# A Riemann-Hurwitz Formula for Number Fields

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   - Background
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- In the late 1970's, Kida proved an analogous formula for CM-fields which is used to compute Iwasawa invariants.
- I'll discuss/compare both formulas, give examples/uses, and sketch the ideas involved in the proof of Kida's result.

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## Definition

Recall that a Riemann surface $R$ is a connected 1-dimensional complex manifold.

In other words, $R$ is a second countable, Hausdorff topological space and $\exists$charts $\{(U_i, \varphi_i)\}_{i \in I}$ such that

- $\{U_i\}_{i \in I}$ forms an open cover of $R$
- $\forall i \in I$ $\varphi_i : U_i \to \mathbb{C}$ is a homeomorphism onto its image
- $\forall i, j \in I$ $\varphi_i \circ \varphi_j^{-1}$ is holomorphic on $\varphi_j(U_i \cap U_j)$.

## Remark

Let $f : R_1 \to R_2$ be a nonconstant holomorphic map between compact Riemann surfaces and fix $p \in R_1$.

$\exists$charts $\varphi : U \to \mathbb{C}$ around $p$ and $\psi : V \to \mathbb{C}$ around $f(p)$ s.t.

- $\psi \circ f \circ \varphi^{-1}$ is holomorphic on $\varphi(U \cap f^{-1}(V))$
- wlog $\varphi(p) = 0 = \psi(f(p))$.

## Remark (continued)

$\therefore$ $\exists$a disk $|z| < r$ on which

$$(\psi \circ f \circ \varphi^{-1})(z) = \sum_{n=e(p)}^{\infty} a_n z^n$$

where $a_{e(p)} \neq 0$ and $e(p) \geq 1$ (independent of the charts $\varphi, \psi$) is called the ramification index of $f$ at $p$. When $e(p) > 1$, we call $p$ a ramification point of $f$.

## Theorem (Riemann-Hurwitz, late 1800's)

*With $f$ as above, $\exists d \in \mathbb{N}$ (= degree of $f$) such that $\forall q \in R_2$*

$$\sum_{p \in f^{-1}(\{q\})} e(p) = d.$$

*Moreover, we have the formula*

$$2g_1 - 2 = d(2g_2 - 2) + \sum_{p \in R_1} (e(p) - 1)$$

*where for $i = 1, 2$ the surface $R_i$ has genus $g_i$ (= # of "holes").*

## Example

Consider the Fermat curve

$$F_3 = \{[x, y, z] \in \mathbb{CP}^2 : x^3 + y^3 + z^3 = 0\}.$$

Note that $F_3$ is a compact ($\because$ it's a closed subset of a compact space) Riemann surface and we have a natural mapping

$$f : F_3 \to \mathbb{CP}^1$$

given by

$$[x, y, z] \mapsto [x, y].$$

## Example (continued)

Fix $[x, y] \in \mathbb{CP}^1$. Then either

1. $[x, y] \in \{[1, -1], [1, -\omega], [1, -\omega^2]\}$ where $\omega = e^{2\pi i/3}$, in which case
$$f^{-1}(\{[x, y]\}) = \{[x, y, 0]\},$$
or
2. $$f^{-1}(\{[x, y]\}) = \{[x, y, -\alpha], [x, y, -\omega\alpha], [x, y, -\omega^2\alpha]\}$$
where $\alpha^3 = x^3 + y^3 \neq 0$.

## Example (continued)

$\therefore$ the degree of $f$ is 3 and $\exists$exactly 3 ramification points

$$[1,-1,0],[1,-\omega,0],[1,-\omega^2,0],$$

each having ramification index 3.

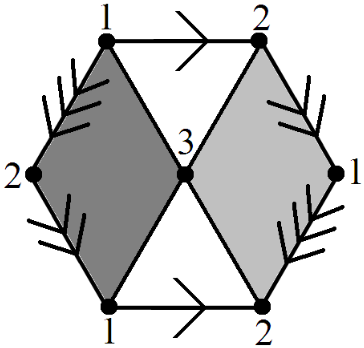
$\because \mathbb{CP}^1 \approx \mathbb{S}^2$ has genus 0 the Riemann-Hurwitz formula gives

$$2g-2=3(2\cdot 0-2)+[(3-1)+(3-1)+(3-1)]=0,$$

whence $F_3$ has genus $g=1$.

