An Introduction to Iwasawa Theory

Jordan Schettler

University of California, Santa Barbara

?/?/2012

Outline

Cyclotomic Fields

Notation

Unless noted otherwise, p is an odd prime: 3, 5, 7, 11, 13, 17, 19 . . .

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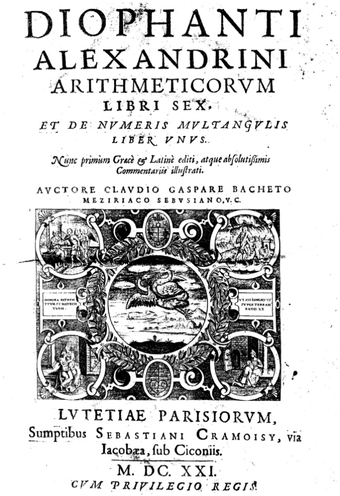
Throughout m, n are positive integers.

Diophantus of Alexandria (≈ AD 207–291)

Arithmetica: integer or rational solutions to polynomial equations w/ integer coeff.s



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AVCTORE CLAVDIO GASPARE BACHETO MEZIRIACO SEBVSIANO, V.C.

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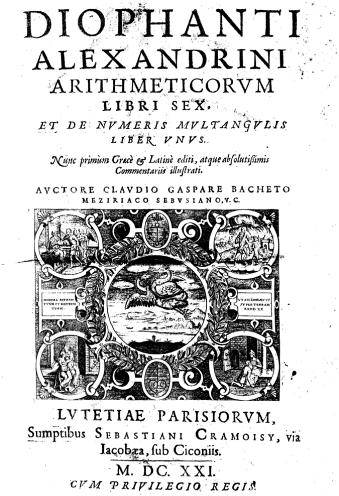

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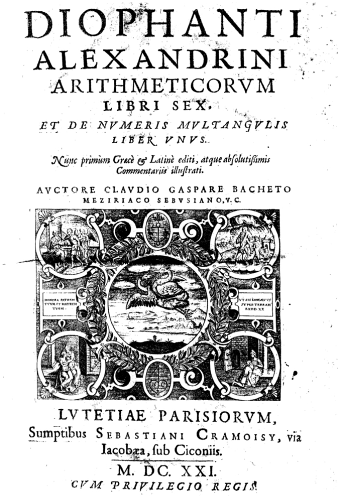
We know $\exists\infty$ly many tuples of relatively prime positive integers (x, y, z) s.t. $x^2 + y^2 = z^2$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Fermat claimed that if $n \geq 3$, there are no tuples of positive integers (x, y, z) s.t. $x^n + y^n = z^n$

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Thus the conjecture boils down to the case $n = p$.

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Suppose (x, y, z) is a tuple of pairwise relatively prime integers s.t. $x^p + y^p = z^p$.

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Note: $\mathbb{Z}[\zeta_p]$ is a Dedekind domain but not necessarily a PID.

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If $p \nmid$ class number of $\mathbb{Q}(\zeta_p)$, then $J = (\alpha)$ is principal,

but $x + \zeta_p y = \varepsilon \alpha^p$ (some unit ε) leads to a contradiction.

Regular versus Irregular Primes

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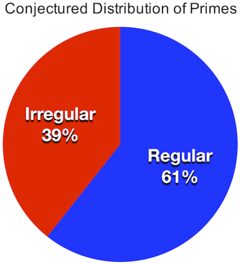

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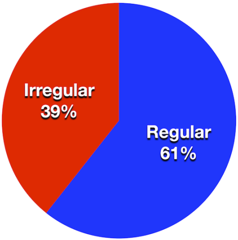

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Theorem (Kummer's Criterion)

An odd prime p is regular if and only if $p \nmid$ *numerator of* B_j *for all* $j = 2, 4, \ldots, p-3$ *where* $\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$.

