HILBERT FUNCTIONS

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1. Introduction

A Hilbert function (so far as we will discuss) is a map from the nonnegative integers to themselves which records the lengths of composition series of each layer in a graded module. In many situations of interest, the Hilbert function attached to a module agrees for sufficiently large inputs with a polynomial, called a Hilbert polynomial. Thus an infinite amount of information is encoded in a finite object. Indeed, the degree and coefficients of a Hilbert polynomial represent important invariants from which one can potentially read off properties of the module.

Of interest to us here is that the coordinate ring of a projective variety is a graded module for which there is an associated Hilbert polynomial that tells us information about the space such as its dimension and degree. Also, there are Hilbert polynomials which capture the dimension and multiplicity at a point in a quasi-projective variety.

First, we'll build up the needed machinery with a review of graded modules. Then a crucial result and corollary about Poincaré series is proved which will allow us to conclude that a wide range of Hilbert functions have the property of eventually behaving like polynomials mentioned above. Next, we will explore the appropriate notion of a Hilbert function for a projective variety. Lastly, Hilbert functions of local rings (in particular, stalks at points in a quasi-projective variety) are introduced and studied.

An effort was made to include proofs in cases where either the result was fundamental or the source was unclear. On the other hand, I've intentionally omitted standard or otherwise unenlightening details which can be found in my references.

2. Graded Modules

Definition 2.1. We define $\underline{\text{CommRing}}$ to be the category whose class of objects $|\underline{\text{CommRing}}|$ consists of all nonzero commutative rings with identity and whose morphisms are ring homomorphisms which preserve 1. Recall that $R \in |\underline{\text{CommRing}}|$ is said to be (nicely) **graded** (by $\mathbb{N}_0$) when

$$R = \bigoplus_{n=0}^{\infty} R_n$$

as abelian groups with $R_0 \neq \{0\}$ and

$$R_i R_j \subseteq R_{i+j}$$

for all $i, j \in \mathbb{N}_0$; an R-module M (assumed to be unitary) is **graded** when

$$M = \bigoplus_{n=0}^{\infty} M_n$$

as abelian groups and

$$R_i M_j \subseteq M_{i+j}$$

for all $i, j \in \mathbb{N}_0$.

Example 2.2. The polynomial ring $S = R[x_0, \ldots, x_N]$ is graded in a natural way whenever $R \in |\underline{\text{CommRing}}|$ by taking S_n equal to the set of homogeneous polynomials in S of degree n along with 0. Here S is said to be **graded by degree**. Note that $S_0 = R$.

Remark 2.3. Let $R \in |\underline{\text{CommRing}}|$ be graded. Then we may write

$$1 = \sum_{n=0}^{\infty} r_n$$

where $r_n \in R_n$ for all $n \in \mathbb{N}_0$ and all but finitely many summands are 0. Thus if $\rho_0 \in R_0$, then

$$\sum_{n=0}^{\infty} \rho_0 r_n = \rho_0 \cdot 1 = \rho_0 \in R_0,$$

so $\rho_0 = \rho_0 r_0$ since each $\rho_0 r_n \in R_n$. Hence r_0 is a multiplicative identity in R_0, so we have $R_0 \in |\underline{\text{CommRing}}|$ because $R_0 R_0 \subseteq R_0 \neq \{0\}$; in particular, R is an R_0-algebra. Also, it's clear that if M is a graded R-module, then M_n is a R_0-module for all $n \in \mathbb{N}_0$.

Lemma 2.4. *Let $R \in |\underline{\text{CommRing}}|$ be graded. Then R is Noetherian $\Leftrightarrow$ (R_0 is Noetherian and R is finitely-generated as an R_0-algebra).*

Sketch. ($\Rightarrow$) Suppose R is Noetherian. Then the irrelevant ideal

$$R_+ := \bigoplus_{n=1}^{\infty} R_n$$

is finitely-generated, wlog, by $r_i \in R_{k(i)}$ for $i = 1, \ldots, s$, and satisfies $R_0 \cong R/R_+$ (thus R_0 is Noetherian). One can use induction to show that $R_n \subseteq R_0[r_1, \ldots, r_s]$ for all $n \geq 0$, so $R = R_0[r_1, \ldots, r_s]$. ($\Leftarrow$) Conversely, suppose R_0 is Noetherian and R is finitely-generated as an R_0-algebra. Then R is Noetherian by the Hilbert basis theorem. $\square$

