Slumdog Millionaire

Jordan Schettler

Department of Mathematics
University of Arizona

Outline

Quartic Formula

Figure: Srinivasa Ramanujan

- born in India on Dec. 22, 1887; contracted smallpox in 1889

Figure: Srinivasa Ramanujan

- born in India on Dec. 22, 1887; contracted smallpox in 1889
- became one of the greatest mathematicians of modern times

Figure: Srinivasa Ramanujan

- born in India on Dec. 22, 1887; contracted smallpox in 1889
- became one of the greatest mathematicians of modern times
- extracted deep results from a single out-of-date textbook published in 1856

Figure: Srinivasa Ramanujan

- born in India on Dec. 22, 1887; contracted smallpox in 1889
- became one of the greatest mathematicians of modern times
- extracted deep results from a single out-of-date textbook published in 1856
- an orthodox Brahmin (Hindu) and a strict vegetarian

- Ramanujan dreamt of blood drops symbolizing Narasimha's murder of a demon and received visions of formulas.

- Ramanujan dreamt of blood drops symbolizing Narasimha's murder of a demon and received visions of formulas.
- He said "An equation for me has no meaning, unless it represents a thought of God,"

- Ramanujan dreamt of blood drops symbolizing Narasimha's murder of a demon and received visions of formulas.
- He said "An equation for me has no meaning, unless it represents a thought of God,"
- but he also remarked that all religions seemed equally true to him.

At the age of 15, Ramanujan was shown how to solve a cubic

$$x^3 + ax^2 + bx + c = 0.$$

At the age of 15, Ramanujan was shown how to solve a cubic

$$x^3 + ax^2 + bx + c = 0.$$

He then solved the quartic equation

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

on his own; one of the roots is given by $x = \ldots$

$$\frac{-a}{4}-\frac{1}{2}\sqrt{\frac{a^2}{4}-\frac{2b}{3}+\frac{2^{\frac{1}{3}}\left(b^2-3ac+12d\right)}{3\left(2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4\left(b^2-3ac+12d\right)^3+\left(2b^3-9abc+27c^2+27a^2d-72bd\right)^2}\right)^{\frac{1}{3}}}}$$

$$\frac{-a}{4}-\frac{1}{2}\sqrt{\frac{a^2}{4}-\frac{2b}{3}+\frac{2^{\frac{1}{3}}\left(b^2-3ac+12d\right)}{3\left(2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4\left(b^2-3ac+12d\right)^3+\left(2b^3-9abc+27c^2+27a^2d-72bd\right)^2}\right)^{\frac{1}{3}}}+\left(\frac{2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4\left(b^2-3ac+12d\right)^3+\left(2b^3-9abc+27c^2+27a^2d-72bd\right)^2}}{54}\right)^{\frac{1}{3}}}$$

$$\frac{-a}{4}-\frac{1}{2}\sqrt{\frac{a^2}{4}-\frac{2b}{3}+\frac{2^{\frac{1}{3}}\left(b^2-3ac+12d\right)}{3\left(2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4\left(b^2-3ac+12d\right)^3+\left(2b^3-9abc+27c^2+27a^2d-72bd\right)^2}\right)^{\frac{1}{3}}}}$$

$$+\left(\frac{2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4\left(b^2-3ac+12d\right)^3+\left(2b^3-9abc+27c^2+27a^2d-72bd\right)^2}}{54}\right)^{\frac{1}{3}}$$

$$-\frac{1}{2}\sqrt{\frac{a^2}{2}-\frac{4b}{3}-\frac{2^{\frac{1}{3}}\left(b^2-3ac+12d\right)}{3\left(2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4\left(b^2-3ac+12d\right)^3+\left(2b^3-9abc+27c^2+27a^2d-72bd\right)^2}\right)^{\frac{1}{3}}}}$$

