of K_0 involving idempotent matrices. Using this approach, one can show the following properties:

(1) (Morita invariance) If $R \in |\underline{\text{Ring}}|$ and $n \in \mathbb{N}$, then
$$K_0(R) \cong K_0(\text{M}_n(R)).$$
(2) If $R, S \in |\underline{\text{Ring}}|$, then
$$K_0(R \times S) \cong K_0(R) \oplus K_0(S).$$
(3) If $\{\theta_{\alpha,\beta} : R_\alpha \to R_\beta\}_{\alpha \leq \beta}$ is a direct system in $\underline{\text{Ring}}$, then
$$\{(\theta_{\alpha,\beta})_* : K_0(R_\alpha) \to K_0(R_\beta)\}_{\alpha \leq \beta}$$
is a direct system of abelian groups and
$$K_0\left(\varinjlim R_\alpha\right) \cong \varinjlim K_0(R_\alpha).$$

2.2. Commutative Rings.

Remark 2.10. Although theorem 2.8 considers Ring, our focus from now on will be in the subcategory CommRing of commutative rings. In particular, if $R \in |\underline{\text{CommRing}}|$, then the tensor product of R-modules is an R-module and, in fact, $K_0(R)$ is a commutative ring with multiplication (on generators) given by

$$[M][N] := [M \otimes_R N].$$

This multiplication is well-defined since again tensor product distributes over direct sums. The multiplicative identity is $[R]$ since

$$[R][M] = [R \otimes_R M] = [M].$$

Corollary 2.11. *K_0 is a functor from CommRing to itself.*

Proof. Clearly, if $f : R \to S$ is a morphism in CommRing, then $f_*([R]) = [S \otimes_R R] = [S]$ (i.e., f_* preserves the multiplicative identity). Hence we only need to check that f_* preserves multiplication on generators. Let $[M], [N] \in K_0(R)$. Then we have S-module isomorphisms

$$\begin{aligned} & (S \otimes_R M) \otimes_S (S \otimes_R N) \cong ((S \otimes_R M) \otimes_S S) \otimes_R N \\ \cong \quad & (S \otimes_R M) \otimes_R N \cong S \otimes_R (M \otimes_R N), \end{aligned}$$

so

$$\begin{aligned} f_*([M][N]) &= f_*([M \otimes_R N]) = [S \otimes_R (M \otimes_R N)] = [(S \otimes_R M) \otimes_S (S \otimes_R N)] \\ &= [S \otimes_R M][S \otimes_R N] = f_*([M])f_*([N]). \end{aligned}$$

(Note that $K_0(R)$ is not the zero ring by theorem 2.14 below.) □

Definition 2.12. Let $R \in |\underline{\text{CommRing}}|$ and ι be the unique morphism from $\mathbb{Z}$ to R ($\mathbb{Z}$ is an initial object in CommRing). Define the **reduced K_0 group** of R (also called the **projective class group** of R) by

$$\widetilde{K}_0(R) = \frac{K_0(R)}{\text{im}(\iota_*)}.$$

Example 2.13. If $R = \mathbb{Z}$, then $\iota : \mathbb{Z} \to \mathbb{Z}$ is the identity map, so $\text{im}(\iota_*) = K_0(\mathbb{Z})$, giving

$$\widetilde{K}_0(\mathbb{Z}) \cong 0.$$

Theorem 2.14. *For each $R \in |\underline{\text{CommRing}}|$ we have*

$$K_0(R) \cong \mathbb{Z} \oplus \widetilde{K}_0(R)$$

as groups.

Proof. We know $\exists \mathfrak{m} \in \operatorname{Max}(R) \neq \varnothing$ since $R \neq 0$, so there is a (canonical) homomorphism $j : R \to F = R/\mathfrak{m}$ of rings where F is a field. Note that j_* is surjective since $K_0(F) = \langle [F] \rangle \cong \mathbb{Z}$ by example 2.5 and

$$j_*([R]) = [F].$$

Thus we have a short exact sequence of abelian groups

$$0 \to \ker(j_*) \hookrightarrow K_0(R) \xrightarrow{j_*} K_0(F) \quad (\cong \mathbb{Z}) \to 0,$$

so

$$K_0(R) \cong \ker(j_*) \oplus \mathbb{Z}.$$

Hence it's enough to show that the canonical composition

$$\varphi : \ker(j_*) \hookrightarrow K_0(R) \twoheadrightarrow \widetilde{K}_0(R)$$

