Dynamics Over Number Fields

Jordan Schettler

Department of Mathematics
University of Arizona

2/23/10

Outline

Introduction

Definition

A **(discrete) dynamical system** (S, ϕ) is a set S and a map

$$\phi : S \to S.$$

The **orbit** of $\alpha \in S$ is

$$\mathrm{Orb}_\phi(\alpha) := \{\phi^n(\alpha) : n \in \mathbb{N}_0\}$$

where

$$\phi^n = \begin{cases} \mathrm{id}_S & \text{for } n = 0 \\ \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}} & \text{for } n \geq 1. \end{cases}$$

Definition (continued)

We categorize points of finite orbit into the following subsets

$\mathrm{Fix}(\phi)$	**fixed**	$\phi(\alpha) = \alpha$
$\cap\vert$		
$\mathrm{Per}(\phi)$	**periodic**	$\phi^n(\alpha) = \alpha$ some $n \geq 1$
$\cap\vert$		
$\mathrm{PrePer}(\phi)$	**preperiodic**	$\phi^{m+n}(\alpha) = \phi^m(\alpha)$ some $m \geq 0$, $n \geq 1$

Points of infinite orbit will (in this talk) be called **wandering.**

Example

Consider

$$\phi : \mathbb{Z} \to \mathbb{Z} : x \mapsto x^2 - 3.$$

Then

$$\text{Fix}(\phi) = \{\},$$

$$\text{Per}(\phi) = \{-2, 1\},$$

$$\text{PrePer}(\phi) = \{-2, -1, 1, 2\}.$$

Usually, S has some additional structure (algebraic, topological, analytic) and ϕ is related to this structure.

For our purposes, S will be $\mathbb{P}^N(F)$ for a field F, and ϕ will be a morphism defined over F.

Ultimately, we'll be interested in morphisms on $\mathbb{P}^N(K)$ for a number field K. Beforehand, we consider rational maps

$$\phi(z) \in K_v(z)$$

on $\mathbb{P}^1(K_v)$ where

$$K_v = \text{completion of } K \text{ at a place } v.$$

If v is archimedean, then $K_v \cong \mathbb{R}$ or $\mathbb{C}$. This is the context of classical dynamics, which we consider first.

If v is nonarchimedean, then K_v is a finite extension of $\mathbb{Q}_p$ for some prime p. This is a natural starting point for arithmetic dynamics, which we'll consider second.

Classical Dynamics

Each $\phi(z) \in \mathbb{C}(z)$ is viewed as a map from

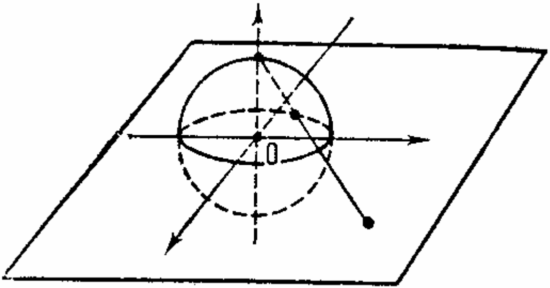
$$\mathbb{P}^1(\mathbb{C}) = \{[z,1] : z \in \mathbb{C}\} \cup \{[1,0]\} = \mathbb{C} \cup \{\infty\}$$

to itself and, as such, is open and Lipschitz with respect to the **chordal metric** $\rho(\cdot,\cdot)$ defined by

$$\rho([x_1, y_1],[x_2, y_2]) := \frac{|x_1 y_2 - x_2 y_1|}{\sqrt{|x_1|^2 + |y_1|^2}\sqrt{|x_2|^2 + |y_2|^2}}.$$

If we identify $\mathbb{P}^1(\mathbb{C}) = \mathbb{S}^2 \subseteq \mathbb{R}^3$ via the stereographic projection from the unit sphere as seen below, then

$$\rho(P_1, P_2) = \frac{1}{2}(\text{length of the chord joining } P_1 \text{ and } P_2)$$



Definition

If $\alpha \in \mathbb{C}$ is a periodic point of $\phi(z) \in \mathbb{C}(z)$, the **multiplier** of α is

$$\lambda_\phi(\alpha) := (\phi^m)'(\alpha)$$

where α has exact period $m = |\mathrm{Orb}_\phi(\alpha)|$.