$$
\frac{-a}{4}-\frac{1}{2}\sqrt{\frac{a^2}{4}-\frac{2b}{3}+\frac{2^{\frac{1}{3}}\left(b^2-3ac+12d\right)}{3\left(2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4\left(b^2-3ac+12d\right)^3+\left(2b^3-9abc+27c^2+27a^2d-72bd\right)^2}\right)^{\frac{1}{3}}}+\left(\frac{2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4\left(b^2-3ac+12d\right)^3+\left(2b^3-9abc+27c^2+27a^2d-72bd\right)^2}}{54}\right)^{\frac{1}{3}}}
$$

$$
-\frac{1}{2}\sqrt{\frac{a^2}{2}-\frac{4b}{3}-\frac{2^{\frac{1}{3}}\left(b^2-3ac+12d\right)}{3\left(2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4\left(b^2-3ac+12d\right)^3+\left(2b^3-9abc+27c^2+27a^2d-72bd\right)^2}\right)^{\frac{1}{3}}}-\left(\frac{2b^3-9abc+27c^2+27a^2d-72bd+\sqrt{-4\left(b^2-3ac+12d\right)^3+\left(2b^3-9abc+27c^2+27a^2d-72bd\right)^2}}{54}\right)^{\frac{1}{3}}}
$$

$$
\begin{aligned}
&\frac{-a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}}
+ \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}} \\
&- \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} - \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}}
- \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}
- \frac{-a^3 + 4ab - 8c}{4\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{2^{\frac{1}{3}}\left(b^2 - 3ac + 12d\right)}{3\left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}\right)^{\frac{1}{3}}} + \left(\frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4\left(b^2 - 3ac + 12d\right)^3 + \left(2b^3 - 9abc + 27c^2 + 27a^2d - 72bd\right)^2}}{54}\right)^{\frac{1}{3}}}}}
\end{aligned}
$$

He tried to solve the quintic equation

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$$

in the following year, but was unsuccessful.

He tried to solve the quintic equation

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$$

in the following year, but was unsuccessful.

Turns out, there is no quintic formula.

He tried to solve the quintic equation

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$$

in the following year, but was unsuccessful.

Turns out, there is no quintic formula.

It's not that we just can't find it... we've proven that no such formula exists!

Nested Expressions

In 1904, he was given a scholarship to his hometown college, but he lost it after one year because he neglected all but math.

Figure: Temple in Kumbakonam

In 1905, Ramanujan ran away from home, became ill the next year, and lived in extreme poverty.

In 1905, Ramanujan ran away from home, became ill the next year, and lived in extreme poverty.

He tried to attend the University of Madras in 1906, but failed every subject on a preliminary exam except math.

In 1905, Ramanujan ran away from home, became ill the next year, and lived in extreme poverty.

He tried to attend the University of Madras in 1906, but failed every subject on a preliminary exam except math.

He became ill again in 1909, and eventually wound up at the *Journal of the Indian Math. Society*:

$$? = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots}}}}$$

He posed the above problem, going unsolved for over 6 months.

Note that

$$n^2 - 1 = (n-1)(n+1),$$

so

$$n = \sqrt{1 + (n-1)(n+1)}$$

Note that

$$n^2 - 1 = (n-1)(n+1),$$

so

$$n = \sqrt{1 + (n-1)(n+1)}$$

In particular,

$$3 = \sqrt{1 + 2 \cdot 4}$$

Note that

$$n^2 - 1 = (n-1)(n+1),$$

so

$$n = \sqrt{1 + (n-1)(n+1)}$$

In particular,

$$3 = \sqrt{1 + 2 \cdot 4}$$

$$= \sqrt{1 + 2\sqrt{1 + 3 \cdot 5}}$$

Note that

$$n^2 - 1 = (n-1)(n+1),$$

so

$$n = \sqrt{1 + (n-1)(n+1)}$$

In particular,

$$3 = \sqrt{1 + 2 \cdot 4}$$

$$= \sqrt{1 + 2\sqrt{1 + 3 \cdot 5}}$$

$$= \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4 \cdot 6}}}$$

Note that

$$n^2 - 1 = (n-1)(n+1),$$

so

$$n = \sqrt{1 + (n-1)(n+1)}$$

In particular,

$$3 = \sqrt{1 + 2 \cdot 4}$$

$$= \sqrt{1 + 2\sqrt{1 + 3 \cdot 5}}$$

$$= \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4 \cdot 6}}}$$

$$= \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + \cdots}}}}$$

Ramanujan was also a master of continued fractions...

