Theory of Classical Modular Forms and Symbols

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Outline

Motivation: Elliptic Curves over $\mathbb{Q}$

Every elliptic curve $E/\mathbb{Q}$ has a Tate-Weierstrass equation

$$y^2 + Axy + By = f(x),$$

with minimal discriminant Δ_E where $A, B \in \mathbb{Z}$ and $f(x) \in \mathbb{Z}[x]$ is monic of degree 3.

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The L-function of E is defined by an Euler product

$$L(E, s) = \prod_p L_p(p^{-s})^{-1}$$

where for $a_p = p + 1 - |\widetilde{E}_p(\mathbb{F}_p)|$

$$L_p(T) = \begin{cases} 1 - a_p T + pT^2 & \text{good reduction} \\ 1 - T & \text{split multiplicative reduction} \\ 1 + T & \text{nonsplit multiplicative reduction} \\ 1 & \text{additive reduction} \end{cases}$$

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In fact, L extends to an analytic function on all of $\mathbb{C}$ and satisfies a functional equation

$$L^*(E, s) = \pm L^*(E, 2 - s)$$

where

$$L^*(E, s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(E, s)$$

with $N =$ conductor of E.

The Birch and Swinnerton-Dyer conjecture states that

$$0 \neq \lim_{s \to 1} \frac{L(E, s)}{(s-1)^r} = \frac{|\text{Ш}| R \prod_v c_v}{|E(\mathbb{Q})_{\text{tors}}|^2} \Omega_+$$

where

$$E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus E(\mathbb{Q})_{\text{tors}}$$

and $\text{Ш} =$ Tate-Shaferavich group, $R =$ regulator of E $c_v =$ Tamagawa factor at the place v, and $\Omega_+ =$ real period of E.

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Here Ω_+ is the positive generator of $\mathbb{R} \cap \Lambda_E$ where Λ_E is the lattice generated by the period integrals of the invariant differential

$$\omega = \frac{dx}{2y + Ax + B}$$

Suppose E has either good ordinary or multiplicative reduction for a prime p. Write the Euler factor at p as

$$L_p(T)^{-1} = \frac{1}{(1 - \alpha_p T)(1 - \beta_p T)}$$

where $\alpha_p \in \mathbb{Z}_p^\times$ and $p | \beta_p$.

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There is a p-adic L-function $L_p(E, s)$ associated to E defined by an interpolation property which, in particular, implies

$$L_p(E, 1) = (1-\alpha_p^{-1})\frac{L(E, 1)}{\Omega_+}$$

If E has split reduction at p, then $L_p(T) = 1 - T$, so $\alpha_p = 1$, $\beta_p = 0$, and

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The Mazur, Tate, Teitelbaum conjecture predicts

$$L_p'(E, 1) = \mathcal{L}_p(E) \frac{L(E, 1)}{\Omega_+}$$

where $\mathcal{L}_p(E) = \log_p(q)/\mathrm{ord}_p(q)$ with $\overline{\mathbb{Q}}_p^\times / q^{\mathbb{Z}} \cong E(\overline{\mathbb{Q}}_p)$

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is a weight 2 cusp form for $\Gamma_0(N)$. In fact, f is an eigenform for Hecke operators $T(p)$ for all primes $p \nmid N$:

$$T(p) f_E = a_p f_E$$

The fact that $f_E(z)$ is a cusp form is part of a series of deep results known as the modularity theorem.

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This states, in particular, that there is a surjective morphism (of curves over $\mathbb{Q}$)

$$\phi\colon X_0(N) \to E$$

such that $\phi^*(\omega)$ is a multiple of $f_E(z)\,dz$.

We have

$$
\begin{aligned}
2\pi i \int_{i\infty}^{0} f_E(z)\, dz &= L(E, 1) \\
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In general,

$$L(E,\chi,1) = \frac{\tau(\chi)}{N} \sum_{a=1}^{N} \overline{\chi}(a) 2\pi i \int_{i\infty}^{-a/N} f_E(z)\, dz$$

Modular Forms: Definitions

For $k \in \mathbb{N}_0$, we have a right action of $\mathrm{GL}_2(\mathbb{Q})$ on functions f from the upper half-plane $\mathcal{H}$ to $\mathbb{C}$:

$$f^{[\gamma]_k}(z) := \det(\gamma)^{k-1}(cz + d)^{-k} f(\gamma(z))$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\gamma(z) = \frac{az+b}{cz+d}$.