## Example (continued)

$\therefore F_3 \approx \mathbb{T}^2$, so the torus may be cut into three spheres as illustrated below where the ramification points are the intersections of multiple sheets.



## Remark

More generally, the Riemann-Hurwitz formula implies that the genus $g$ of the Fermat curve

$$F_d := \{[x, y, z] \in \mathbb{CP}^2 : x^d + y^d + z^d = 0\}$$

is given by

$$g = \frac{(d-1)(d-2)}{2}.$$

## Remark

Assume $E$ is a $\mathbb{Z}_p$-extension of a number field $F$ for some prime $p$, i.e. $E/F$ is Galois and

$$G := \mathrm{Gal}(E/F) \cong \mathbb{Z}_p.$$

The nontrivial closed subgroups of $\mathbb{Z}_p$ are $p^n\mathbb{Z}_p$ for each $n \in \mathbb{N}_0$, so the extensions $F_\alpha$ of $F$ contained in $E$ form a tower

$$F = F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq E$$

such that

$$\forall n \in \mathbb{N}_0 \quad \mathrm{Gal}(F_n/F) \cong \mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}.$$

**Theorem (Iwasawa's Growth Formula, 1973)**

*Let $E/F$ be as above. Then $\exists \lambda, \mu, \nu \in \mathbb{Z}$ with $\lambda, \mu \geq 0$ such that the exponent $e_n$ of $p$ occurring in the class number $h(F_n)$ is given by*

$$e_n = \lambda n + \mu p^n + \nu$$

*for all sufficiently large $n$.*

## Remark

Let $E/F$ be as above. Then the only primes of $F$ which ramify in $E/F$ lie over $p$, and one such prime must ramify. Moreover, if

$$p \nmid h(F)$$

and exactly one prime of $F$ ramifies in $E/F$, then

$$\forall n \in \mathbb{N}_0 \quad p \nmid h(F_n),$$

so

$$\lambda = \mu = \nu = 0;$$

in particular, this is always the case for $F = \mathbb{Q}$.

## Remark

For given $F$ and $p$, there may be infinitely many $\mathbb{Z}_p$-extensions, but $\exists!$ $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$, so there's a canonical choice

$$F_\infty := F\mathbb{Q}_\infty$$

called the cyclotomic $\mathbb{Z}_p$-extension of $F$.

$\therefore$ by taking $E = F_\infty$ in the growth formula we may define the Iwasawa invariants of $F$ as

$$\lambda_p(F) = \lambda,\ \mu_p(F) = \mu,\ \nu_p(F) = \nu.$$

## Definition

A CM-field $K$ is a totally complex quadratic extension of a totally real number field $K_+$. In other words, $[K : K_+] = 2$, $[K_+ : \mathbb{Q}] < \infty$, and

$$\forall \text{embeddings } \iota : K \rightarrowtail \mathbb{C} \quad \text{im}(\iota) \not\subseteq \mathbb{R}$$

while

$$\forall \text{embeddings } \iota : K_+ \rightarrowtail \mathbb{C} \quad \text{im}(\iota) \subseteq \mathbb{R}.$$

## Remark

Given a CM-field $K$ and a prime $p$, the exponent $e_n^-$ of $p$ occurring in the relative class number $h^-(K_n) = h(K_n)/h(K_{n,+})$ is given by

$$e_n^- = \lambda_p^-(K)n + \mu_p^-(K)p^n + \nu_p^-(K)$$

for all sufficiently large $n$ where for $\gamma \in \{\lambda, \mu, \nu\}$ we take

$$\gamma_p^-(K) := \gamma_p(K) - \gamma_p(K_+).$$

## Theorem (Kida, 1979)

*Suppose $L/K$ is a finite $p$-extension of CM-fields with $p$ an odd prime & $\mu_p^-(K) = 0$. Then $\mu_p^-(L) = 0$ and we have the formula*

$$2\lambda_p^-(L) - 2\delta = [L_\infty : K_\infty](2\lambda_p^-(K) - 2\delta) + \sum_{\mathfrak{P} \in S(L)} (e(\mathfrak{P}) - 1)$$