If $\infty \in \mathrm{Per}(\phi)$, we take

$$\lambda_\phi(\infty) := \lambda_{\phi(z^{-1})^{-1}}(0).$$

Definition (continued)

With α, ϕ as above, note that

$$|\phi^m(z) - \alpha| \approx |\lambda_\phi(\alpha)| \cdot |z - \alpha|$$

for z in a small neighborhood of α, so we say $\beta \in \mathrm{Per}(\phi)$ is...

$$\begin{cases} \textbf{superattracting} & \text{if } |\lambda_\phi(\beta)| = 0 \\ \textbf{attracting} & \text{if } |\lambda_\phi(\beta)| < 1 \\ \textbf{neutral} & \text{if } |\lambda_\phi(\beta)| = 1 \\ \textbf{repelling} & \text{if } |\lambda_\phi(\beta)| > 1. \end{cases}$$

Definition (continued)

With ϕ as above, we define the **Fatou set** of ϕ as

$$\mathcal{F}(\phi) := \text{maximal open set on which } \{\phi^n : n \in \mathbb{N}\} \text{ is equicontinuous,}$$

and we define the **Julia set** of ϕ as

$$\mathcal{J}(\phi) := \mathbb{P}^1(\mathbb{C}) \backslash \mathcal{F}(\phi).$$

Remark

Points in $\mathcal{F}(\phi)$ which are close together tend to stay close together under iterates of ϕ. In fact,

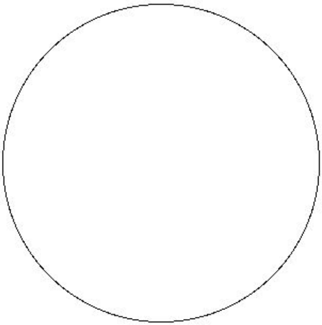
$$\{\text{attracting periodic points}\} \subseteq \mathcal{F}(\phi) = \phi(\mathcal{F}(\phi))$$

Points in $\mathcal{J}(\phi)$ which are close together tend to drift apart under iterates of ϕ. In fact,

$$\{\text{repelling periodic points}\} \subseteq \mathcal{J}(\phi) = \phi(\mathcal{J}(\phi)).$$

Example

When $\phi(z) = z^2$, the Julia set $\mathcal{J}(\phi)$ is the unit circle...

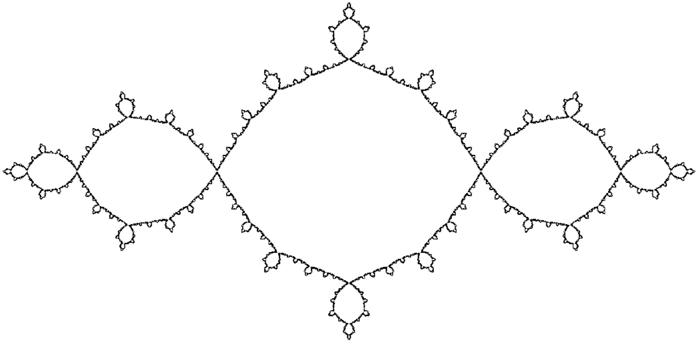


Example

When $\phi(z) = z^2 - 2$, the Julia set $\mathcal{J}(\phi)$ is a line segment...

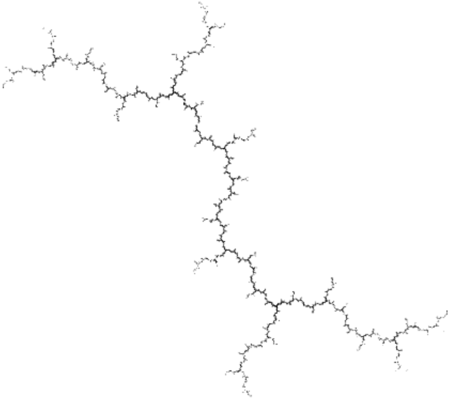
Example

When $\phi(z) = z^2 - 1$, the Julia set $\mathcal{J}(\phi)$ is fractal-like...



Example

When $\phi(z) = z^2 + i$, the Julia set $\mathcal{J}(\phi)$ is again fractal-like...