Lemma 2.5. *Let $R \in |\underline{\text{CommRing}}|$ be graded and Noetherian, and suppose M is a finitely-generated graded R-module. Then M_n is a finitely-generated R_0-module for all $n \in \mathbb{N}_0$.*

Proof. We have $R = R_0[r_1, \dots, r_s]$ by lemma 2.4 and $M = Rm_1 + \cdots + Rm_t$ where wlog $r_i \in R_{k(i)}$ for all $i = 1, \dots, s$ and $m_i \in M_{l(i)}$ for all $i = 1, \dots, t$. Thus fixing $n \in \mathbb{N}_0$ and $m \in M_n$, we have

$$m = \sum_{i=1}^{t} \rho_i m_i$$

for some $\rho_1, \dots, \rho_t \in R$. Since $m \in M_n$ we may assume $\rho_i \in R_{n-l(i)}$ for all $i = 1, \dots, t$ where we use the convention that $R_k = \{0\}$ whenever $0 > k \in \mathbb{Z}$. Thus each ρ_i is of the form

$$\rho_i = \sum_{j=1}^{J(i)} f_j(r_1, \dots, r_s)$$

where every $f_j \in R_0[x_1, \dots, x_s]$ is a sum of monomials of bounded degree (with a bound that may be taken independent of i). Therefore M_n is a finitely-generated R_0-module. □

3. Poincaré Series

Definition 3.1. Let $R \in |\underline{\text{CommRing}}|$. A function λ from a subclass $\mathscr{S}$ of the class all R-modules to $\mathbb{Z}$ is said to be **additive** on $\mathscr{S}$ if

$$\lambda(B) = \lambda(A) + \lambda(C)$$

whenever

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of R-modules in $\mathscr{S}$.

Remark 3.2. Let $R \in |\underline{\text{CommRing}}|$ be graded and Noetherian, and suppose λ is an additive function with values in $\mathbb{Z}$ on the class $\mathscr{S}$ all finitely-generated R_0-modules. Then by lemma 2.5 we may define the **Poincaré series** of a finitely-generated graded R-module M as

$$P_M(x) = \sum_{n=0}^{\infty} \lambda(M_n) x^n \in \mathbb{Z}[[x]].$$

Theorem 3.3 (Hilbert-Serre). *Let R, λ, M be as in remark 3.2. Then*

$$P_M(x) = \frac{f(x)}{\prod_{i=1}^{s}(1 - x^{k(i)})}$$

for some $f(x) \in \mathbb{Z}[x]$ and some $k(1), \dots, k(s) \in \mathbb{N}_0$.

Proof. By lemma 2.4 we may write $R = R_0[r_1, \dots, r_s]$ where wlog $r_i \in R_{k(i)}$ for all $i = 1, \dots, s$. We'll use induction on s. If $s = 0$, then $R = R_0$, so M is a finitely generated R_0-module, say

$$M = R_0 m_1 + \cdots + R_0 m_t$$

where wlog $m_i \in M_{l(i)}$ for all $i = 1, \dots, t$, but then $M_n = \{0\}$ for $n > \max\{l(1), \dots, l(t)\}$, whence $P_M(x)$ is a polynomial because $\lambda(M_n) = 0$ for all sufficiently large n by additivity (apply λ to the SES consisting entirely of 0's). Now assume $s > 0$ and that the statement holds for $s - 1$. For every $n \in \mathbb{N}_0$, consider the multiplication map

$$\varphi_n : M_n \longrightarrow M_{n+k(s)} : m \mapsto r_s m,$$

which gives rise to short exact sequences of R_0-modules

$$\begin{array}{rcccl} \ker(\varphi_n) & \hookrightarrow & M_n & \twoheadrightarrow & r_s M_n \\ r_s M_n & \hookrightarrow & M_{n+k(s)} & \twoheadrightarrow & M_{n+k(s)}/r_s M_n, \end{array}$$

so additivity gives

$$\lambda(M_{n+k(s)}) - \lambda(M_n) = \lambda(M_{n+k(s)}/r_s M_n) - \lambda(\ker(\varphi_n)).$$