*where $\delta$ is 1 or 0 if $\zeta_p \in K$ or $\zeta_p \notin K$, respectively, $e(\mathfrak{P})$ is the ramification index of $\mathfrak{P}$ in $L_\infty/K_\infty$, and*

$$S(L) := \{\textit{prime ideals } \mathfrak{P} \nmid p \textit{ in } L_\infty : \mathfrak{P} \textit{ splits in } L_\infty/L_{\infty,+}\}.$$

## Remark

The hypothesis $\mu_p^-(K) = 0$ holds in many useful contexts.

## Theorem (Ferrero-Washington, 1979)

*Let $F$ be an abelian number field and $p$ be a prime. Then*

$$\mu_p(F) = 0.$$

## Conjecture (Iwasawa)

*Let $F$ be a number field and $p$ be a prime. Then $\mu_p(F) = 0$.*

## Remark

We may be able to replace the relative Iwasawa invariants $\lambda^-, \mu^-$ with $\lambda, \mu$ in Kida's formula, as the following conjecture suggests.

## Conjecture (Greenberg)

*Let $K_+$ be a totally real number field and $p$ be a prime. Then*

$$\lambda_p(K_+) = \mu_p(K_+) = 0.$$

### Theorem (Iwasawa, 1980)

*Suppose $L/K$ is a $\mathbb{Z}/p\mathbb{Z}$-extension of number fields with $p$ prime, $L_\infty/K_\infty$ unramified at every infinite place of $K_\infty$, & $\mu_p(K) = 0$. Then $\mu_p(L) = 0$ and we have the formula*

$$\lambda_p(L) + h_2 - h_1 = [L_\infty : K_\infty](\lambda_p(K) + h_2 - h_1) + \sum_{w \nmid p}(e(w) - 1)$$

*where $w$ ranges over all non-$p$-places of $L_\infty$, $e(w)$ is the ramification index of $w$ in $L_\infty/K_\infty$, and for $i = 1, 2$*

$$h_i = \text{rank of } H^i(L_\infty/K_\infty, \mathcal{O}_{L_\infty}^\times).$$

## Example

$$
\begin{array}{c}
\mathbb{Q}(\zeta_{13}) \\
\big| \; 2 \\
\mathbb{Q}(\cos(2\pi/13)) \\
\big| \; 2 \\
F \\
\big| \; 3 \\
\mathbb{Q}
\end{array}
$$

Consider the ! field $F$ in the tower. Then $F/\mathbb{Q}$...

- is totally real
- is Galois
- has $\mathrm{Gal}(F/\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$
- is unramified outside 13
- is totally ramified at 13
- remains prime at 3.

## Example (continued)

Note that $F(i)/\mathbb{Q}(i)$ is a $\mathbb{Z}/3\mathbb{Z}$-extension of CM-fields and

$$\lambda_3^-(\mathbb{Q}(i)) = \mu_3^-(\mathbb{Q}(i)) = 0$$

$\because$ 3 remains prime in $\mathbb{Q}(i)/\mathbb{Q}$ and $\mathbb{Q}(i)$ has class number 1.

Using Kida's formula with $p = 3$, $K = \mathbb{Q}(i)$, and $L = F(i)$, gives

$$2\lambda_3^-(L) - 2 \cdot 0 = [L_\infty : K_\infty](2 \cdot 0 - 2 \cdot 0) + \sum_{\mathfrak{P} \in S(L)} (e(\mathfrak{P}) - 1).$$

## Example (continued)

Also, $[L_\infty : K_\infty] = [L : K] = 3$, so we get

$$\begin{aligned}
\lambda_3^-(L) &= \frac{1}{2} \sum_{\mathfrak{P} \in S(L)} (e(\mathfrak{P}) - 1) \\
&= \#\{\mathfrak{P} \in S(L) : e(\mathfrak{P}) > 1\} \\
&= \#\{\text{prime ideals } \mathfrak{P} \text{ in } L_\infty : \mathfrak{P} | 13\} \\
&= 2.
\end{aligned}$$

## Remark

In fact, using a more general construction, Kida's formula can be used to prove the following result.