Theorem

Suppose $\phi(z) \in \mathbb{C}(z)$ has degree $d \geq 2$. Then

- $\mathcal{J}(\phi) = \overline{\{\textit{repelling periodic points}\}} \neq \{\}$
- $\mathcal{J}(\phi) = \mathbb{P}^1(\mathbb{C}) \Leftrightarrow \mathcal{J}(\phi)^\circ \neq \{\}$
- $\exists \leq 2d - 2$ *non-repelling periodic orbits in* $\mathbb{P}^1(\mathbb{C})$.

Arithmetic Dynamics

For the remainder of the talk we fix the following notation:

- K is a number field
- $\mathfrak{p}$ is a maximal ideal of $\mathcal{O}_K$
- $|\cdot|_{\mathfrak{p}}$ is the standard absolute value
- $K_{\mathfrak{p}}$ is the completion of K w.r.t. $|\cdot|_{\mathfrak{p}}$
- $\mathbb{F}_{\mathfrak{p}}$ is the residue field of $\mathcal{O}_{K_{\mathfrak{p}}}$
- $p = \mathrm{char}(\mathbb{F}_{\mathfrak{p}})$

Again, each $\phi(z) \in K_{\mathfrak{p}}(z)$ is viewed as a map from

$$\mathbb{P}^1(K_{\mathfrak{p}}) = \{[z,1] : z \in K_{\mathfrak{p}}\} \cup \{[1,0]\} = K_{\mathfrak{p}} \cup \{\infty\}$$

to itself and, as such, is Lipschitz with respect to the **$\mathfrak{p}$-adic chordal metric** $\rho_{\mathfrak{p}}(\cdot,\cdot)$ defined by

$$\rho_{\mathfrak{p}}([x_1,y_1],[x_2,y_2]) := \frac{|x_1y_2 - x_2y_1|_{\mathfrak{p}}}{\max\{|x_1|_{\mathfrak{p}},|y_1|_{\mathfrak{p}}\}\max\{|x_2|_{\mathfrak{p}},|y_2|_{\mathfrak{p}}\}}.$$

Definition

If $\alpha \in K_\mathfrak{p}$ is periodic for $\phi(z) \in K_\mathfrak{p}(z)$, the **multiplier** is again

$$\lambda_\phi(\alpha) := (\phi^m)'(\alpha)$$

where $m = |\mathrm{Orb}_\phi(\alpha)|$. Again we say $\beta \in \mathrm{Per}(\phi)$ is...

$$\begin{cases} \textbf{superattracting} & \text{if } |\lambda_\phi(\beta)|_\mathfrak{p} = 0 \\ \textbf{attracting} & \text{if } |\lambda_\phi(\beta)|_\mathfrak{p} < 1 \\ \textbf{neutral} & \text{if } |\lambda_\phi(\beta)|_\mathfrak{p} = 1 \\ \textbf{repelling} & \text{if } |\lambda_\phi(\beta)|_\mathfrak{p} > 1. \end{cases}$$

Let

$$\mathcal{O}_{K_\mathfrak{p}} \twoheadrightarrow \mathbb{F}_\mathfrak{p} : \alpha \mapsto \widetilde{\alpha}$$

denote the natural projection, and define

$$\mathbb{P}^1(K_\mathfrak{p}) \to \mathbb{P}^1(\mathbb{F}_\mathfrak{p}) : P \mapsto \widetilde{P}$$

by

$$[x, y] \mapsto \left[\widetilde{x/a}, \widetilde{y/a}\right]$$

where $|a|_\mathfrak{p} = \max\{|x|_\mathfrak{p}, |y|_\mathfrak{p}\}$.

Every $\phi(z) \in K_{\mathfrak{p}}(z)$ can be written as

$$\phi(z) = f(z)/g(z)$$

where

$$f(z), g(z) \in \mathcal{O}_{K_{\mathfrak{p}}}[z]$$

and $\widetilde{f}(z) \neq 0$ or $\widetilde{g}(z) \neq 0$ in $\mathbb{F}_{\mathfrak{p}}[z]$. Thus

$$\widetilde{\phi}(z) := \widetilde{f}(z)/\widetilde{g}(z)$$

is a rational map on $\mathbb{P}^1(\mathbb{F}_{\mathfrak{p}})$.