Note that (assuming wlog $k(s) > 0$)

$$R' := R_0[r_1, \dots, r_{s-1}] \cong R/r_s R = \bigoplus_{n=0}^{\infty} (R_n + r_s R)/r_s R$$

is a graded element of $|\underline{\text{CommRing}}|$ with the last equality on the right as abelian groups. In fact,

$$K := \bigoplus_{n=0}^{\infty} \ker(\varphi_n) \cong \ker(\varphi_0 \oplus \varphi_1 \oplus \cdots)$$

and

$$L := \bigoplus_{n=0}^{\infty} M_n/(M_n \cap r_s M) \cong \bigoplus_{n=0}^{\infty} (M_n + r_s M)/r_s M = M/r_s M$$

are finitely-generated graded R'-modules. Thus the induction hypothesis implies that there are $f_L(x), f_K(x) \in \mathbb{Z}[x]$ such that

$$\begin{aligned}
(1-x^{k(s)})P_M(x) &= \sum_{n=0}^{\infty} \lambda(M_n)x^n - \sum_{n=0}^{\infty} \lambda(M_n)x^{n+k(s)} \\
&= g(x) + \sum_{n=0}^{\infty} (\lambda(M_{n+k(s)}) - \lambda(M_n))x^{n+k(s)} \\
&= g(x) + \sum_{n=0}^{\infty} (\lambda(M_{n+k(s)}/r_s M_n) - \lambda(\ker(\varphi_n)))x^{n+k(s)} \\
&= P_L(x) - x^{k(s)}P_K(x) \\
&= \frac{f_L(x) - x^{k(s)} f_K(x)}{\prod_{i=1}^{s-1}(1-x^{k(i)})} \\
&= \frac{f(x)}{\prod_{i=1}^{s-1}(1-x^{k(i)})}
\end{aligned}$$

where

$$\begin{aligned}
g(x) &= \sum_{n=0}^{k(s)-1} \lambda(M_n)x^n \in \mathbb{Z}[x] \\
f(x) &= f_L(x) - x^{k(s)} f_K(x) \in \mathbb{Z}[x].
\end{aligned}$$

□

Corollary 3.4. *Let R, λ, M be as in 3.2. If $R = R_0[r_1, \ldots, r_s]$ for some $r_1, \ldots, r_s \in R_1$, then there exists an $h_M(x) \in \mathbb{Q}[x]$ called the* ***Hilbert polynomial*** *of M such that*

$$\lambda(M_n) = h_M(n)$$

for all sufficiently large $n \in \mathbb{N}_0$.

Proof. By theorem 3.3 we have

$$P_M(x) = \frac{f(x)}{(1-x)^s} = \frac{g(x)}{(1-x)^w}$$

where $g(x) \in \mathbb{Z}[x]$ and $g(1) \neq 0$. Thus writing

$$g(x) = \sum_{n=0}^{N} a_n x^n$$

and noting that (with the convention $\binom{n}{-1} = 0 = 1 - \binom{-1}{-1}$ for $n \in \mathbb{N}_0$)

$$(1-x)^{-w} = \sum_{n=0}^{\infty} \binom{w+n-1}{w-1} x^n,$$

we find $N \leq n \in \mathbb{N}_0$ implies

$$\lambda(M_n) = \sum_{i=0}^{N} a_i \binom{w+n-i-1}{w-1}$$

which is a polynomial in n with leading term

$$\frac{g(1)}{(w-1)!} n^{w-1}.$$

□

4. Hilbert Functions

Remark 4.1. Recall that if $R \in |\underline{\text{CommRing}}|$ is Artinian and M is a finitely-generated R-module, then every chain of submodules of length n

$$M = M_0 > M_1 > \ldots > M_n = \{0\}$$

can be extended to a composition series (i.e., a maximal chain) and that every such composition series has the same length. We write $\ell_R(M)$ for the common length of all composition series. Note that ℓ_R is additive on the class of all finitely-generated R-modules. To see this, suppose

$$A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$$

is a short exact sequence of finitely-generated R-modules. Let

$$A_0 > \ldots > A_a$$

be a composition series for A and

$$C_0 > \ldots > C_c$$

be a composition series for C. Then

$$B = g^{-1}(C_0) > \ldots > g^{-1}(C_c) = f(A_0) > \ldots > f(A_a) = \{0\}$$

is a composition series for B of length $a + c$.

