Brauer's Theorems and the Meromorphicity of L-Functions

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Outline

$$\begin{array}{ccccccc} & & & & & \overline{\mathbb{Q}} & \\ & & & & & | & \\ 0 \neq & \mathfrak{P} & \overset{\text{prime}}{\subseteq} & \mathcal{O}_E & \subseteq & E & \\ & | & & | & & | & \text{finite, normal} \\ 0 \neq & \mathfrak{p} & \overset{\text{prime}}{\subseteq} & \mathcal{O}_F & \subseteq & F & \\ & \| & & & & | & \text{finite} \\ & \mathfrak{P} \cap \mathcal{O}_F & & & & \mathbb{Q} & \end{array}$$

Let $G = \mathrm{Gal}(E/F)$ and fix a $\mathbb{C}$-representation

$$T : G \to \mathrm{GL}(V)$$

with character $\chi : g \mapsto \mathrm{tr}(\widehat{T}(g))$.

Then $V^{I(\mathfrak{P})}$ is $T|_{G(\mathfrak{P})}$-invariant where $I(\mathfrak{P}) =$ inertia & $G(\mathfrak{P}) =$ decomposition, so $\exists$rep

$$T_{\mathfrak{P}} : G(\mathfrak{P})/I(\mathfrak{P}) \to \mathrm{GL}(V^{I(\mathfrak{P})})$$

On the other hand, E/F is Galois, so $\exists$isomorphism

$$\frac{G(\mathfrak{P})}{I(\mathfrak{P})} \to \mathrm{Gal}\left(\frac{\mathcal{O}_E/\mathfrak{P}}{\mathcal{O}_F/\mathfrak{p}}\right) = \langle \mathrm{Frob}_{N(\mathfrak{p})} \rangle$$

with

$$\sigma_{\mathfrak{P}} := \left[\frac{E/F}{\mathfrak{P}}\right] \mapsto \mathrm{Frob}_{N(\mathfrak{p})}$$

where $\left[\frac{E/F}{\mathfrak{P}}\right]$ is the Artin symbol of E/F at $\mathfrak{P}$.

Fact

If S is an equivalent representation of T and $\mathfrak{Q} \neq 0$ is another prime in $\mathcal{O}_E$ lying over $\mathfrak{p}$, then for all $s \in \mathbb{C}$

$$\det\left(I - \widehat{T}_{\mathfrak{P}}\left(\sigma_{\mathfrak{P}}\right) N(\mathfrak{p})^{-s}\right) = \det\left(I - \widehat{S}_{\mathfrak{Q}}\left(\sigma_{\mathfrak{Q}}\right) N(\mathfrak{p})^{-s}\right).$$

Suppose

$$\mathfrak{p}\mathcal{O}_E = \mathfrak{P}_1^e \cdots \mathfrak{P}_g^e$$

is a prime decomposition in $\mathcal{O}_K$.

If $\mathfrak{p}$ is unramified in E/F (i.e., $e = 1$, true for all but finitely many $\mathfrak{p}$), then $I(\mathfrak{P})$ is trivial (and conversely), so we may view

$$\sigma_{\mathfrak{P}} \in G(\mathfrak{P}) \subseteq G$$

and

$$T_{\mathfrak{P}} = T|_{G(\mathfrak{P})}.$$

Definition

We now define the Artin L-function of χ as

$$L(s, \chi, E/F) = \prod_{0 \neq \mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_F)} \det(I - \widehat{T}_{\mathfrak{P}}(\sigma_{\mathfrak{P}}) N(\mathfrak{p})^{-s})^{-1}.$$

We write $L(s, \chi)$ for $L(s, \chi, E/F)$ if the extension is understood.

Note: When $\dim(V^{I(\mathfrak{P})}) = 0$, the factor corresponding to $\mathfrak{p}$ is 1.