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

Ramanujan was also a master of continued fractions...

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

Note that

$$\varphi = 1 + \frac{1}{\varphi}$$

Ramanujan was also a master of continued fractions...

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

Note that

$$\varphi = 1 + \frac{1}{\varphi}$$

so

$$\varphi^2 - \varphi - 1 = 0.$$

Ramanujan was also a master of continued fractions...

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$$

Note that

$$\varphi = 1 + \frac{1}{\varphi}$$

so

$$\varphi^2 - \varphi - 1 = 0.$$

Thus

$$\varphi = \frac{1 + \sqrt{5}}{2} = 1.61803398\ldots$$

$\varphi =$

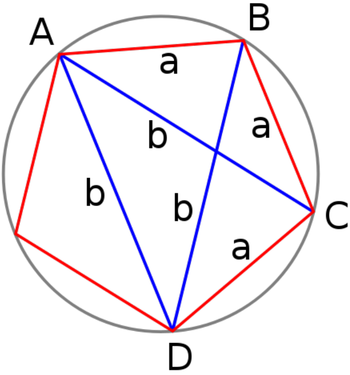


Figure: Ptolemy's Pentagon

$$\varphi = \frac{b}{a}$$

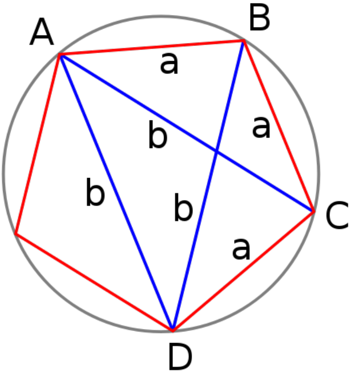


Figure: Ptolemy's Pentagon

$$\varphi = \frac{b}{a} = \lim_{n\to\infty} \frac{F_n}{F_{n-1}}$$

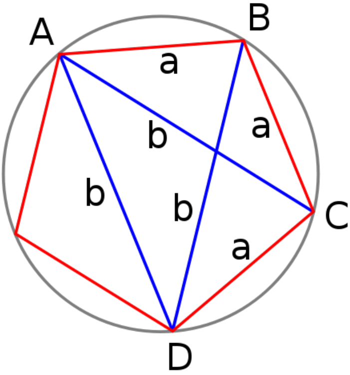


Figure: Ptolemy's Pentagon

$$\varphi = \frac{b}{a} = \lim_{n\to\infty} \frac{F_n}{F_{n-1}} = 3 - 4\sin^2(\pi/5)$$

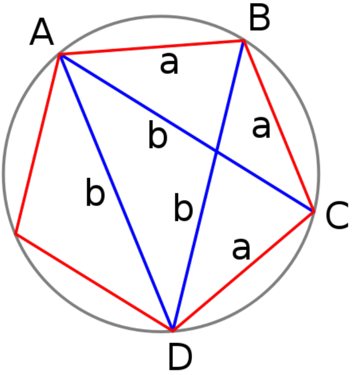


Figure: Ptolemy's Pentagon

$$\begin{aligned}\varphi &= \frac{b}{a} \\ &= \lim_{n\to\infty} \frac{F_n}{F_{n-1}} \\ &= 3 - 4\sin^2(\pi/5) \\ &= \sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}\end{aligned}$$

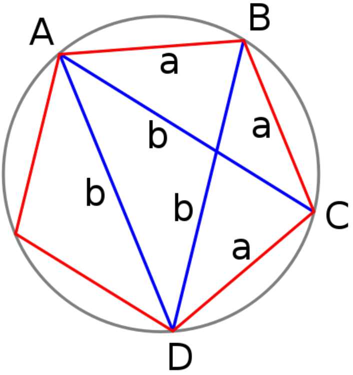


Figure: Ptolemy's Pentagon

Ramanujan found a relationship between φ, π, and e:

$$\frac{e^{-2\pi/5}}{\sqrt{\varphi\sqrt{5}-\varphi}} = 1 + \cfrac{e^{-2\pi}}{1 + \cfrac{e^{-4\pi}}{1 + \cfrac{e^{-6\pi}}{1 + \cfrac{e^{-8\pi}}{1 + \cdots}}}}$$