Bernoulli Numbers: $\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$

$B_0 = 1, B_1 = -\frac{1}{2}$ while $B_{2n+1} = 0$ for all n and

$$B_2 = \frac{1}{6}, \qquad B_4 = -\frac{1}{30}, \qquad B_6 = \frac{1}{42},$$

$$B_8 = -\frac{1}{30}, \qquad B_{10} = \frac{5}{66}, \qquad B_{12} = -\frac{691}{2730}$$

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Theorem (Bernoulli's/Faulhaber's Formula)

$$1^n + 2^n + 3^n + \cdots + m^n = \frac{m^{n+1}}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} \frac{B_j}{(-m)^j}$$

In lowest terms,

$$B_{2n} \text{ has denominator} = \prod_{p-1|2n} p$$

More About Bernoulli Numbers

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If $n \equiv m \not\equiv -1 \pmod{p-1}$,

$$\frac{B_{n+1}}{n+1} \equiv \frac{B_{m+1}}{m+1} \pmod{p}$$

Zeta Values

In 1735, Euler showed

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

In general,

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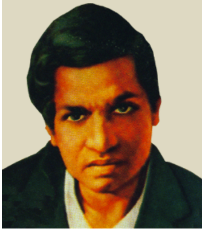

In 1913, Ramanujan suggested

$$1 + 2 + 3 + \cdots = -\frac{1}{12}$$

In general,

$$1^n + 2^n + 3^n + \cdots = -\frac{B_{n+1}}{n+1}$$

Bernhard Riemann (1826–1866)

For $\Re(s) > 1$,

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$\exists!$meromorphic continuation of ζ s.t.

$$\frac{\zeta(s)}{(-s)!} = (2\pi)^{s-1} 2 \sin(s\pi/2) \zeta(1-s)$$

Restatement of Kummer Criterion

Let A denote the p-primary part of the class group of $\mathbb{Q}(\zeta_p)$.

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E.g., we have $691 | \zeta(-11)$. What does the -11 mean here?

$\exists$isomorphism $\phi\colon \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \to (\mathbb{Z}/(p))^\times$ given by

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The Teichmüller Character

$\exists$isomorphism $\phi\colon \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \to (\mathbb{Z}/(p))^\times$ given by

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Define $\omega\colon \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \to \mathbb{Z}_p$ via

$$\omega(\sigma) \equiv \phi(\sigma) \pmod{p} \quad \text{and} \quad \omega(\sigma)^{p-1} = 1$$

$G := \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ acts on $A =$ p-primary part of class grp.

‘Eigenespace’ Decomposition

$G := \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ acts on $A = p$-primary part of class grp.

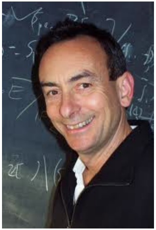
Regard A as a $\mathbb{Z}_p G$-module (written additively):

$$A = \bigoplus_{n=1}^{p-1} A^{\omega^{-n}}$$

where

$$\alpha \in A^{\omega^{-n}} \Leftrightarrow \sigma\alpha = \omega(\sigma)^{-n}\alpha \quad \forall \sigma \in G$$

J. Herbrand (1908–1931), K. Ribet (1948–)



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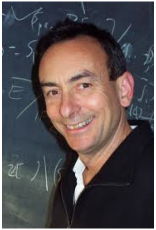

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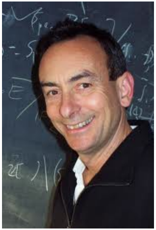

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Conjecturally, $A^{\omega^{-n}} = 0$ for all p and all even n.