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For an integer $N \geq 1$, we define

$$\Gamma(N) := \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod{N}\}.$$

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E.g., $\Gamma_0(N)$ (upper triangular modulo N) and $\Gamma_1(N)$ (upper uni-triangular modulo N) are congruence subgroups.

The *space of weight k modular forms* $M_k(\Gamma)$ is the set of functions $f\colon \mathcal{H} \to \mathbb{C}$ such that $f^{[\gamma]_k} = f$ for all $\gamma \in \Gamma$ and f is holomorphic on the extended upper half plane $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ where we view $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{i\infty\}$ (the set of cusps).

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Note: For k even, $\dim_{\mathbb{C}} M_k(\Gamma(1) = \lfloor k/12 \rfloor$ (resp. $\lfloor k/12 \rfloor + 1$) if $k \equiv 2 \pmod{12}$ (resp. $k \not\equiv 2 \pmod{12}$). Also, $\dim_{\mathbb{C}} S_k(\Gamma(1)) = \dim_{\mathbb{C}} M_k(\Gamma(1)) - 1$.

The *nonnormalized Eisenstein series of weight* $2k$ ($k > 1$) is

$$G_{2k}(z) = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(mz+n)^{2k}}$$

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The *normalized Eisenstein series* of weight $2k$ ($k > 1$) is

$$E_{2k} = \frac{(2k-1)!}{2(2\pi i)^{2k}} G_{2k} = -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

For $\frac{g_2}{60} = G_4$, $\frac{g_3}{140} = G_6$, there is an elliptic curve $y^2 = g(x)$ where the cubic $g(x) = 4x^3 - g_2 x - g_3$ has discriminant

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Actually, a modular form of any weight for $\Gamma(1)$ is in $\mathbb{C}[G_4, G_6]$.

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$$

where $\tau(n)\colon \mathbb{N} \to \mathbb{Z}$ is Ramanujan's tau function, which turns out to be multiplicative, i.e., $\tau(mn) = \tau(m)\tau(n)$ whenever $(m, n) = 1$, and satisfies interesting congruences like

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}$$

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In fact, $\Delta(z) = (\eta(z))^{24}$ where

$$\eta(z) = q^{1/12} \prod_{n=1}^{\infty} (1 - q^n)$$

is Dedekind's eta function.

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Also, note that $f_E(z) \in S_2(\Gamma_0(N))$ where $N =$ conductor of E.

Modular Symbols: Weight 2

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If $N = p \equiv -1 \pmod{12}$ is a prime, then $g = (p+1)/12$.

Thus $S_2(\Gamma_0(11)) = \mathbb{C}(\eta(z)\eta(11z))^2$.

We define the period integrals of $f \in S_2(\Gamma)$ for Γ a congruence subgroup by

$$I(r,s) := 2\pi i \int_r^s f(z)\,dz$$

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(Drinfeld-Manin) Each element in Λ_{f_E} is of the form

$$a\Omega_+ + b\Omega_-$$

for some $a, b \in \mathbb{Q}$ with bounded denominators where Ω_- is the positive imaginary generator of $\Lambda_E \cap i\mathbb{R}$.

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In fact, the Γ-invariance of $f(z)\,dz$ shows that ψ_f is in the space of $\mathbb{C}$-*valued modular symbols*

$$\text{Symb}_\Gamma(\mathbb{C}) := \text{Hom}_\Gamma(\text{Div}_0, \mathbb{C})$$

where Γ acts trivially on $\mathbb{C}$.

For $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$, there is an involution ι on $\mathrm{Symb}_\Gamma(\mathbb{C})$ given by

$$\iota(\varphi)(D) = \varphi(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) D)$$

since here $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ normalizes Γ.

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Thus, in this case, each ψ_f gives two modular symbols ψ_f^+, ψ_f^- with $\psi_f = \psi_f^+ + \psi_f^-$.