Definition 4.2. Let $R \in |\underline{\text{CommRing}}|$ be graded and finitely-generated as an R_0-algebra with R_0 Artinian, and suppose M is a finitely-generated graded R-module. Then R_0 is Noetherian by Hopkin's theorem, so R is Noetherian by 2.4, whence 2.5 and 4.1 allow us to define the **Hilbert function** of M by

$$H_M(n) := \ell_{R_0}(M_n)$$

for all $n \in \mathbb{N}_0$.

Remark 4.3. Suppose $R \in \underline{|\text{CommRing}|}$ is Artinian and let $S := R[x_0, \ldots, x_N]$ be graded by degree. Then since $x_0, \ldots, x_N \in S_1$ remark 4.1 and corollary 3.4 imply that for every finitely-generated graded S-module M there is a Hilbert polynomial $h_M(x) \in \mathbb{Q}[x]$ such that

$$h_M(n) = H_M(n)$$

for all sufficiently large $n \in \mathbb{N}_0$. In fact, if w is the order of the pole at $x = 1$ in $P_M(x)$, then by 3.4 we have $\deg(h_M) = w - 1$.

Definition 4.4. Let $X \subseteq \mathbb{P}^N$ be a projective variety. Recall that $k[X] = S/I(X)$ where $S = k[x_0, \ldots, x_N]$ for an algebraically closed field k and $I(X)$ is the homogeneous radical ideal in S consisting of those polynomials which vanish on all of X. Then $k[X]$ is a finitely-generated graded S-module (S graded by degree) with

$$k[X]_n = (S_n + I(X))/I(X)$$

and $S_0 = k$ Artinian, so we may define the **Hilbert function** of X by

$$H_X(n) := H_{k[X]}(n) = \ell_k(k[X]_n) = \dim_k((S_n + I(X))/I(X))$$

for all $n \in \mathbb{N}_0$, which by 4.3 agrees for sufficiently large n with the **Hilbert polynomial** of X given by

$$h_X(x) := h_{k[X]}(x) \in \mathbb{Q}[x].$$

Example 4.5. Consider $X = \mathbb{P}^N$. Then $k[X] = S$, so

$$H_{\mathbb{P}^N}(n) = \dim_k(S_n) = \#\text{monic monomials of degree } n = \binom{N+n}{N},$$

giving

$$h_{\mathbb{P}^N}(x) = \frac{1}{N!}(x+N)(x+N-1)\cdots(x+1) = \frac{1}{N!}x^N + \frac{N+1}{2(N-1)!}x^{N-1} + \cdots + 1.$$

As we'll see below, this implies the "degree" of $\mathbb{P}^N$ is 1.

Theorem 4.6. *Let $X \subseteq \mathbb{P}^N$ be a projective variety with $\dim(X) = D$. Then $\deg(h_X) = D$ (with the convention $\deg(0) = -1$), and if $X \neq \varnothing$, then the leading term of $h_X(x)$ is*

$$\frac{dx^D}{D!}$$

for some $d \in \mathbb{N}$ called the ***degree*** *of X. Moreover, if $X = V(F)$ is a hypersurface, then $d = \deg(F)$.*

Proof. To show h_X has degree D, we use induction on D. If $D = -1$, then $X = \varnothing$, so $I(X) = S$ and $k[X]_n = (S_n + S)/S \cong \{0\}$ for all $n \in \mathbb{N}_0$, giving $\ell_k(k[X]_n) \equiv 0$, whence $h_X = 0$ as needed. Now suppose $D \geq 0$ and that $\deg(h_Y) = \dim(Y)$ whenever $Y \subseteq \mathbb{P}^N$ has $\dim(Y) < D$. We may assume X is irreducible (see [3], pg.s 50-51, for this reduction). Then $I(X) = \mathfrak{p}$ is a homogeneous prime ideal. Note that $x_i \notin \mathfrak{p}$ for some $i \in \{0, \ldots, N\}$ since otherwise $\varnothing = V(\mathfrak{p}) = V(I(X)) = X$. Hence $Y := X \cap V(x_i) = V(\mathfrak{p} + x_i S)$ has $\dim(Y) = D - 1$. Then $x_i \in S_1$ is not a zero divisor of $k[X]$ since $S/\mathfrak{p}$ is an integral domain and $x_i \notin \mathfrak{p}$, so using the notation in the proof of theorem 3.3 we have maps (which are now injective)