is an isomorphism. Observe that

$$\operatorname{im}(\iota_*) = \iota_*(\langle [\mathbb{Z}] \rangle) = \langle \iota_*([\mathbb{Z}]) \rangle = \langle [R] \rangle,$$

so

$$\ker(\varphi) = \ker(j_*) \cap \operatorname{im}(\iota_*) = 0$$

since $0 = j_*(n[R]) = n[F]$ implies $n = 0$. Thus φ is injective. Next, let $[M] \in K_0(R)$ be a generator. We know

$$j_*([M]) = n[F]$$

for some $n \in \mathbb{Z}$. Taking $X = [M] - n[R] \in K_0(R)$, we find

$$j_*(X) = j_*([M]) - n j_*([R]) = n[F] - n[F] = 0$$

with

$$\varphi(X) = [M] - n[R] + \langle [R] \rangle = [M] + \langle [R] \rangle,$$

so φ is surjective. Therefore φ is an isomorphism, as needed. □

2.3. **Local Rings, PIDs, and Dedekind Domains.**

Theorem 2.15. *Suppose $R \in |\underline{CommRing}|$ is a local ring or a PID. Then $K_0(R) \cong \mathbb{Z}$ as rings.*

Proof. We've seen that a finitely generated module over a commutative local ring is projective if and only if it's free, and the same statement holds when the ring is a PID by the structure theorem proved in the first-year algebra sequence. Hence $K_0(R) = \langle [R] \rangle = \operatorname{im}(\iota_*)$, so $\widetilde{K}_0(R) \cong 0$. Therefore

$$K_0(R) \cong \mathbb{Z} \oplus \widetilde{K}_0(R) \cong \mathbb{Z} \oplus 0 \cong \mathbb{Z}$$

as groups by theorem 2.14. The ring structure is apparent since if $m, n \in \mathbb{N}$, then

$$(m[R])(n[R]) = [R^m \otimes_R R^n] = [R^{mn}].$$

□

Example 2.16. Fix $1 < n \in \mathbb{N}$ and let

$$n = p_1^{a_1} \cdots p_k^{a_k}$$

be a factorization into distinct primes $p_1, \dots, p_k \in \mathbb{N}$ where $a_1, \dots, a_k \in \mathbb{N}$. Then by the Chinese remainder theorem we have

$$\mathbb{Z}/\langle n \rangle \cong (\mathbb{Z}/\langle p_1^{a_1} \rangle) \times \cdots \times (\mathbb{Z}/\langle p_k^{a_k} \rangle)$$

as rings since the $p_i^{a_i}$ are pairwise coprime. Thus by part (2) of remark 2.9, we have

$$K_0(\mathbb{Z}/\langle n \rangle) \cong K_0(\mathbb{Z}/\langle p_1^{a_1} \rangle) \oplus \cdots \oplus K_0(\mathbb{Z}/\langle p_k^{a_k} \rangle).$$

Moreover, each $\mathbb{Z}/\langle p_i^{a_i} \rangle$ is a local ring with maximal ideal $\langle p_i \rangle / \langle p_i^{a_i} \rangle$, so by theorem 2.15 we get

$$K_0(\mathbb{Z}/\langle n \rangle) \cong \mathbb{Z}^k.$$

Definition 2.17. Recall that a **Dedekind domain** is a Noetherian, integrally closed domain in which every nonzero prime ideal is maximal. For a Dedekind domain D a **fractional ideal** of D is a non-zero finitely-generated D-submodule of the field of fractions of D. Ideals in D may be regarded as fractional ideals of D, and such ideals are called **integral ideals** of D. A fractional ideal of D is called **principal** if it's cyclic as a D-module.

Remark 2.18. The set of fractional ideals of a Dedekind domain forms a free abelian group, which we denote by $F(D)$, generated by the nonzero prime ideals of D under a multiplication given by

$$IJ := \left\{ \sum_{k=1}^{m} i_k j_k : i_k \in I, j_k \in J, \forall k = 1, \dots, m \right\},$$

which is intended to mimic ideal multiplication in rings. The set of principal fractional ideals, which we denote by $P(D)$, forms a subgroup of $F(D)$.

Definition 2.19. For a Dedekind domain D define the **ideal class group** $C(D)$ of D to be the quotient group

$$C(D) = \frac{F(D)}{P(D)}.$$

Example 2.20. For a number field L, its ring of integers

$$\mathcal{O}_L = \{\alpha \in L : \exists \text{monic } f \in \mathbb{Z}[x] \text{ with } f(\alpha) = 0\}$$

is a Dedekind domain and the familiar class group of L studied in algebraic number theory is just $C(\mathcal{O}_L)$. It's a nontrivial (although fundamental) result that the class number $h_L := \#C(\mathcal{O}_L)$ is finite.