Define $R(q)$ by

$$\frac{q^{1/5}}{R(q)} = 1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cfrac{q^4}{1 + \cdots}}}}$$

Define $R(q)$ by

$$\frac{q^{1/5}}{R(q)} = 1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cfrac{q^4}{1 + \cdots}}}}$$

Ramanujan's identity amounts to

$$R(e^{-2\pi}) = \sqrt{\varphi\sqrt{5}} - \varphi$$

Let $q = e^{2\pi i\tau}$ for $\mathrm{Im}(\tau) > 0$. Then $R(q)$ satisfies the identity

$$\frac{1}{R(q)} - 1 - R(q) = \left(\frac{\Delta(\tau/5)}{\Delta(5\tau)}\right)^{1/24}$$

Let $q = e^{2\pi i\tau}$ for $\mathrm{Im}(\tau) > 0$. Then $R(q)$ satisfies the identity

$$\frac{1}{R(q)} - 1 - R(q) = \left(\frac{\Delta(\tau/5)}{\Delta(5\tau)}\right)^{1/24}$$

where

$$\begin{aligned}\Delta(\tau) &= \text{discriminant of the elliptic curve} \\ &\quad E_\tau : y^2 = 4x^3 - g_2(\tau)x - g_3(\tau) \\ &\quad \text{associated to the lattice } \mathbb{Z} + \tau\mathbb{Z} \\ &= g_2(\tau)^3 - 27g_3(\tau)^2\end{aligned}$$

If $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$, then

$$\alpha^6 \Delta(\alpha i/\pi) = \beta^6 \Delta(\beta i/\pi)$$

If $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$, then

$$\alpha^6 \Delta(\alpha i/\pi) = \beta^6 \Delta(\beta i/\pi)$$

From this it follows that

$$\left(\frac{\Delta(i/5)}{\Delta(5i)}\right)^{1/24} = \left(\frac{5\pi}{\pi/5}\right)^{6/24} = \sqrt{5}$$

If $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$, then

$$\alpha^6 \Delta(\alpha i/\pi) = \beta^6 \Delta(\beta i/\pi)$$

From this it follows that

$$\left(\frac{\Delta(i/5)}{\Delta(5i)}\right)^{1/24} = \left(\frac{5\pi}{\pi/5}\right)^{6/24} = \sqrt{5}$$

Ramanujan's identity then follows by solving $1/x - 1 - x = \sqrt{5}$.

Notebooks and Nonsense

"A short uncouth figure... walked in with a frayed notebook under his arm. He was miserably poor... He opened his book and began to explain some of his discoveries... but my knowledge did not permit me to judge whether he talked sense or nonsense." —Ramachandra Rao

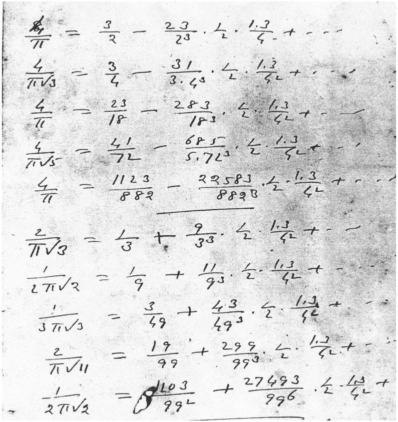


Figure: A Page in the Notebooks

As paper was scare, Ramanujan wrote only his final results on paper.

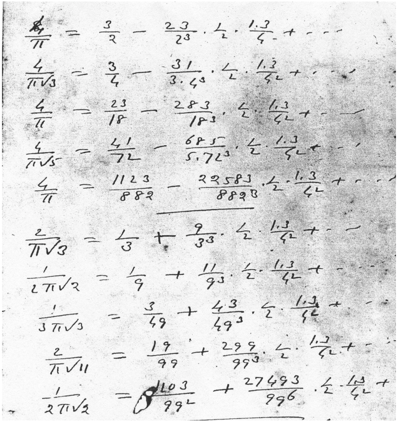


Figure: A Page in the Notebooks

As paper was scare, Ramanujan wrote only his final results on paper.