$$\varphi_n = k[X]_n \to k[X]_{n+1} : m \mapsto x_i m$$

for all $n \in \mathbb{N}_0$. Injectivity implies $K \cong \{0\}$, so $f_K(x) = 0$ and

$$(1-x)P_{k[X]}(x) = P_L(x),$$

whence the proof of corollary 3.4 implies

$$\text{leading term of } h_X(x) = \frac{d}{(w-1)!}x^{w-1}$$

where $d := g(1) \in \mathbb{N}$ (since $g(x) \in \mathbb{Z}[x]$ and $h_X(n) = \ell_k(k[X]_n) \geq 0$ for sufficiently large $n \in \mathbb{N}_0$). Thus

$$\begin{aligned} \deg(h_X) + 1 &= w \\ &= \text{order of pole at 1 in } P_{k[X]}(x) \\ &= 1 + \text{order of pole at 1 in } P_L(x) \\ &= 2 + \deg(h_L). \end{aligned}$$

On the other hand,

$$L \cong \bigoplus_{n=0}^{\infty} \frac{k[X]_n + x_i k[X]}{x_i k[X]} = \bigoplus_{n=0}^{\infty} \frac{(S_n + x_i S + \mathfrak{p})/\mathfrak{p}}{(x_i S + \mathfrak{p})/\mathfrak{p}} \cong \bigoplus_{n=0}^{\infty} k[Y]_n = k[Y],$$

so the induction hypothesis gives

$$\deg(h_X) = 1 + \deg(h_L) = 1 + \dim(Y) = D.$$

Now assume $X = V(F)$ is a hypersurface. Then if $n \geq \delta := \deg(F)$, we have

$$\frac{S_n + (F)}{(F)} \cong \frac{S_n}{S_n \cap (F)} = S_n/FS_{n-\delta},$$

so

$$\begin{aligned} H_{V(F)}(n) &= \dim_k(S_n/FS_{n-\delta}) \\ &= \dim_k(S_n) - \dim_k(S_{n-\delta}) \\ &= \binom{N+n}{N} - \binom{N+n-\delta}{N} \\ &= \frac{1}{N!}n^N + \frac{N+1}{2(N-1)!}n^{N-1} \cdots - \frac{1}{N!}n^N - \frac{-2\delta+N+1}{2(N-1)!}n^{N-1} \cdots . \end{aligned}$$

Thus

$$\begin{aligned} \text{leading term of } h_{V(F)}(x) &= \left(\frac{N+1}{2(N-1)!} - \frac{-2\delta+N+1}{2(N-1)!} \right) x^{N-1} \\ &= \frac{\delta}{(N-1)!}x^{N-1}, \end{aligned}$$

giving $d = \delta = \deg(F)$ (and $\dim(X) = N-1$, as expected). ☐

5. Local Rings and Multiplicity

Definition 5.1. Given $R \in |\underline{\text{CommRing}}|$ and a proper ideal I of R, we define the **associated graded ring** of R with respect to I by

$$\mathrm{gr}_I(R) := \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$$

as abelian groups with $I^0 = R$ where for $a \in I^i$, $b \in I^j$ we have a well-defined product given by

$$(a + I^{i+1})(b + I^{j+1}) := ab + I^{i+j+1} \in I^{i+j}/I^{i+j+1}.$$

If M is an R-module, then an I**-filtration** $\mathcal{F}$ of M is a decreasing chain of submodules

$$M = M_0 \geq M_1 \geq M_2 \geq \dots$$

such that

$$IM_n \subseteq M_{n+1}$$

for all $n \in \mathbb{N}_0$; if, in addition, $IM_n = M_{n+1}$ for all sufficiently large $n \in \mathbb{N}_0$, we say $\mathcal{F}$ is **stable**. We define the **associated graded module** of M with respect to $\mathcal{F}$ by

$$\mathrm{gr}_{\mathcal{F}}(M) := \bigoplus_{n=0}^{\infty} M_n/M_{n+1}$$

as abelian groups. This becomes a $\mathrm{gr}_I(R)$-module by taking

$$(r + I^{i+1})(m + M_{j+1}) := rm + M_{i+j+1}$$

whenever $r \in I^i$, $m \in M_j$.