Theorem 2.21. *For a Dedekind domain D we have*

$$C(D) \cong \widetilde{K}_0(D)$$

as groups.

Sketch. Each finitely generated projective D-module of rank $n \geq 1$ is isomorphic to $D^{n-1} \oplus I$ for some $I \in F(D)$ where $I \cdot P(D) \in C(D)$ is uniquely determined. In fact, the map

$$K_0(D) \to C(D)$$

given by

$$[D^{n-1} \oplus I] \mapsto I \cdot P(D)$$

is a surjective group homomorphism with kernel $\langle [D] \rangle = \mathrm{im}(\iota_*)$. $\square$

Corollary 2.22. *A Dedekind domain D is a PID if and only if $K_0(D) \cong \mathbb{Z}$ as groups.*

Example 2.23. Consider the number field $L = \mathbb{Q}(\sqrt{15})$. Then the ring of integers $\mathcal{O}_L$ is $\mathbb{Z}[\sqrt{15}]$ (since $15 \equiv 3 \pmod 4$). We claim that $C(\mathbb{Z}[\sqrt{15}]) \cong \mathbb{Z}/\langle 2 \rangle$. We'll use a little basic algebraic number theory. The Minkowski bound is

$$\frac{2!}{2^2}\left(\frac{4}{\pi}\right)^0 \sqrt{|\delta_L|} = \frac{2}{4}\sqrt{4 \cdot 15} = \sqrt{15} < 4,$$

which means every ideal class contains an integral ideal of norm less than 4, so we only need to consider primes lying above $2, 3$. We have

$$2\mathcal{O}_L = \langle 2, 1+\sqrt{15} \rangle^2 \quad \text{and} \quad 3\mathcal{O}_L = \langle 3, \sqrt{15} \rangle^2.$$

Note that $\mathfrak{p} = \langle 2, 1+\sqrt{15} \rangle$ is not principal by the following norm argument: if $\mathfrak{p} = \langle a + b\sqrt{15} \rangle$ for some $a, b \in \mathbb{Z}$, then either

$$2 = a^2 - 15b^2 \equiv a^2 \pmod 3 \qquad \text{or} \qquad 2 = 15b^2 - a^2 \equiv (2a)^2 \pmod 5$$

both of which are contradictions since 2 is not a square modulo 3 nor modulo 5. Thus it suffices to show that $\mathfrak{p}$ and $\mathfrak{q}$ differ by a principal fractional ideal since then there will be exactly 2 equivalence classes of fractional ideals. We'll show that

$$\mathfrak{p} = \left\langle 1 + \frac{1}{3}\sqrt{15} \right\rangle \mathfrak{q}.$$

We have

$$\begin{aligned} \left\langle 1 + \frac{1}{3}\sqrt{15} \right\rangle \mathfrak{q} &= \left\langle 1 + \frac{1}{3}\sqrt{15} \right\rangle \langle 3, \sqrt{15} \rangle = \langle 3+\sqrt{15}, 5+\sqrt{15} \rangle \\ &= \langle 2+1+\sqrt{15}, 2 \cdot 2 + 1 + \sqrt{15} \rangle \subseteq \langle 2, 1+\sqrt{15} \rangle = \mathfrak{p}; \end{aligned}$$

also, the reverse inclusion follows since

$$\begin{aligned} \mathfrak{p} = \langle 2, 1+\sqrt{15} \rangle &= \langle 5+\sqrt{15} - (3+\sqrt{15}), 2(3+\sqrt{15}) - (5+\sqrt{15}) \rangle \\ &\subseteq \langle 3+\sqrt{15}, 5+\sqrt{15} \rangle = \left\langle 1 + \frac{1}{3}\sqrt{15} \right\rangle \mathfrak{q}. \end{aligned}$$

Therefore

$$K_0(\mathbb{Z}[\sqrt{15}]) \cong \mathbb{Z} \oplus (\mathbb{Z}/\langle 2 \rangle).$$

3. K_1

3.1. Definitions and Basic Properties.

Remark 3.1. Let $m \leq n$ be positive integers and pick $R \in |\underline{\text{Ring}}|$. Then there is a canonical group homomorphism

$$\phi_{m,n} : \mathrm{GL}_m(R) \to \mathrm{GL}_n(R)$$

given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_{n-m} \end{pmatrix}$$

where I_{n-m} is the $(n-m) \times (n-m)$ identity matrix and the 0's are zero matrices of appropriate sizes. These morphisms form a directed system of groups.