Theorem (Higher Kummer Congruences)

If $n \equiv m \not\equiv -1 \pmod{p-1}$ *and* $n \neq m$

$$\left| \frac{\zeta(-n)}{(1-p^n)^{-1}} - \frac{\zeta(-m)}{(1-p^m)^{-1}} \right|_p < |n-m|_p$$

where $|\cdot|_p$ *is the normalized* p*-adic absolute value.*

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Theorem (Kubota and Leopoldt, 1964)

$\exists!$*continuous function* $L_p(s, \omega^j)$ *from* $\mathbb{Z}_p$ *to* $\mathbb{Q}_p$ *s.t.*

$$L_p(-n, \omega^j) = \frac{\zeta(-n)}{(1-p^n)^{-1}}$$

whenever $n \equiv j-1 \pmod{p-1}$*.*

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$\rightsquigarrow$ For n odd, $L_p(s, \omega^{n+1})$ contains information about $A^{\omega^{-n}}$

$\mathbb{Z}_p$-Extensions

First Examples of $\mathbb{Z}_p$-Extensions

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$$\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) \cong (\mathbb{Z}/(p^n))^\times$$

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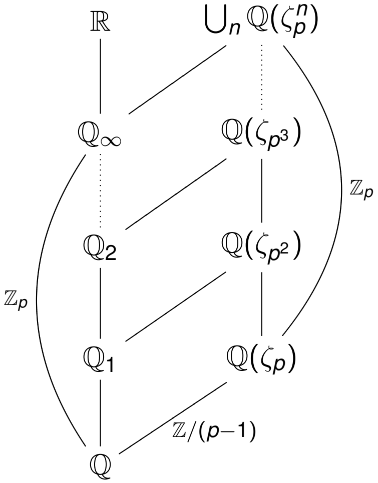

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Note: If $\zeta_p \in F$, then $F\mathbb{Q}_\infty = F(\zeta_p, \zeta_{p^2}, \ldots)$.

General $\mathbb{Z}_p$-Extensions

Now let p be any prime, and suppose F is a number field s.t.

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The subfields of F_∞ which contain F lie in a tower

$$F \subset F_1 \subset F_2 \subset \ldots \subset F_\infty$$

s.t. for all n

$$\text{Gal}(F_n/F) \cong \mathbb{Z}/(p^n)$$

Kenkichi Iwasawa (1917–1998)

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Theorem (Iwasawa's Growth Formula)

Suppose $F \subset F_1 \subset F_2 \dots$ is a $\mathbb{Z}_p$-extension of number fields. Let A_n denote the p-primary part of the class group of F_n. Then $\exists$ integers λ, μ, ν s.t.

$$|A_n| = p^{\lambda n + \mu p^n + \nu}$$

for all sufficiently large n.

Trivial Iwasawa Invariants λ, μ, ν

Theorem

Let F be a number field with exactly one prime above p. If $p \nmid$ class number of F, then $\lambda = \mu = \nu = 0$ for any $\mathbb{Z}_p$-extension F_∞/F.

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In fact, $p \nmid$ class number of F_n for all n in this case.

E.g., consider $F = \mathbb{Q}$, or $F = \mathbb{Q}(\zeta_p)$ for a regular prime p.

Theorem

Suppose F is a number field in which p splits completely. Then $\lambda \geq r_2$ for the cyclotomic $\mathbb{Z}_p$-extension $F\mathbb{Q}_\infty/F$ where $r_2 = \#$ complex primes of F.

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E.g., consider $F = \mathbb{Q}(\sqrt{-1})$. If $p \equiv 1 \pmod 4$, then p splits in $F/\mathbb{Q}$, so here $\lambda \geq 1$ for $\mathbb{Q}_\infty(i)/\mathbb{Q}(i)$.

Nontrivial Iwasawa Invariants λ, μ, ν

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What about $r_2 = 0$ (F is totally real)? Can we have $\lambda > 0$?

Iwasawa Modules

Iwasawa's Idea to Derive Growth Formula

Let $L_n = p$-Hilbert class field of F_n. Take $L_\infty = \bigcup_n L_n$.