Example

Let $E = \mathbb{Q}(e^{2\pi i/3}, \sqrt[3]{2})$ and $F = \mathbb{Q}$. Then we may identify $G = S_3$. Consider $\widehat{T} : G \to \mathrm{GL}(\mathbb{C}^{3\times 3})$ given by

$$(1\ \ 2) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$(1\ \ 2\ \ 3) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Example (continued)

For $\mathfrak{p} = 3\mathbb{Z}$, we have $I(\mathfrak{P}) = G(\mathfrak{P}) = G$, so $V^{I(\mathfrak{P})} = \mathbb{C}\begin{pmatrix}1\\1\\1\end{pmatrix}$,

giving

$$\widehat{T}_{\mathfrak{P}}(\sigma_{\mathfrak{P}}) = 1$$

Therefore the local factor at $\mathfrak{p}$ is

$$\frac{1}{1-3^{-s}}$$

Example (continued)

For $\mathfrak{p} = 5\mathbb{Z}$, we have $I(\mathfrak{P}) = \{1\}$ and $G(\mathfrak{P}) = \langle(1 \;\; 2)\rangle$, so $V^{I(\mathfrak{P})} = V$ and $\sigma_{\mathfrak{P}} = (1 \;\; 2)I(\mathfrak{P})$, giving

$$\widehat{T}_{\mathfrak{P}}(\sigma_{\mathfrak{P}}) = \widehat{T}((1 \;\; 2)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore the local factor at $\mathfrak{p}$ is

$$\begin{vmatrix} 1 & -5^{-s} & 0 \\ -5^{-s} & 1 & 0 \\ 0 & 0 & 1-5^{-s} \end{vmatrix}^{-1} = \frac{1}{(1-5^{-s})(1-5^{-2s})}$$

Example (continued)

Similarly, it can be shown that the local factor at $3\mathbb{Z}$ is

$$\frac{1}{1-3^{-s}},$$

and the local factor at $7\mathbb{Z}$ is

$$\frac{1}{1-7^{-3s}}$$

- $L(s, \chi)$ is analytic in s on the half plane where $\Re(s) > 1$
- If $\chi = \chi_1 + \chi_2$ for some characters χ_1, χ_2, then

$$L(s, \chi) = L(s, \chi_1) L(s, \chi_2)$$

- In particular, if $\chi_1, \ldots, \chi_r$ are the irreducible characters, then

$$L(s, \chi) = \prod_{j=1}^{r} L(s, \chi_j)^{(\chi, \chi_j)},$$

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$$(\chi, \psi) = \frac{1}{|G|} \sum_{g \in G} \chi(g) \psi(g^{-1})$$

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If $\deg(\chi) = \chi(1) = 1$ (always true when both G is abelian and $\chi \in \mathrm{Irr}(G)$ = set of irreducible characters in G), then

$$L(s,\chi) = \prod_{0 \neq \mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_F)} \frac{1}{1 - \chi(\mathfrak{p}) N(\mathfrak{p})^{-s}} = \sum_{\text{ideals } J \neq 0} \frac{\chi(J)}{N(J)^s}$$

where

$$\chi(J) = \prod_{\mathfrak{p}^e || J} \widehat{T}_{\mathfrak{P}}(\sigma_{\mathfrak{P}})^e \in \mathbb{C}$$

Example

Let $E = \mathbb{Q}(\sqrt{2})$, $F = \mathbb{Q}$. Then $G = \{1, g\}$ where $g(\sqrt{2}) = -\sqrt{2}$.

Consider $\chi = \widehat{T} : G \to \mathbb{C}^\times = \mathrm{GL}(\mathbb{C})$ given by $g \mapsto -1$.

For $\mathfrak{p} = 2\mathbb{Z}$, we have $\mathfrak{p}\mathcal{O}_E = (\sqrt{2}\mathcal{O}_E)^2$, so $I(\mathfrak{P}) = G(\mathfrak{P}) = G$, giving $V^{I(\mathfrak{P})} = \{0\}$. Thus the local factor at $\mathfrak{p}$ is 1.

Example (continued)

Now suppose $p \in \mathbb{N}$ is an odd prime. Then $I(\mathfrak{P}) = \{1\}$. If $(2|p) = 1$, then $G(\mathfrak{P}) = \{1\}$, and if $(2|p) = -1$, then $G(\mathfrak{P}) = G$.