Lemma 5.2. *Let $R, I, M, \mathcal{F}$ be as in definition 5.1 with R Noetherian. Suppose M is finitely generated over R and that $\mathcal{F}$ is stable. Then $gr_{\mathcal{F}}(M)$ is a finitely-generated $gr_I(R)$-module.*

Proof. See Proposition 10.22 in [1]. □

Remark 5.3. Let $R \in |\underline{\text{CommRing}}|$ be local and Noetherian with maximal ideal $\mathfrak{m}$, and suppose M is a finitely generated R-module. Then there's a natural stable $\mathfrak{m}$-filtration $\mathcal{F}$ of M given by

$$M = \mathfrak{m}^0 M \geq \mathfrak{m}M \geq \mathfrak{m}^2 M \geq \dots.$$

Hence $\mathrm{gr}_{\mathcal{F}}(M)$ is a finitely generated $\mathrm{gr}_{\mathfrak{m}}(R)$-module by 5.2. Also $\mathrm{gr}_{\mathfrak{m}}(R)_0 = R/\mathfrak{m}$ is a field (so Artinian) and if $m_1, \dots, m_s$ generate $\mathfrak{m}$ as an R-module, then

$$\mathrm{gr}_{\mathfrak{m}}(R) = (R/\mathfrak{m})[m_1 + \mathfrak{m}^2, \dots, m_s + \mathfrak{m}^2].$$

In particular, $\mathrm{gr}_{\mathfrak{m}}(R)$ is finitely generated as an $R/\mathfrak{m}$-algebra. Thus in the context of definition 4.2 we may define the **Hilbert function** of M by

$$H_M(n) := H_{\mathrm{gr}_{\mathcal{F}(M)}}(n) = \dim_{R/\mathfrak{m}}(\mathfrak{m}^n M/\mathfrak{m}^{n+1} M)$$

for all $n \in \mathbb{N}_0$. As in remark 4.3, we have $m_i + \mathfrak{m}^2 \in \mathrm{gr}_{\mathfrak{m}}(R)_1$ for all $i = 1, \ldots, s$, so again by corollary 3.4 there is a **Hilbert polynomial** h_M of M with

$$h_M(n) = H_M(n)$$

for all sufficiently large $n \in \mathbb{N}_0$.

Definition 5.4. Let $X \subseteq \mathbb{P}^N$ be a quasi-projective variety and fix $p \in X \neq \varnothing$. Then the stalk $\mathcal{O} := \mathcal{O}_{X,p}$ of X at p is a local Noetherian ring with, say, maximal ideal $\mathfrak{m}$, so remark 5.3 allows us to define **Hilbert function** of X at p as

$$H_{X,p}(n) := \dim_k(\mathcal{O}/\mathfrak{m}^n) = \sum_{i=0}^{n-1} \dim_k(\mathfrak{m}^i\mathcal{O}/\mathfrak{m}^{i+1}\mathcal{O}) = \sum_{i=0}^{n-1} H_{\mathcal{O}}(i)$$

for all $n \in \mathbb{N}_0$ where $k = \mathcal{O}/\mathfrak{m}$.

Theorem 5.5. *Let $X, p, \mathcal{O}, \mathfrak{m}$ be as in 5.4 with $D_p = \dim_p(X)$. There there is an $h_{X,p} \in \mathbb{Q}[x]$ called the **Hilbert polynomial** of X at p such that*

$$h_{X,p}(n) = H_{X,p}(n)$$

for all sufficiently large $n \in \mathbb{N}_0$. In fact, the leading term of $h_{X,p}$ is

$$\frac{m}{D_p!}x^{D_p}$$

for some $m \in \mathbb{N}$. Moreover, if $X = V(f) \subseteq \mathbb{A}^N$ is a hypersurface, then m is the multiplicity $m_p(X)$ of p on X.