Definition 3.2. For $R \in |\underline{\text{Ring}}|$ define

$$\mathrm{GL}(R) = \varinjlim \mathrm{GL}_n(R)$$

to be the direct limit of the above directed system viewed as a disjoint union where for $m \leq n$ we identify $A \in \mathrm{GL}_m(R)$ and $B \in \mathrm{GL}_n(R)$ if and only if $B = \phi_{m,n}(A)$. Also, take

$$\mathrm{E}(R) = \varinjlim \mathrm{E}_n(R) \subseteq \mathrm{GL}(R)$$

where for each $n \in \mathbb{N}$ the set $\mathrm{E}_n(R) \subseteq \mathrm{GL}_n(R)$ is the subgroup generated by the $n \times n$ **elementary matrices**, i.e., matrices with 1's along the main diagonal and at most one nonzero entry off the main diagonal. For $i, j \in \{1, \ldots, n\}$ with $i \neq j$, let $e_{i,j}(a)$ denote the $n \times n$ elementary matrix with 1's along its main diagonal, a in the (i,j)th entry, and 0's elsewhere.

Lemma 3.3. *Let $R \in |\underline{Ring}|$. Then every triangular matrix in $GL_n(R)$ with 1's along its main diagonal is in $E_n(R)$ and*

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \in E_{2n}(R)$$

whenever $A \in GL_n(R)$.

Proof. See [4]. □

Lemma 3.4 (Whitehead)**.** *For $R \in |\underline{Ring}|$ we have that $E(R)$ is the commutator subgroup of $GL(R)$.*

Sketch. For each $a \in R$ and each triple i, j, k of distinct positive integers we have

$$\begin{aligned}[e_{i,k}(a), e_{k,j}(1)] &= e_{i,k}(a)e_{k,j}(1)e_{i,k}(a)^{-1}e_{k,j}(1)^{-1} \\ &= e_{i,k}(a)e_{k,j}(1)e_{i,k}(-a)e_{k,j}(-1) = e_{i,j}(a)\end{aligned}$$

where the last equality is best seen by starting with the identity matrix and then performing the appropriate sequence of elementary row operations corresponding to matrices of the form $e_{l,m}(b)$ (i.e., adding b times the mth row to the lth row). Hence

$$\mathrm{E}(R) = [\mathrm{E}(R), \mathrm{E}(R)] \subseteq [\mathrm{GL}(R), \mathrm{GL}(R)].$$

Next, if $A, B \in \mathrm{GL}_n(R)$, then

$$\begin{pmatrix} AB & 0 \\ 0 & (AB)^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} [A, B] & 0 \\ 0 & 1 \end{pmatrix},$$

but by 3.3

$$\begin{pmatrix} C & 0 \\ 0 & C^{-1} \end{pmatrix} \in \mathrm{E}_{2n}(R)$$

whenever $C \in \mathrm{GL}_n(R)$, so $[A, B] \in \mathrm{E}(R)$. Thus we also get the opposite inclusion

$$[\mathrm{GL}(R), \mathrm{GL}(R)] \subseteq \mathrm{E}(R).$$

□

Definition 3.5. For each $R \in |\underline{\mathrm{Ring}}|$ define the **Whitehead group** of R by

$$K_1(R) = \frac{\mathrm{GL}(R)}{\mathrm{E}(R)}.$$

Remark 3.6. Let $f : R \to S$ be a morphism in $\underline{\mathrm{Ring}}$. Then there are canonical group homomorphisms $f_n : \mathrm{GL}_n(R) \to \mathrm{GL}_n(S)$ given by $(a_{i,j}) \mapsto (f(a_{i,j}))$. These maps are indeed homomorphisms since

$$\begin{aligned} f_n((a_{i,j})(b_{i,j})) &= f_n\left(\left(\sum_{k=1}^n a_{i,k}b_{k,j}\right)\right) = \left(f\left(\sum_{k=1}^n a_{i,k}b_{k,j}\right)\right) \\ &= \left(\sum_{k=1}^n f(a_{i,k})f(b_{k,j})\right) = (f(a_{i,j}))(f(b_{i,j})) \\ &= f_n((a_{i,j}))f_n((b_{i,j})). \end{aligned}$$