Thus the local factor at $\mathfrak{p} = p\mathbb{Z}$ is

$$\frac{1}{1 - (2|p)p^{-s}}$$

Example (continued)

Therefore

$$L(s,\chi) = \prod_{\text{odd primes } p\in\mathbb{N}} \frac{1}{1-(2|p)p^{-s}} = \sum_{\text{odd } n\in\mathbb{N}} \frac{(2|n)}{n^s}$$

$$= \sum_{\text{odd } n\in\mathbb{N}} \frac{(-1)^{(n^2-1)/8}}{n^s}$$

For the trivial character $1_G \in \mathrm{Irr}(G)$, we have

$$L(s, 1_G) = \sum_{\text{ideals } J \neq 0} \frac{1}{N(J)^s} = \zeta_F(s),$$

which is the Dedekind zeta function of the number field F.

Also, if $F = \mathbb{Q}$ and $E = \mathbb{Q}(e^{2\pi i/m})$ while $\chi \in \mathrm{Irr}(G)$ is arbitrary, then $G \cong (\mathbb{Z}/m\mathbb{Z})^\times$ is abelian and

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n\mathbb{Z})}{|\mathbb{Z}/n\mathbb{Z}|^s},$$

which is a Dirichlet L-function.

In particular, if $m = 1$ then $E = F = \mathbb{Q}$, so $\mathrm{Irr}(G) = \{1_G\}$ and

$$L(s, 1_G) = \sum_{n=1}^{\infty} \frac{1}{|\mathbb{Z}/n\mathbb{Z}|^s} = \zeta(s),$$

which is the Riemann zeta function.

In the abelian case, L-functions have been used in the proofs of

- prime number theorem (Riemann zeta function)
- prime ideal theorem (Dedekind zeta functions)
- primes in arith. prog. theorem (Dirichlet L-functions)

In the nonabelian case, L-functions can be used to prove the following theorem

Theorem (Chebotarev, 1922)

Let E, F, G be as above and C be a conjugacy class in G. Then the set of unramified primes $\mathfrak{p} \neq 0$ in $\mathcal{O}_F$ such that $\sigma_{\mathfrak{P}} \in C$ for some prime $\mathfrak{P}$ in $\mathcal{O}_E$ with $\mathfrak{P} \cap \mathcal{O}_F = \mathfrak{p}$ has density $|C|/|G|$.

Now let G be an arbitrary finite group.

Definition

Take $\mathrm{Char}(G) = \mathbb{Z}$-module generated by $\mathrm{Irr}(G)$, and take $\mathrm{cf}(G) =$ set of functions from G to $\mathbb{C}$ which are constant on conjugacy classes (i.e., class functions).

Then $\mathrm{cf}(G) \supseteq \mathrm{Char}(G)$ is a $\mathbb{C}$-algebra.

Definition

Suppose $H \leq G$ and $\varphi \in \text{cf}(H)$. The induced class function $\varphi^G \in \text{cf}(G)$ is given by

$$\varphi^G(x) = \frac{1}{|H|} \sum_{g \in G} \dot{\varphi}(g^{-1}xg)$$

where $\dot{\varphi}$ is 0 on $G \backslash H$ and φ on H.

- Induction is a $\mathbb{C}$-linear transformation from cf(H) to cf(G)
- Induction is transitive
- If φ is a character of $H \leq G$, then φ^G is a character of G

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- (Frobenius Reciprocity) If $\varphi \in \mathrm{cf}(H)$ and $\theta \in \mathrm{cf}(G)$, then

$$(\varphi^G, \theta)_G = (\varphi, \theta|_H)_H$$

- In the context of $G = \mathrm{Gal}(E/F)$, let K be a field between F and E. Then if φ is a character of $H := \mathrm{Gal}(E/K)$, we have

$$L(s, \varphi^G, E/F) = L(s, \varphi, E/K)$$

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Definition

An elementary (resp. hyper-elementary) subgroup of G is a subgroup of the form $C \times P$ (resp. $C \rtimes P$) where P is a p-group for some prime p and C is cyclic with $(|C|, p) = 1$.

Notation: (1) Let $\mathcal{E}$ (resp. $\mathcal{H}$) be the set of elementary (resp. hyper-elementary) subgroups of G.

(2) Let $\mathcal{R}$ be the ring of all class functions of G which restrict to generalized characters on H for all $H \in \mathcal{E}$.

(3) Let $\mathcal{I}$ be the $\mathbb{Z}$-module generated by all characters of the form λ^G where λ is a linear character of some $H \in \mathcal{E}$.

Theorem (Brauer's Induction and Characterization Theorem)

We have $\mathcal{I} = Char(G) = \mathcal{R}$.