His notebooks have been the source of many articles and books.

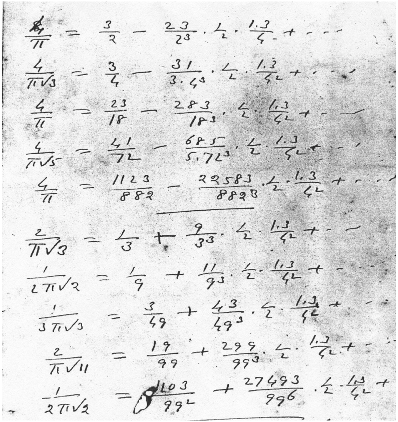


Figure: A Page in the Notebooks

As paper was scare, Ramanujan wrote only his final results on paper.

His notebooks have been the source of many articles and books.

There are some results which took mathematicians many years after his death to verify.

The last line of the notebook page above is

$$\frac{1}{2\pi\sqrt{2}} = \frac{1103}{99^2} + \frac{27493}{99^6} \cdot \frac{1}{2} \cdot \frac{1 \cdot 3}{4^2} + \cdots$$

The last line of the notebook page above is

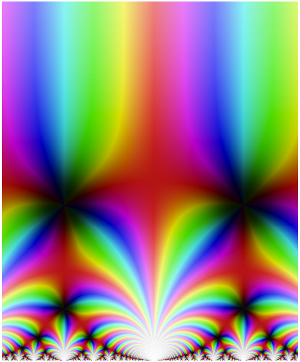
$$\frac{1}{2\pi\sqrt{2}} = \frac{1103}{99^2} + \frac{27493}{99^6} \cdot \frac{1}{2} \cdot \frac{1 \cdot 3}{4^2} + \cdots$$

Which suggested the now famous series

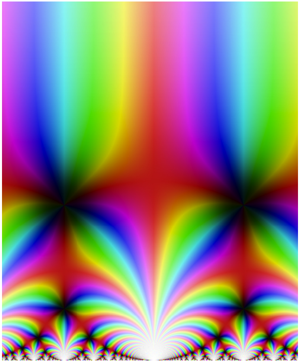
$$\frac{1}{\pi} = 2\sqrt{2} \sum_{n=0}^{\infty} \frac{1103 + 26390n}{(99)^{4n+2}} \cdot \frac{(4n)!}{(n!)^4 4^{4n}}$$

For $\mathrm{Im}(\tau) > 0$, define the Klein J-invariant

$$J(\tau) = \frac{g_2(\tau)^3}{\Delta(\tau)}$$



For $\text{Im}(\tau) > 0$, define the Klein J-invariant



$$J(\tau) = \frac{g_2(\tau)^3}{\Delta(\tau)}$$

There's a $q = e^{2\pi i\tau}$-expansion with integer coefficients:

$$(12)^3 J(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots$$

Suppose F is an imaginary quadratic number field with ring of integers $\mathbb{Z}[\tau]$. Then $F(J(\tau))$ is the Hilbert class field of F.

Suppose F is an imaginary quadratic number field with ring of integers $\mathbb{Z}[\tau]$. Then $F(J(\tau))$ is the Hilbert class field of F.

From the q-expansion we get ‘almost integers’

$$e^{\pi\sqrt{163}} \approx 12^3(231^2 - 1)^3 + 744$$
$$e^{\pi\sqrt{58}} \approx 396^4 - 104$$

In February of 1913, Ramanujan wrote to Cambridge mathematician G. H. Hardy and said...

In February of 1913, Ramanujan wrote to Cambridge mathematician G. H. Hardy and said...

“If I tell you

$$1 + 2 + 3 + 4 + 5 + 6 + \cdots =$$

In February of 1913, Ramanujan wrote to Cambridge mathematician G. H. Hardy and said...