Proof. Note that

$$H_{X,p}(n+1) - H_{X,p}(n) = H_{\mathcal{O}}(n) = h_{\mathcal{O}}(n)$$

for all sufficiently large $n \in \mathbb{N}_0$. This implies $H_{X,p}$ is a polynomial function $h_{X,p}$ with rational coefficients for large n. In fact, if the leading term of $h_{X,p}(x)$ is αx^β, then the above relation along with the proof of corollary 3.4 implies

$$\begin{aligned} \frac{m}{(w-1)!}x^{w-1} &= \text{leading term of } h_{\mathcal{O}} \\ &= \alpha\beta x^{\beta-1} \end{aligned}$$

where w is the order of pole at $x = 1$ in $P_{\mathrm{gr}_{\mathcal{F}(\mathcal{O})}}(x)$ and $m = g(1) \in \mathbb{N}$ (again positivity follows since $H_{X,p} \geq 0$), so $\beta = w$ and $\alpha = m/w!$. The Krull dimension theorem implies that

$$\begin{aligned} w &= \text{least \#generators of } \mathfrak{m} \\ &= \dim(\mathcal{O}) \\ &= \dim_p(X) \\ &= D_p \end{aligned}$$

(see [1], pg.s 119-121). Now assume $X = V(f) \subseteq \mathbb{A}^N$ is a hypersurface and denote $\mu = m_p(X)$. Then wlog $p = (0, \ldots, 0)$, so setting $I = (x_1, \ldots, x_N)$ and $R = k[x_1, \ldots, x_N]$, we have

$$\mathcal{O}/\mathfrak{m}^n \cong R/(I^n, f)$$

for all $n \in \mathbb{N}_0$. Also, for $n \geq \mu$ there's a short exact sequence of k-vector spaces

$$\begin{array}{ccccc} R/I^{n-\mu} & \rightarrowtail & R/I^n & \twoheadrightarrow & R/(I^n, f) \\ r + I^{n-\mu} & \mapsto & fr + I^n & & \\ & & r + I^n & \mapsto & r + (I^n, f) \end{array}$$

where injectivity of the first map can be seen as follows, while exactness elsewhere is clear. Let $r \in R$, and write

$$f = f_\mu + f_{\mu+1} + \cdots \qquad \text{and} \qquad r = r_\nu + r_{\nu+1} + \cdots$$

where $f_i, r_i \in R_i$ for all i and $f_\mu \neq 0 \neq r_\nu$. Hence if $fr \in I^n$, then $\mu + \nu \geq n$, so $\nu \geq n - \mu$, giving $r \in I^{n-\mu}$. Note that R/I^n is generated as a k-vector space by all monic monomials having degree less than n, so

$$\begin{aligned} \dim_k(R/I^n) &= \sum_{j=0}^{n-1} \#\text{monic monomials of degree } j \\ &= \sum_{j=0}^{n-1} \binom{N-1+j}{N-1} \\ &= \sum_{j=N-1}^{N+n-2} \binom{j}{N-1} \\ &= \binom{N+n-1}{N}. \end{aligned}$$

Therefore

$$\begin{aligned} H_{X,p}(n) &= \dim_k(\mathcal{O}/\mathfrak{m}^n) \\ &= \dim_k(R/(I^n, f)) \\ &= \dim_k(R/I^n) - \dim_k(R/I^{n-\mu}) \\ &= \binom{N+n-1}{N} - \binom{N+n-\mu-1}{N} \\ &= \frac{(n+N-1)\cdots n - (n-\mu+N-1)\cdots(n-\mu)}{N!} \\ &= \frac{1}{N!}\left(n^N + \binom{N}{2} n^{N_1} \cdots - n^N - \left(-N\mu + \binom{N}{2}\right) n^{N-1} \cdots\right) \\ &= \frac{\mu}{(N-1)!} n^{N-1} + \cdots \end{aligned}$$

for all sufficiently large $n \in \mathbb{N}_0$, giving

$$\text{leading term of } h_{X,p}(x) = \frac{\mu}{(N-1)!} x^{N-1},$$

whence $m = \mu = m_p(X)$. □

References

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