Moreover, if $m \leq n$, then it's clear that $f_n \circ \phi^R_{m,n} = \phi^S_{m,n} \circ f_m$, so by the UMP of direct limits we have well-defined homomorphism $\mathrm{GL}(R) \to \mathrm{GL}(S)$ given by sending the equivalence class $[A]$ of $A \in \mathrm{GL}_n(R)$ to $[f_n(A)]$. This in turn induces a homomorphism on the maximal abelian quotients

$$f^1_* : K_1(R) \to K_1(S)$$

given by

$$[A]\mathrm{E}(R) \to [f_n(A)]\mathrm{E}(S).$$

Theorem 3.7. *$K_1 : \underline{Ring} \to \underline{Ab}$ is a functor where K_1 sends morphisms f to f^1_*.*

Proof. Suppose $f : R \to S$ and $g : S \to T$ are morphisms in $\underline{\text{Ring}}$. Then for $A \in \mathrm{GL}_n(R)$ we have

$$\begin{aligned}(g \circ f)^1_*([A]\mathrm{E}(R)) &= [(g \circ f)_n(A)]\mathrm{E}(T) = [g_n(f_n(A))]\mathrm{E}(T) \\ &= g^1_*([f_n(A)]\mathrm{E}(S)) = g^1_*(f^1_*([A]\mathrm{E}(R))).\end{aligned}$$

Also, if $i : R \to R$ is an identity morphism in $\underline{\text{Ring}}$, then

$$i^1_*([A]\mathrm{E}(\mathrm{R})) = [i_n(A)]\mathrm{E}(R) = [A]\mathrm{E}(R).$$

□

Remark 3.8. As for K_0, we have the following properties for K_1:

(1) (Morita invariance) If $R \in |\underline{\text{Ring}}|$ and $n \in \mathbb{N}$, then
$$K_1(R) \cong K_1(\mathrm{M}_n(R)).$$
(2) If $R, S \in |\underline{\text{Ring}}|$, then
$$K_1(R \times S) \cong K_1(R) \oplus K_1(S).$$
(3) If $\{\theta_{\alpha,\beta} : R_\alpha \to R_\beta\}_{\alpha \le \beta}$ is a direct system in $\underline{\text{Ring}}$, then
$$\{(\theta_{\alpha,\beta})_* : K_1(R_\alpha) \to K_1(R_\beta)\}_{\alpha \le \beta}$$
is a direct system of abelian groups and
$$K_1\left(\varinjlim R_\alpha\right) \cong \varinjlim K_1(R_\alpha).$$

3.2. **Commutative Rings.**

Definition 3.9. For $R \in |\underline{\text{CommRing}}|$ define

$$\mathrm{SL}(R) = \varinjlim \mathrm{SL}_n(R)$$

and

$$SK_1(R) = \frac{\mathrm{SL}(R)}{\mathrm{E}(R)}.$$

Example 3.10. Suppose $F \in |\underline{\text{CommRing}}|$ is a field and let $A \in \mathrm{GL}_n(F)$. Using only the elementary row operation of adding a multiple of one row to another we may reduce A to a diagonal matrix $D = E_1 A$ where $E_1 \in \mathrm{GL}_n(F)$ is a product of matrices of the form $e_{i,j}(a)$. Write $D = \mathrm{diag}(d_1, \ldots, d_n)$. Then each $d_i \neq 0$ since $\det(D) = \det(E_1 A) = \det(A) \neq 0$, so, in particular,

$$\begin{pmatrix} d_n & 0 \\ 0 & d_n^{-1} \end{pmatrix} \in \mathrm{E}_2(F)$$

by lemma 3.3, whence

$$D_n = \mathrm{diag}(1, \ldots, 1, d_n, d_n^{-1}) \in \mathrm{E}_n(F).$$

Likewise,

$$D_{n-1} = \mathrm{diag}(1, \ldots, 1, d_{n-1}d_n, d_{n-1}^{-1}d_n^{-1}, 1) \in \mathrm{E}_n(F)$$

and so on. Thus

$$E_2 = D_1 D_2 \cdots D_n \in \mathrm{E}_n(F)$$

with

$$E_2E_1A = E_2D = \operatorname{diag}(d, 1, \ldots, 1)$$

where

$$d = d_1d_2\cdots d_n = \det(D) = \det(A).$$

Thus $A \in \mathrm{SL}_n(F) \Rightarrow A = E_1^{-1}E_2^{-1} \in \mathrm{E}_n(F)$, so $\mathrm{SL}_n(F) \subseteq \mathrm{E}_n(F)$, giving $\mathrm{E}_n(F) = \mathrm{SL}_n(F)$ and $SK_1(F) \cong 0$.