Definition

We say $\phi(z) \in K_{\mathfrak{p}}(z)$ has **good reduction** when

$$\deg(\phi) = \deg(\widetilde{\phi});$$

equivalently, there are no solutions $[x, y] \in \mathbb{P}^1(\overline{\mathbb{F}}_{\mathfrak{p}})$ to

$$y^{\deg(f)}\widetilde{f}(x/y) = y^{\deg(g)}\widetilde{g}(x/y) = 0$$

Theorem

Suppose $\phi(z) \in K_{\mathfrak{p}}(z)$ has good reduction. Then

- $\rho_{\mathfrak{p}}(\phi(P_1), \phi(P_2)) \leq \rho_{\mathfrak{p}}(P_1, P_2)$
- $\widetilde{\phi^n(P_1)} = \tilde{\phi}^n(\widetilde{P_1})$

for all $P_1, P_2 \in \mathbb{P}^1(K_{\mathfrak{p}})$ and all $n \in \mathbb{N}_0$.

Corollary

Suppose $\phi(z) \in K_{\mathfrak{p}}(z)$ has good reduction. Then

- $\mathcal{J}(\phi) = \{\}$
- $\widetilde{Per(\phi)} \subseteq Per(\tilde{\phi})$.

In fact, we have the following more precise statement.

Theorem

Suppose $\phi(z) \in K_{\mathfrak{p}}(z)$ has good reduction and degree $d \geq 2$. Let $P \in Per(\phi)$. Then

$$|Orb_{\phi}(P)| = |Orb_{\tilde{\phi}}(\widetilde{P})| \cdot |\{\lambda_{\tilde{\phi}}(\widetilde{P}), \lambda_{\tilde{\phi}}(\widetilde{P})^2, \ldots\}| \cdot p^n$$

for some $n \in \mathbb{N}_0$.

Example

Let $f(x) \in \mathbb{Q}[x]$ have degree $d \geq 2$.

Suppose $f(x)$ has good reduction in both $\mathbb{Q}_2(x)$ and $\mathbb{Q}_3(x)$.

Consider a periodic point $\alpha \in \mathbb{Q}$ of $f(x)$.

Example (continued)

On the one hand, $f(x)$ has good reduction in $\mathbb{Q}_2(x)$.

Hence

$$|\mathrm{Orb}_f(\alpha)| = 2^m$$

for some $m \in \mathbb{N}_0$ since

$$|\mathrm{Orb}_{\tilde{f}}(\tilde{\alpha})| = 1 \text{ or } 2 \quad \text{and} \quad |\{\lambda_{\tilde{f}}(\tilde{\alpha}), \lambda_{\tilde{f}}(\tilde{\alpha})^2, \ldots\}| = 1.$$

Example (continued)

On the other hand, $f(x)$ has good reduction in $\mathbb{Q}_3(x)$.

Hence

$$|\mathrm{Orb}_f(\alpha)| = 2^r \cdot 3^n$$

for some $r \in \{0, 1, 2\}$, $n \in \mathbb{N}_0$ since

$$|\mathrm{Orb}_{\widetilde{f}}(\widetilde{\alpha})| = 1, 2, \text{ or } 3 \quad \text{and} \quad |\{\lambda_{\widetilde{f}}(\widetilde{\alpha}), \lambda_{\widetilde{f}}(\widetilde{\alpha})^2, \ldots\}| = 1 \text{ or } 2.$$

Example (continued)

Thus

$$|\mathrm{Orb}_f(\alpha)| = 1, 2, \text{ or } 4.$$

Each of these possibilities is realized for $\alpha = 0$ and...

- $f(x) = x^2$
- $f(x) = x^2 - 1$
- $f(x) = x^4 + \frac{348}{35}x^3 + \frac{123}{7}x^2 - \frac{1243}{35}x + 1$

Theorem (Narkiewicz)

Let $f(x) \in \mathbb{Z}[x]$ be monic of degree $d \geq 2$. Suppose $P \in \mathbb{P}^1(\mathbb{Q})$ is a periodic point for f. Then

$$|Orb_f(P)| = 1 \text{ or } 2.$$

The above example generalizes over an arbitrary number field.