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Also, any $\varphi \in \mathrm{Hom}_\Gamma(\mathrm{Div}, \mathbb{C})$ will restrict $\varphi|_{\mathrm{Div}_0}$, but such a φ is just a function which is constant on the cusps $\mathbb{P}^1(\mathbb{Q})/\Gamma$.

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Moreover, $\psi \in \mathrm{Symb}_\Gamma(\mathbb{C})$ is determined by its values on generators of Div_0 as a $\mathbb{Z}[\Gamma]$-module.

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This follows from Manin's continued fraction trick: if p_n, q_n are the numerator and denominator (resp.) of the nth convergent in the continued fraction expansion of some real number,

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For example,

$$[4,7] - [0,1] =$$
$$[4,7] - [-1,-2] + [1,2] - [1,1] + [-1,-1] - [-1,0] + [1,0] - [0,-1]$$
$$= \left[\begin{pmatrix} -1 & 4 \\ -2 & 7 \end{pmatrix}\right] + \left[\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\right] + \left[\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}\right] + \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]$$

In fact, if $\alpha, \beta \in \Gamma(1)$, then $\alpha([\beta]) = [\alpha\beta]$. In other words, the divisor of the product $\alpha\beta$ is the divisor obtained by applying α to the divisor $[\beta]$ via Möbius transformation.

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$$[\Gamma(1) : \Gamma_0(N)] = N \prod_{p|N} (1 + p^{-1}).$$

Hecke Operators on Modular Forms

Conceptual Definition for Level 1

We can view modular forms $f \in M_k(\Gamma(1))$ as functions F on lattices $\Lambda \subset \mathbb{C}$. (Here $F(\mathbb{Z} + z\mathbb{Z}) = f(z)$.)

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Then for each $n \in \mathbb{N}$ we define

$$(T(n)F)(\Lambda) = \sum_{\substack{\Lambda' \leq \Lambda \\ [\Lambda:\Lambda'] = n}} F(\Lambda')$$

For each integer $n \geq 1$, define $X_n \subseteq \mathrm{M}_2(\mathbb{Z})$ to be the matrices of determinant n in Hermite normal form, i.e.,

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \text{ with } ad = n \text{ and } d > b \geq 0$$

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If $n = p$ is prime,

$$T(p)f = f^{\left[\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\right]_k} + \sum_{a=0}^{p-1} f^{\left[\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right)\right]_k}$$

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For a prime p,

$$T(p^n) = T(p^{n-1})T(p) - p^{k-1}T(p^{n-2})$$

When we have a Fourier series

$$f = \sum_{n=0}^{\infty} c_n q^n,$$

for $q = e^{2\pi iz}$ and the Hecke action can be described by

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so for $n = p$ prime

$$T(p)f = \sum_{m=0}^{\infty} \left(c_{mp} + p^{k-1} c_{m/p} \right) q^m$$

where $c_{m/p} = 0$ if $p \nmid m$. ($T(n)$ preserves $M_k(\Gamma(1))$, $S_k(\Gamma(1))$.)

Conceptual Definition for $S_2(\Gamma_0(N))$

The complex points of $Y_0(N) = \mathcal{H}/\Gamma_0(N)$ are in natural bijection with pairs (E, C) where $E/\mathbb{C}$ is an elliptic curve and $\langle P \rangle \leq E(\mathbb{C})$ is cyclic of order N.

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$$\left(\frac{\mathbb{C}}{\mathbb{Z} + \lambda\mathbb{Z}}, \frac{(1/N)\mathbb{Z} + \lambda\mathbb{Z}}{\mathbb{Z} + \lambda\mathbb{Z}} \right)$$

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When $(n, N) = 1$ there are two natural maps

$$\pi_1 \colon Y_0(n \cdot N) \to Y_0(N) \colon (E, \langle Q \rangle) \mapsto (E, \langle Q^n \rangle)$$
$$\pi_2 \colon Y_0(n \cdot N) \to Y_0(N) \colon (E, \langle Q \rangle) \mapsto (E/\langle Q^N \rangle, \langle Q \rangle / \langle Q^N \rangle)$$

These maps extend uniquely to maps on X_0:

$$\begin{array}{ccc} & X_0(n \cdot N) & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ X_0(N) & & X_0(N) \end{array}$$

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$\exists$action on $S_2(\Gamma_0(N)) \cong$ holomorphic 1-forms on $X_0(N)$ given by

$$T(n)f =: \pi_{1*}(\pi_2^*(f(z)dz))$$

and there is a similar definition when $(n, N) \neq 1$.