Theorem 3.11. *For each $R \in |\underline{CommRing}|$ we have*

$$K_1(R) \cong R^\times \oplus SK_1(R).$$

Proof. If $m \leq n$, the determinant maps $\det_i : \mathrm{GL}_n(R) \to R^\times$ satisfy

$$(\det{}_n \circ \phi_{m,n})(A) = \det{}_n\left(\begin{pmatrix} A & 0 \\ 0 & I_{n-m}\end{pmatrix}\right) = \det{}_m(A)\det{}_{n-m}(I_n - m) = \det{}_m(A),$$

so by the UMP of direct limits there's a homomorphism $\det : \mathrm{GL}(R) \to R^\times$ given by sending the equivalence class $[A]$ of $A \in \mathrm{GL}_n(R)$ to $\det_n(A)$. This induces a map $\det^1 : K_1(R) \to R^\times$ since $\mathrm{E}(R) \subseteq \mathrm{SL}(R) \subseteq \ker(\det)$. We also have a canonical injection

$$\iota^1 : R^\times = \mathrm{GL}_1(R) \rightarrowtail \mathrm{GL}(R) \twoheadrightarrow K_1(R),$$

so we may consider the endomorphism $\psi := \iota^1 \circ \det^1$ of $K_1(R)$. Note that ψ is idempotent since for $A \in \mathrm{GL}_n(R)$

$$\begin{aligned}\psi^2([A]\mathrm{E}(R)) &= \psi([\det{}_n(A)]\mathrm{E}(R)) = \iota^1(\det{}_1(\det{}_n(A))) \\ &= \iota^1(\det{}_n(A)) = [\det{}_n(A)]\mathrm{E}(R) \\ &= \psi([A]\mathrm{E}(R)).\end{aligned}$$

Therefore

$$K_1(R) \cong \operatorname{im}(\psi) \oplus \ker(\psi) \cong R^\times \oplus SK_1(R)$$

since

$$[A]\mathrm{E}(R) \in \ker(\psi) \Leftrightarrow [\det{}_n(A)] \in \mathrm{E}(R) \Leftrightarrow \det{}_n(A) = 1 \Leftrightarrow A \in \mathrm{SL}_n(R) \Leftrightarrow [A] \in \mathrm{SL}(R).$$

□

Corollary 3.12. *If $F \in |\underline{CommRing}|$ is a field, then*

$$K_1(F) \cong F^\times.$$

Remark 3.13. In fact, we have the following theorem (whose proof is omitted), which is a vast generalization of corollary 3.12.

Theorem 3.14. *If $R \in |\underline{CommRing}|$ is a local ring or Euclidean domain, then*

$$K_1(R) \cong R^\times.$$

Example 3.15. Fix $1 < n \in \mathbb{N}$ and let

$$n = p_1^{a_1} \cdots p_k^{a_k}$$

be a factorization into distinct primes $p_1, \ldots, p_k \in \mathbb{N}$ where $a_1, \ldots, a_k \in \mathbb{N}$. Then by the Chinese remainder theorem we have

$$\mathbb{Z}/\langle n \rangle \cong (\mathbb{Z}/\langle p_1^{a_1} \rangle) \times \cdots \times (\mathbb{Z}/\langle p_k^{a_k} \rangle)$$

as rings since the $p_i^{a_i}$ are pairwise coprime. Thus by part (2) of remark 3.8, we have

$$K_1(\mathbb{Z}/\langle n \rangle) \cong K_1(\mathbb{Z}/\langle p_1^{a_1} \rangle) \oplus \cdots \oplus K_1(\mathbb{Z}/\langle p_k^{a_k} \rangle).$$

Moreover, each $\mathbb{Z}/\langle p_i^{a_i} \rangle$ is a local ring with maximal ideal $\langle p_i \rangle / \langle p_i^{a_i} \rangle$, so by theorem 3.14 we get

$$K_1(\mathbb{Z}/\langle n \rangle) \cong (\mathbb{Z}/\langle p_1^{a_1} \rangle)^\times \oplus \cdots \oplus (\mathbb{Z}/\langle p_k^{a_k} \rangle)^\times \cong (\mathbb{Z}/\langle n \rangle)^\times.$$

