Special Riemann-Hurwitz Formulas in Iwasawa Theory

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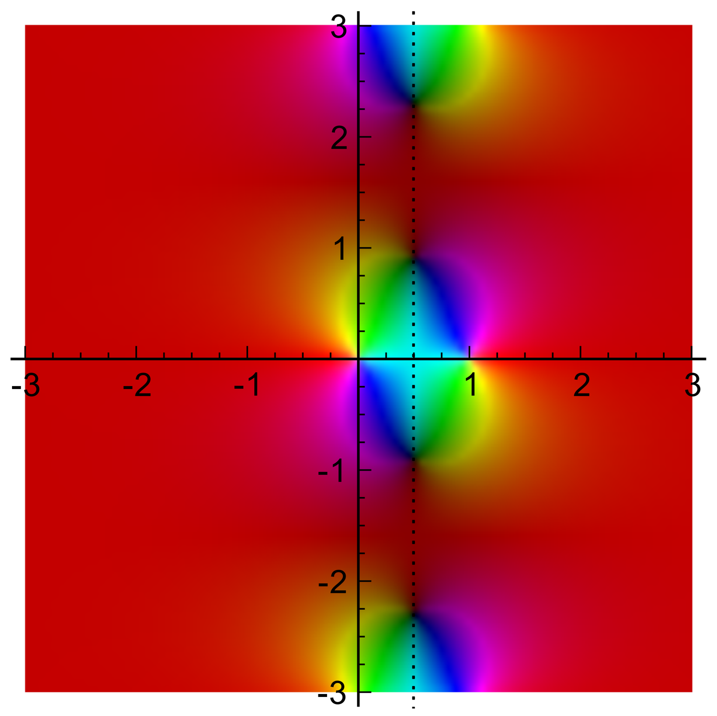
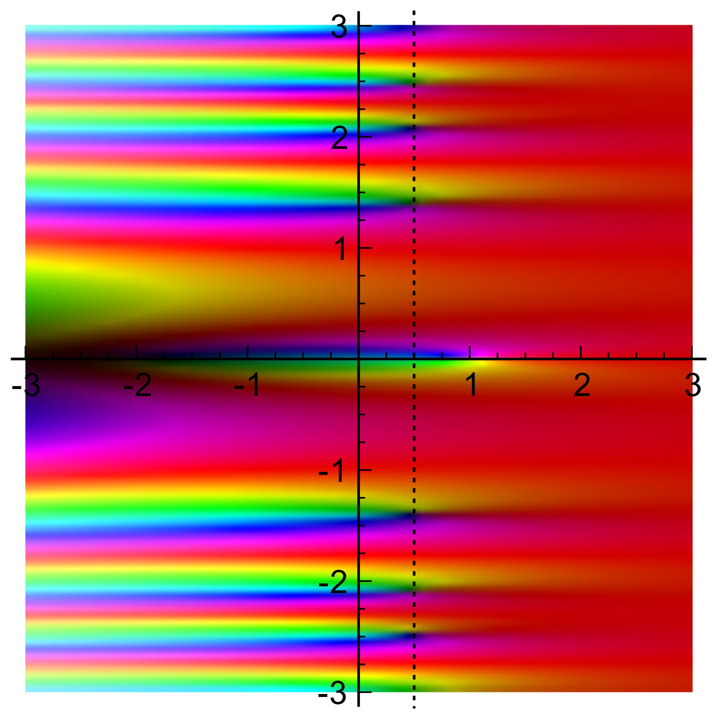
Analogies