- The first equality says, in particular, that every irreducible character can be written as the $\mathbb{Z}$-linear combination of characters induced from linear characters of elementary subgroups.
- The second equality says that a class function of G is a generalized character of G if and only if it restricts to generalized characters for all elementary subgroups.

Sketch of Proof of Brauer's Theorems.

STEP 1: $\mathcal{I} \subseteq \text{Char}(G) \subseteq \mathcal{R}$ and $\mathcal{I}$ is an ideal in $\mathcal{R}$

STEP 2: $1_G = \sum_{H \in \mathcal{H}} a_H (1_H)^G$ with $a_H \in \mathbb{Z}$ by Banaschewski

STEP 3: Assume $G = C \rtimes P \in \mathcal{H}$ by transitivty of induction

STEP 4: If $G \in \mathcal{E}$, we're done; otherwise $G \neq N := N_G(P) \in \mathcal{E}$

Sketch Continued.

STEP 5: $(1_N^G, 1_G)_G = (1_N, 1_N)_N = 1$ by Frobenius

STEP 6: $1_G = 1_N^G - \sum_{1_G \neq \chi_i} a_i \chi_i$ with $a_i \in \mathbb{Z}$

STEP 7: Suppose $\chi_i(1) = 1$ for some i

STEP 8: $P \in \mathrm{Syl}_p(K)$ and $N \leq K$ with $K := \ker(\chi_i) \trianglelefteq G$

STEP 9: $G = KN \subseteq K \neq G$ by Frattini □

Theorem (Meromorphicity of L-Functions)

In the context of $G = Gal(E/F)$, there are intermediate fields $K_1, \ldots, K_m$ of E/F and integers n_j such that

$$L(s, \chi, E/F) = \prod_{j=1}^{m} L(s, \xi_j, E/K_j)^{n_j}$$

for some linear characters ξ_j of $Gal(E/K_j)$; in particular, Artin L-functions are meromorphic.

Example

Let $E = \mathbb{Q}(e^{2\pi i/3}, \sqrt[3]{2})$, $F = \mathbb{Q}$, $\widehat{T}$ be as in the first example. Set $H = \langle(2\ \ 3)\rangle = \mathrm{Stab}_G(1)$ and take K to be the fixed field of H. Then inducing the trivial character on H gives us the permutation character χ of $\widehat{T}$. Thus

$$L(s, \chi, E/F) = L(s, 1_H, E/K) = \zeta_K(s)$$

is actually well-understood.

Proof of Meromorphicity.

We may assume $\chi \in \mathrm{Irr}(G)$. Then by Brauer's induction theorem there are elementary subgroups $H_1, \ldots, H_m$ and linear characters λ_j of H_j with

$$\chi = \sum_{j=1}^{m} n_j \lambda_j^G$$

for some $n_1, \ldots, n_m \in \mathbb{Z}$. The result now follows from Galois correspondence and the properties of L-functions above. □

Conjecture (Artin)

If $\chi \in Irr(G)$, then $L(s, \chi)$ is an entire function unless χ is the trivial character, in which case the only pole of $L(s, \chi)$ is a simple pole at $s = 1$.

Note: Artin's conjecture is known to be true in the cases where $[G, G]$ is abelian, G is a p-group, or $|G|$ is squarefree.

Example

Let E be the splitting field of $x^4 + 3x + 3$ over $F = \mathbb{Q}$.

- $G = \mathrm{Gal}(E/F) \cong D_8 = \langle a, b \mid a^4 = b^2 = 1, ab = ba^{-1} \rangle$
- $H_1 := Z(G)$, $H_2 := \langle a \rangle$, K_i is the fixed field of H_i for $i = 1, 2$

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Example (continued)

- $G/H_1 \cong C_2 \times C_2$ has 4 irreducible characters $\psi_1, \ldots, \psi_4$, which inflate to 4 irreducible characters $\chi_j = \inf_G^{G/H_1}(\psi_j)$ of G; turns out that

$$L(s, \chi_j, E/F) = L(s, \psi_j, K_1/F)$$

- The remaining character χ_5 is induced from a character φ_3 of H_2, so

$$L(s, \chi_5, E/F) = L(s, \varphi_3, E/K_2)$$

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