We can be more specific. Recall that if p_i is odd, then

$$(\mathbb{Z}/\langle p_i^{a_i} \rangle)^\times \cong \mathbb{Z}/\langle p_i^{a_i - 1}(p_i - 1) \rangle,$$

while

$$(\mathbb{Z}/\langle 2^a \rangle)^\times \cong \begin{cases} 0 & \text{if } a = 1 \\ \mathbb{Z}/\langle 2 \rangle & \text{if } a = 2 \\ (\mathbb{Z}/\langle 2 \rangle) \oplus (\mathbb{Z}/\langle 2^{a-2} \rangle) & \text{if } a \geq 3. \end{cases}$$

Example 3.16. Consider $R = F[x]/\langle x^n \rangle$ for some field F and some $n \in \mathbb{N}$. Then R is local with maximal ideal $\langle x \rangle / \langle x^n \rangle$ (since $F[x]$ is a PID), so

$$K_1(R) \cong R^\times$$

by 3.14. Note that $f(x) + \langle x^n \rangle$ is a unit in $R \Leftrightarrow (f(x), x^n) = 1 \Leftrightarrow (f(x), x) = 1 \Leftrightarrow x \nmid f(x) \Leftrightarrow f(0) \neq 0$, so

$$R^\times = \{f(x) + \langle x^n \rangle : f(0) \neq 0\} = \{a_1 x^{n-1} + \cdots + a_n + \langle x^n \rangle : a_n \neq 0\}.$$

3.3. **The Mayer-Vietoris Sequence and Rim's Theorem.**

Theorem 3.17 (Mayer-Vietoris Sequence)**.** *Suppose the following diagram of ring homomorphisms is commutative*

$$\begin{array}{ccc} R & \xrightarrow{i} & S \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} \\ T & \xrightarrow{l} & U \end{array}$$

Also, assume k and l are surjective and that if $k(b) = l(c)$, then there is a unique $a \in R$ such that $i(a) = b$ and $j(a) = c$. Then there is an exact sequence

$$\begin{array}{lcl} K_1(R) & \xrightarrow{(i_*^1, j_*^1)} & K_1(S) \oplus K_1(T) \xrightarrow{k_*^1 - l_*^1} K_1(U) \xrightarrow{\partial} \\ K_0(R) & \xrightarrow{(i_*, j_*)} & K_0(S) \oplus K_0(T) \xrightarrow{k_* - l_*} K_0(U) \end{array}$$

for some connecting homomorphism ∂.

Proof. See [3]. ∎

Theorem 3.18 (Rim)**.** *Let p be prime and consider the square of natural ring homomorphisms*

$$\begin{array}{ccc} \mathbb{Z}\mu_p & \xrightarrow{i} & \mathbb{Z}[\zeta_p] \\ \downarrow j & & \downarrow k \\ \mathbb{Z} & \xrightarrow{l} & \mathbb{F}_p \end{array}$$

where $\zeta_p = e^{2\pi i/p}$, μ_p = group of pth roots of unity, and

$$i(\zeta_p) = \zeta_p, \qquad j(\zeta_p) = 1, \qquad k(\zeta_p) = 1.$$

Then

$$i_* : K_0(\mathbb{Z}\mu_p) \to K_0(\mathbb{Z}[\zeta_p])$$

is an isomorphism.

Proof. We'll first check that the square commutes and satisfies the hypotheses of theorem 3.17. Commutativity follows by observing that

$$k(i(\zeta_p)) = k(\zeta_p) = 1 = l(1) = l(j(\zeta_p)).$$

Clearly, l and k are surjective since $\mathbb{F}_p$ is generated by 1 as an abelian group. Next, suppose $k(b) = l(c)$. Write

$$b = \sum_{n=0}^{p-2} b_n \zeta_p^n$$

for some $b_0, \ldots, b_{p-2} \in \mathbb{Z}$. Then

$$\sum_{n=0}^{p-2} b_n + p\mathbb{Z} = k(b) = l(c) = c + p\mathbb{Z},$$

so

$$\sum_{n=0}^{p-2} b_n = c - mp$$

for some $m \in \mathbb{Z}$. Suppose there exists an $a \in \mathbb{Z}\mu_p$ such that $i(a) = b$ and $j(a) = c$. Write

$$a = \sum_{n=0}^{p-1} a_n \zeta_p^n$$

for some $a_0, \ldots, a_{p-1} \in \mathbb{Z}$. Then

$$\sum_{n=0}^{p-2} b_n \zeta_p^n = i(a) = \sum_{n=0}^{p-1} a_n \zeta_p^n,$$

so

$$a_{p-1}\zeta_p^{p-1} + (a_{p-2} - b_{p-2})\zeta_p^{p-2} + \cdots + (a_0 - b_0) = 0.$$