Function Fields k over $\mathbb{F}_q$	Number Fields k
$[k:\mathbb{F}_q(t)]<\infty$, $\overline{\mathbb{F}}_q\cap k=\mathbb{F}_q$	$[k:\mathbb{Q}]<\infty$
k is the function field of a projective curve X_k over $\mathbb{F}_q$ (regular, integral)	k is the function field of $X_k:=\mathrm{Spec}(\mathcal{O}_k)$, $\dim=1$ (regular, integral)
$\exists\zeta_k(s)$, Riemann Hyp.	$\exists\zeta_k(s)$, Riemann Hyp.
 Figure: $Z(7^{-s})$ for a curve$/\mathbb{F}_7$	 Figure: Riemann zeta $\zeta(s)$
$h(k)=\lvert\mathrm{Pic}^0(X_k)\rvert<\infty$	$h(k)=\lvert\mathrm{Pic}(X_k)\rvert<\infty$
$\bigcup_{n\geq 0}k\mathbb{F}_{q^{p^n}}$ — k_∞ — k_n — k; $\mathrm{Gal}(k_\infty/k)$: $\mathbb{Z}_p$, $\mathrm{Gal}(k_n/k)$: $\frac{\mathbb{Z}}{(p^n)}$	$\bigcup_{n\geq 0}k(\zeta_{p^n})$ — k_∞ — k_n — k; $\mathrm{Gal}(k_\infty/k)$: $\mathbb{Z}_p$, $\mathrm{Gal}(k_n/k)$: $\frac{\mathbb{Z}}{(p^n)}$
Growth Formula: $\exists\lambda,\nu\in\mathbb{Z}$ s.t. $\forall n\gg 0$	Growth Formula: $\exists\lambda,\mu,\nu\in\mathbb{Z}$ s.t. $\forall n\gg 0$
$\mathrm{ord}_p h(k_n)=\lambda n+\nu$	$\mathrm{ord}_p h(k_n)=\lambda n+\mu p^n+\nu$

Remark:
Iwasawa conjectured that $\mu=0$ for the $\mathbb{Z}_p$-extensions k_∞/k as above. $\boxed{\text{Here we assume } \mu=0 \text{ for such extensions.}}$ A field $K=k_\infty$ for some number field k is called a $\mathbb{Z}_p$**-field**; equivalently, $[K:\mathbb{Q}_\infty]<\infty$. We take λ_K to be the λ in the growth formula for K/k since it does not depend on k.

Known Formulas

Let $[K:\overline{\mathbb{F}}_q(t)]<\infty$ with $p\nmid q$. Then K is the function field of a projective curve X_K (regular, integral) over $\overline{\mathbb{F}}_q$ and

$$\mathrm{Pic}^0(X_K)[p^\infty]\cong(\mathbb{Q}_p/\mathbb{Z}_p)^{2g_K}$$

(Hurwitz): $\mathrm{Gal}(L/K)\cong\mathbb{Z}/(p)\Rightarrow$

$$2g_L=p2g_K-(p-1)2+\sum_{x\in X_L}(e_x-1)$$

Let $[K:\mathbb{Q}_\infty]<\infty$ with $\mu=0$. Then K is the function field of $X_K=\mathrm{Spec}(\mathcal{O}_K[1/p])$ (regular, integral) with $\dim=1$ and

$$\mathrm{Pic}(X_K)[p^\infty]\cong(\mathbb{Q}_p/\mathbb{Z}_p)^{\lambda_K}$$

(Iwasawa): $\mathrm{Gal}(L/K)\cong\mathbb{Z}/(p)\Rightarrow$

$$\lambda_L=p\lambda_K-(p-1)\chi(G,P_L)+\sum_{x\in X_L}(e_x-1)$$

Special Formulas for $\mathbb{Z}_p$-Fields

Notation:
Let $[K_0:\mathbb{Q}_\infty]<\infty$ with $\mu=0$. Consider a tower:

$$K_0\subset K_1\subset\cdots\subset K_i\subset\cdots\subset K_{n-1}\subset K_n$$

$$\frac{\mathbb{Z}}{(p^{n-i})}\cong\mathrm{Gal}(K_n/K_i)=:N_i \qquad \boxed{\chi(G,M)=\mathrm{ord}_p\frac{\lvert H^2(G,M)\rvert}{\lvert H^1(G,M)\rvert}}$$