Theorem

Let $\phi(z) \in K(z)$ have degree $d \geq 2$. Suppose $\phi(z)$ has good reduction in $K_{\mathfrak{p}}(z)$ and $K_{\mathfrak{q}}(z)$ with $\mathfrak{q} \nmid p$. Then $\forall \alpha \in Per(\phi)$

$$|Orb_{\phi}(\alpha)| \leq (N(\mathfrak{p})^2 - 1)(N(\mathfrak{q})^2 - 1).$$

Now we turn to morphisms $\phi : \mathbb{P}^N(\overline{K}) \to \mathbb{P}^N(\overline{K})$, i.e.

$$\phi(P) = [f_0(P), \ldots, f_N(P)] \quad \text{where} \quad f_0, \ldots, f_N \in \overline{K}[x_0, \ldots, x_N]$$

are homogeneous polynomials of the same degree s.t.

$$f_0(X) = \cdots = f_N(X) = 0$$

has no solutions in $\mathbb{P}^N(\overline{K})$.

Say ϕ is **defined over** K when we can take $f_0, \ldots, f_N \in K[X]$.

Definition

We define the **relative height** of $P = [x_0, \ldots, x_N] \in \mathbb{P}^N(K)$ by

$$H_K(P) := \prod_v \max\{|x_0|_v, \ldots, |x_N|_v\}^{[K_v:\mathbb{Q}_v]}$$

where the product ranges over all places v on K.

We write

$$h_K(P) := \log(H_K(P)).$$

Definition

More generally, we define the **absolute height** of $P \in \mathbb{P}^N(\overline{\mathbb{Q}})$ as

$$H(P) := H_L(P)^{1/[L:\mathbb{Q}]}$$

for any number field L such that $P \in \mathbb{P}^1(L)$.

We write

$$h(P) := \log(H(P)).$$

Lemma

For every $C \in \mathbb{R}$

$$|\{P \in \mathbb{P}^N(K) : H_K(P) \leq C\}| < \infty,$$

so also

$$|\{P \in \mathbb{P}^N(K) : h_K(P) \leq C\}| < \infty.$$

Lemma

Let $\phi : \mathbb{P}^N(\overline{K}) \to \mathbb{P}^N(\overline{K})$ be a morphism. Then $\exists C_1, C_2 > 0$ s.t.

$$C_1 H(P)^d \leq H(\phi(P)) \leq C_2 H(P)^d$$

for all $P \in \mathbb{P}^N(\overline{K})$ where $d = \deg(\phi)$.

Theorem (Northcott)

Let $\phi : \mathbb{P}^N(\overline{K}) \to \mathbb{P}^N(\overline{K})$ be a morphism of degree $d \geq 2$ defined over K. Then

$$PrePer(\phi)$$

is a set of bounded height. In particular,

$$|PrePer(\psi)| < \infty$$

where

$$\psi = \phi|_{\mathbb{P}^N(K)} : \mathbb{P}^N(K) \to \mathbb{P}^N(K).$$

Proof.

There is a constant C s.t. for every $Q \in \mathbb{P}^N(\overline{K})$ we have

$$dh(Q) - C \leq h(\phi(Q)),$$

so if $\phi^m(P)$ has exact period n, then induction gives

$$d^m(h(P) - C) \leq h(\phi^m(P))$$

and

$$d^n(h(\phi^m(P)) - C)) \leq h(\phi^n(\phi^m(P))) = h(\phi^m(P)).$$

Proof continued.

Therefore

$$h(P) \leq \frac{1}{d^m} h(\phi^m(P)) + C \leq \frac{1}{d^m} \cdot \frac{d^n}{d^n - 1} C + C \leq 3C.$$

∎

Remark

We need the morphism assumption in Northcott's theorem. To see why, consider the rational map

$$\phi : \mathbb{P}^2(\overline{\mathbb{Q}}) \dashrightarrow \mathbb{P}^2(\overline{\mathbb{Q}}) : [x, y, z] \mapsto [x^2, y^2, xz].$$

Then

$$\mathrm{Fix}(\phi) \cap \mathbb{P}^2(\mathbb{Q}) \supseteq \{[1, 0, 1], [1, 0, 2], \ldots\}$$

is infinite.