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If $p|N$, the we have

$$T(p)f = \sum_{a=0}^{p-1} f^{\left[\begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}\right]_k}$$

Modular Symbols: Weight k

For $f \in S_k(\Gamma)$ with $\Gamma = \Gamma_1(N)$, we have a Fourier series

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$$= (2\pi)^s \Gamma(s)^{-1} \int_0^{\infty} f(it) t^s \frac{dt}{t}$$

If we change variables to $z = it$, we get special values of L

$$2\pi i \int_{i\infty}^{0} f(z) z^j \, dz = \frac{j!}{(-2\pi i)^j} L(f, j+1)$$

for $0 \leq j \leq k-2$.

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for $0 \leq j \leq k-2$.

We want a target space to encode the period integrals

$$I_j(r, s) = 2\pi i \int_{r}^{s} f(z) z^j \, dz$$

for the $k-1$ values of $j \in \{0, 1, \ldots, k-2\}$.

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$$\psi_f \colon \mathrm{Div}_0 \to V_{k-2}(\mathbb{C})$$

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This is given by $\mathbb{Z}$-linearly extending

$$s - r \mapsto 2\pi i \int_r^s f(z)(zX + Y)^{k-2}\, dz$$

There is a right action of $\mathrm{GL}(\mathbb{Q})$ on $V_g(\mathbb{C})$ given by

$$(P|\gamma)(X,Y) = P(dX - cY, -bX + aY)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

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With this action, ψ_f is in the *space of $V_{k-2}(\mathbb{C})$-valued modular symbols*

$$\mathrm{Symb}_\Gamma(V_{k-2}(\mathbb{C})) = \mathrm{Hom}_\Gamma(\mathrm{Div}_0, V_{k-2}(\mathbb{C}))$$

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If $\varphi \in \mathrm{Hom}(\mathrm{Div}_0, V_{k-2}(\mathbb{C}))$, then φ is a modular symbol if

$$\varphi(\gamma D)|\gamma = \varphi(D)$$

for all $D \in \mathrm{Div}_0$ and all $\gamma \in \Gamma$.

In fact, the $\mathrm{GL}_2(\mathbb{Q})$ action on $V_{k-2}(\mathbb{C})$ was defined so that the association $f \mapsto \psi_f$ is equivariant:

$$\psi_{f[\gamma]_k}(D) = \psi_f(D)|\gamma$$

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We have a right $\mathrm{GL}_2(\mathbb{Q})$-action on modular symbols φ:

$$(\varphi|\gamma)(D) = \varphi(\gamma D)|\gamma$$

(Note: $\varphi|\gamma = \varphi$ for all $\gamma \in \Gamma$)

For a prime p, we define the action of a Hecke operator $T(p)$ on $\mathrm{Symb}_\Gamma(V_{k-2}(\mathbb{C}))$ via the double coset:

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma = \coprod_{\gamma \in X} \Gamma\gamma$$

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Note: for $\Gamma = \Gamma_0(N)$ and $p \nmid N$, this action is

$$T(p)\varphi = \varphi| \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) + \sum_{a+0}^{p-1} \varphi| \left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right)$$

Eichler-Shimura Theorem

For $\Gamma = \Gamma_1(N)$, there is an isomorphism

$$\mathrm{Symb}_\Gamma(V_{k-2}(\mathbb{C})) \cong M_k(\Gamma) \oplus S_k(\Gamma)$$

which respects the Hecke action on both sides.

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Note that $M_k(\Gamma) = E_k(\Gamma) \oplus S_k(\Gamma)$, so there are two copies of the cusp forms in the modular symbols, which correspond to ψ_f^+ and ψ_f^- obtained from the involution $\iota = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

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The Eisenstein series $E_k(\Gamma)$ correspond to those symbols obtained from taking a Γ-invariant homomorphism on Div and then restricting to Div_0.