Thus in $\mathbb{Q}[x]$ we have

$$x^{p-1} + x^{p-2} + \cdots + 1 | a_{p-1}x^{p-1} + (a_{p-2} - b_{p-2})x^{p-2} + \cdots + (a_0 - b_0),$$

so

$$a_{p-1} = a_{p-2} - b_{p-2} = \ldots = a_0 - b_0.$$

Hence

$$a_{p-1}p = a_{p-1} + \sum_{n=0}^{p-2}(a_n - b_n) = j(a) - (c - mp) = c - c + mp = mp,$$

so

$$m = a_{p-1} = a_{p-2} - b_{p-2} = \ldots = a_0 - b_0.$$

Therefore

$$a = m\zeta_p^{p-1} + \sum_{n=0}^{p-2}(b_n + m)\zeta_p^n;$$

moreover, we may check that for this value of a we have

$$i(a) = m\zeta_p^{p-1} + \sum_{n=0}^{p-2}(b_n + m)\zeta_p^n = \sum_{n=0}^{p-2} b_n\zeta_p^n + m\sum_{n=0}^{p-1}\zeta_p^n = b + m \cdot 0 = b$$

and

$$j(a) = m + \sum_{n=0}^{p-2}(b_n + m) = mp + \sum_{n=0}^{p-2} b_n = mp + c - mp = c.$$

Hence we have an exact sequence

$$K_1(\mathbb{Z}\mu_p) \xrightarrow{(i_*^1, j_*^1)} K_1(\mathbb{Z}[\zeta_p]) \oplus K_1(\mathbb{Z}) \xrightarrow{k_*^1 - l_*^1} K_1(\mathbb{F}_p) \xrightarrow{\partial}$$
$$K_0(\mathbb{Z}\mu_p) \xrightarrow{(i_*, j_*)} K_0(\mathbb{Z}[\zeta_p]) \oplus K_0(\mathbb{Z}) \xrightarrow{k_* - l_*} K_0(\mathbb{F}_p).$$

Let $x \in K_0(\mathbb{Z}[\zeta_p])$. Note that $l_* : K_0(\mathbb{Z}) \to K_0(\mathbb{F}_p)$ is an isomorphism since $l_*(1) = 1$ and $K_0(\mathbb{Z}) \cong \mathbb{Z} \cong K_0(\mathbb{F}_p)$, so since

$$(k_* - l_*)(x, l_*^{-1}(k_*(x))) = 0$$

we have

$$(x, l_*^{-1}(k_*(x))) \in \ker(k_* - l_*) = \operatorname{im}((i_*, j_*)).$$

Thus i_* is surjective. Now suppose $x \in \ker(i_*)$. Then

$$(x, 0) \in \ker((i_*, j_*)) = \operatorname{im}(\partial),$$

so it's enough to show $\partial = 0$ since then $x = 0$ and i_* is injective. Thus it suffices to prove

$$K_1(\mathbb{F}_p) = \ker(\partial) = \operatorname{im}(k_*^1 - l_*^1).$$

We'll show $k_*^1 : K_1(\mathbb{Z}[\zeta_p]) \to K_1(\mathbb{F}_p)$ is surjective. By corollary 3.12 we have

$$K_1(\mathbb{F}_p) = \{[m + p\mathbb{Z}]\mathrm{E}(\mathbb{F}_p) | m + p\mathbb{Z} \in \mathbb{F}_p^\times = \mathrm{GL}_1(\mathbb{F}_p)\}.$$

Now fix $m \in \{1, \ldots, p-1\}$ and consider

$$u := \frac{\zeta_p^m - 1}{\zeta_p - 1} = 1 + \zeta_p + \cdots \zeta_p^{m-1} \in \mathbb{Z}[\zeta_p].$$

Since $(m, p) = 1$ there is an $n \in \{1, \ldots, p-1\}$ such that $mn \equiv 1 \pmod p$, so

$$u^{-1} = \frac{\zeta_p^{mn} - 1}{\zeta_p^m - 1} = 1 + \zeta_p^m + \cdots + \zeta_p^{m(n-1)} \in \mathbb{Z}[\zeta_p].$$

Therefore $u \in \mathbb{Z}[\zeta_p]^\times = \mathrm{GL}_1(\mathbb{Z}[\zeta_p])$ with

$$k_*^1([u]\mathrm{E}(\mathbb{Z}[\zeta_p])) = [k(u)]\mathrm{E}(\mathbb{F}_p) = [m + p\mathbb{Z}]\mathrm{E}(\mathbb{F}_p),$$

whence k_*^1 is surjective as claimed. □

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