$$G_i:=\mathrm{Gal}(K_i/K_0)\cong\frac{\mathbb{Z}}{(p^i)}$$

$\boxed{I_{K_i}=\text{invertible ideals of }\mathcal{O}_{K_i}}$

$\boxed{P_{K_i}=\text{principal invertible ideals}}$

$\boxed{C_{K_i}=I_{K_i}/P_{K_i}}$

Main Result:

$$\frac{\lambda_{K_n}-p^n\lambda_{K_0}}{p-1}=p^{n-1}\chi(G_n,C_{K_n})-\sum_{i=1}^{n-1}\varphi(p^i)\chi(G_i,C_{K_i})$$

$$=\frac{p^n}{np-n+1}\chi(N_0,C_{K_n})+\sum_{i=1}^{n-1}\frac{p^i(p-1)\chi(N_{n-i},C_{K_n})}{(ip-i+p)(ip-i+1)}$$

Applications of the Special Formulas

Extending Ferrero's and Kida's Computations:
Let K be the cyclotomic $\mathbb{Z}_2$-extension of the first layer k in the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ where p is 2 or a Fermat prime and $h(k)$ is odd (e.g., $p=2,3,5,17,257,\ldots$), and let L be the cyclotomic $\mathbb{Z}_2$-extension of $k(\sqrt{-d})$ with $d\in\mathbb{Z}$ squarefree and $d>2\geq(d,p)$. Then

$$\lambda_L=\lvert S\rvert-1$$

where S is the set of finite places of L not lying above 2 which are ramified in L/K.

A Vanishing Criterion:
Let L/K be a cyclic p-extension of $\mathbb{Z}_p$-fields which is unramified at every infinite place. Suppose $K=k_\infty$ for a number field k having exactly one place above p with $p\nmid h(k)$. Then $\lambda_L=0$ if and only if $\mathrm{ord}_p\lvert(I_L^G P_L)/(I_K P_L)\rvert=0$ where $G=\mathrm{Gal}(L/K)$. Note that $(I_L^G P_L)/(I_K P_L)$ is generated by the cosets of certain products of finite places of L not lying above p and ramified in L/K.

Congruences:

$$\lambda_{K_n}\equiv\lambda_{K_i}\pmod{\varphi(p^{i+1})}\qquad\text{for all } i=0,\ldots,n$$

$$\lambda_{K_n}\equiv -p^{n-1}\chi(G_n,C_{K_n})-(p-1)\sum_{i=1}^{n-1}\varphi(p^i)\chi(G_i,C_{K_i})\pmod{p^n}$$

and if $p\nmid n-1$

$$\lambda_{K_n}\equiv\sum_{i=1}^{n-1}\frac{p^i(p-1)^2\chi(N_{n-i},C_{K_n})}{(ip-i+p)(ip-i+1)}\pmod{p^n}$$

Note: The first congruence above can be thought of as an analog to the corresponding congruences for twice the genera of function fields in cyclic p-extensions. For example, the genus of the Fermat curve $x^d+y^d=z^d$ over $\mathbb{C}$ is $(d-1)(d-2)/2$, so we can see a special case of the congruence quite easily by noticing that

$$2\frac{(p^n-1)(p^n-2)}{2}\equiv 2\frac{(p^i-1)(p^i-2)}{2}\pmod{\varphi(p^{i+1})}.$$

Inequalities:

$$\mathrm{ord}_p\lvert H^2(G_n,P_{K_n})\rvert\leq n\lambda_{K_0}+\mathrm{ord}_p\lvert H^1(G_n,P_{K_n})\rvert+\chi(G_n,I_{K_n})$$

and if $\lambda_{K_0}=0$

$$\mathrm{ord}_p\lvert H^2(G_n,P_{K_n})\rvert\leq\chi(G_n,I_{K_n})$$