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$g \in \Gamma := \mathrm{Gal}(F_\infty/F)$ acts on $x \in X := \mathrm{Gal}(L_\infty/F_\infty)$ as

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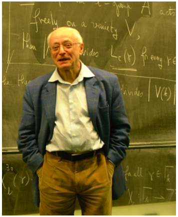
$$g \cdot x = \tilde{g} x \tilde{g}^{-1} \text{ where } \tilde{g} \text{ extends } g \text{ to } L_\infty$$

If F has only one prime above p, and this prime is totally ramified in F_∞/F, then

$$X/(\gamma^{p^n} - 1)X \cong A_n \text{ where } \overline{\langle\gamma\rangle} = \Gamma$$

Jean-Pierre Serre (1926–)



Serre's 1959 Seminaire Bourbaki: let T act on X as $\gamma - 1$. Then X becomes a finitely generated, torsion Λ-module where $\Lambda = \mathbb{Z}_p[[T]]$.

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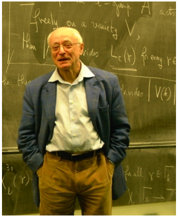

Serre's 1959 Seminaire Bourbaki: let T act on X as $\gamma - 1$. Then X becomes a finitely generated, torsion Λ-module where $\Lambda = \mathbb{Z}_p[[T]]$.

Then use the structure theory for Λ-modules to compute

$$|X/((T+1)^{p_n} - 1)X|$$

Structure Theorem for Λ-Modules

$\Lambda = \mathbb{Z}_p[[T]]$ is a local Noetherian UFD, but *not* a PID.

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Theorem

Suppose M is a finitely gen'd Λ-module. Then $\exists$homomorphism with finite kernel and cokernel

$$M \longrightarrow \Lambda^r \oplus \bigoplus_{i=1}^{s} \frac{\Lambda}{(p^{m_i})} \oplus \bigoplus_{j=1}^{t} \frac{\Lambda}{(f_i(T)^{n_j})}$$

where each $f_j(T) \equiv x^{\deg(f_j)} \pmod{p}$ is an irred. poly.

Cyclotomic Fields
$\mathbb{Z}_p$-Extensions
Iwasawa Modules
The Main Conjecture

In particular, ∃homom. with finite kernel and cokernel

$$X \longrightarrow \bigoplus_{i=1}^{s} \frac{\Lambda}{(p^{m_i})} \oplus \bigoplus_{j=1}^{t} \frac{\Lambda}{(f_i(T)^{n_j})}$$

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where each $f_j(T) \equiv x^{\deg(f_j)} \pmod{p}$ is an irred. poly.

We have a well-defined 'characteristic polynomial'

$$\text{char}(X) = \prod_{i=1}^{s} p^{m_i} \prod_{j=1}^{t} f_j(T)^{n_j}$$

In fact, if λ, μ, ν are as in the growth formula for F_∞/F,

$$\mu = \mathrm{ord}_p(\mathrm{char}(X)) = m_1 + \cdots + m_s$$

$$\lambda = \deg(\mathrm{char}(X)) = n_1 \deg(f_1) + \cdots + n_t \deg(f_t)$$

The Main Conjecture

'Eigenspaces' of X

If $F = \mathbb{Q}(\zeta_p)$, then $G = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ acts on X.

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and each $X^{\omega^{-n}}$ is a finitely gen’d torsion Λ-module.

Again, $L_p(s, \omega^{n+1})$ contains info about $X^{\omega^{-n}}$.

Main Conjecture

There is a power series $\tilde{L}_p(T, \omega^{n+1}) \in \Lambda = \mathbb{Z}_p[[T]]$ s.t.

$$\tilde{L}_p((1+p)^s - 1, \omega^{n+1}) = L_p(s, \omega^{n+1})$$

for all $s \in \mathbb{Z}_p$.

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Theorem (Mazur and Wiles, 1984)

Suppose n is odd and $n \not\equiv 1 \pmod{p-1}$. Then

$$(\mathrm{char}(X^{\omega^{-n}})) = (\tilde{L}_p(T, \omega^{n+1}))$$

as ideals in Λ.

Thank You!