## Theorem (Fujii-Ohgi-Ozaki, 2004)

*Let* $p \in \{3,5\}$ *and* $n \in \mathbb{N}_0$*. Then* $\exists$*CM-field* $L$ *such that* $\mu_p(L) = \mu_p^-(L) = 0$ *and*

$$\lambda_p(L) = \lambda_p^-(L) = n.$$

## Theorem (Special Case)

*Suppose $L/K$ is a $\mathbb{Z}/p\mathbb{Z}$-extension of CM-fields with $p$ an odd prime & $\mu_p^-(K) = 0$ such that*

- $[L_\infty : K_\infty] = p$
- *$\exists$prime ideal $\mathfrak{Q} \nmid p$ in $L$ which ramifies in $L/K$.*

*Then $\mu_p^-(L) = 0$ and we have the formula*

$$\lambda_p^-(L) = p\lambda_p^-(K) + (p-1)(s_\infty - \delta)$$

*where $\delta$ is as above and $2s_\infty = \#\{\mathfrak{P} \in S(L) : e(\mathfrak{P}) > 1\}$.*

**Sketch.**

$\forall n \in \mathbb{N}_0$ $L_{n+1}/K_n$ is a type $(p,p)$-extension, so $\exists p+1$ proper intermediate fields $L_{n,i}$. The analytic class number formula $\Rightarrow$

$$\frac{\#A^-(L_n)}{\#A^-(K_n)} = \prod_{i=0}^{p} \frac{\#A^-(L_{n,i})}{\#A^-(K_n)}$$

where $A(F) \cong A^-(F) \oplus A(F_+)$ is the Sylow-$p$ subgroup of the class group $\mathrm{Cl}(F)$.

## Sketch Continued.

- $\mu_p^-(K) = 0 \Rightarrow \mu_p^-(L) = 0$ and for $F = K, L$

$$\lambda_p^-(F) = \log_p(\#A^-(F_{n+1})/\#A^-(F_n)) = d^{(p)}(A^-(F_n))$$

for sufficiently large $n$ where $d^{(p)}(X) = \dim_{\mathbb{F}_p}(X/pX)$.

- ($p$ odd and $\mathfrak{Q} \nmid p$ ramified in $L/K$) $\Rightarrow$

$$A^-(K_n), A^-(L_{n,i}) \text{ naturally embed into } A^-(L_{n+1})$$

## Sketch Continued.

The above two facts can be used to show

$$d^{(p)}(A^{-}(L_{n,i})^{G_{n,i}}) = \lambda^{-}(K) + s_n - \delta$$

for sufficiently large $n$ where $G_{n,i} = \mathrm{Gal}(L_{n,i}/K_n)$ and

$$2s_n = \#\{\mathfrak{P} \nmid p \text{ in } L_n : \text{ramifies in } L_n/K_n, \text{splits in } L_n/L_{n,+}\}.$$

## Sketch Continued.

Now $L_n = L_{n,j}, K_{n+1} = L_{n,k}$ for some $j, k$, so for large $n$

$$\begin{aligned}
\lambda_p^-(L) &= d^{(p)}(A^-(L_n)) \\
&\leq (p-1)d^{(p)}(A^-(L_n)^{G_{n,j}}) + d^{(p)}(A^-(K_n)) \\
&= (p-1)(\lambda^-(K) + s_n - \delta) + \lambda^-(K) \\
&= p\lambda^-(K) + (p-1)(s_\infty - \delta)
\end{aligned}$$

## Sketch Continued.

and

$$\begin{aligned} \lambda_p^-(L) &= \lambda^-(K) + \sum_{i \neq j,k} \log_p \left( \#A^-(L_{n,i}) / \#A^-(K_n) \right) \\ &\geq \lambda^-(K) + \sum_{i \neq j,k} d^{(p)}(A^-(L_{n,i})^{G_{n,i}}) \\ &= \lambda^-(K) + (p-1)(\lambda^-(K) + s_n - \delta) \\ &= p\lambda^-(K) + (p-1)(s_\infty - \delta). \end{aligned}$$

□