"If I tell you

$$1+2+3+4+5+6+\cdots = -\frac{1}{12},$$

you will at once point out to me the lunatic asylum as my goal."

In February of 1913, Ramanujan wrote to Cambridge mathematician G. H. Hardy and said...

“If I tell you

$$1 + 2 + 3 + 4 + 5 + 6 + \cdots = -\frac{1}{12},$$

you will at once point out to me the lunatic asylum as my goal.”

Hardy remarked that Ramanujan’s theorems “must be true, because, if they were not true, no one would have the imagination to invent them.”

Ramanujan wrote

$$\sum_{k=1}^{x} f(k) = C + \int_0^x f(t)\,dt + \frac{1}{2}f(x) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x)$$

Ramanujan wrote

$$\sum_{k=1}^{x} f(k) = C + \int_0^x f(t)\,dt + \frac{1}{2}f(x) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x)$$

and said “The constant [C] of a series has some mysterious connection with the given infinite series and it is like the centre of gravity of a body. Mysterious because we may substitute it for the divergent series.”

Hardy helped arrange for Ramanujan to come to England in 1914 so they could collaborate on research.

Unfortunately, the outbreak of World War I made specialty vegetarian food scare. This along with stress made Ramanujan fell ill with tuberculosis and a vitamin deficiency.

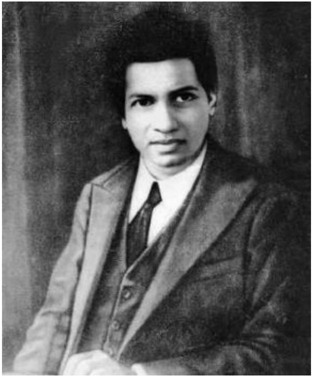


Figure: Photo of Ramanujan

Hardy later wrote “I remember once going to see [Ramanujan] when he was ill... I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one...

Hardy later wrote “I remember once going to see [Ramanujan] when he was ill... I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one...

‘No,’ he replied, ‘it is a very interesting number; it is the smallest number expressible as the sum of two [positive] cubes in two different ways’.”

$$1729 = 1^3 + 12^3 = 9^3 + 10^3$$

The Riemann Zeta Function $\zeta(s)$

For $s > 1$ define

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

For $s > 1$ define

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This function is **intimately** connected with prime numbers p:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

For $s > 1$ define

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This function is **intimately** connected with prime numbers p:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

We can extend $\zeta(s)$ analytically to all of $\mathbb{C}$ except for a simple pole at $s = 1$ with residue 1.

Table: Special Values of $\zeta(s)$

s	$\zeta(s)$	significance

Table: Special Values of $\zeta(s)$

s	$\zeta(s)$	significance
4	$\pi^4/90$	Stefan-Boltzmann law

Table: Special Values of $\zeta(s)$

s	$\zeta(s)$	significance
4	$\pi^4/90$	Stefan-Boltzmann law
3	no simple form known	Apéry's constant

Table: Special Values of $\zeta(s)$

s	$\zeta(s)$	significance
4	$\pi^4/90$	Stefan-Boltzmann law
3	no simple form known	Apéry's constant
2	$\pi^2/6$	probability of coprime pairs

Table: Special Values of $\zeta(s)$

s	$\zeta(s)$	significance
4	$\pi^4/90$	Stefan-Boltzmann law
3	no simple form known	Apéry's constant
2	$\pi^2/6$	probability of coprime pairs
1	∞	only pole

Table: Special Values of $\zeta(s)$

s	$\zeta(s)$	significance
4	$\pi^4/90$	Stefan-Boltzmann law
3	no simple form known	Apéry's constant
2	$\pi^2/6$	probability of coprime pairs
1	∞	only pole
-1	$-1/12$	bosonic string theory

The Riemann Hypothesis: One of six remaining Millennium Problems whose solution will result in a 1,000,000 dollar prize. The problem is to prove that all non-trivial zeros of $\zeta(s)$ lie on the vertical line $x = 1/2$ in the complex $z = (x + iy)$-plane.

The Riemann Hypothesis: One of six remaining Millennium Problems whose solution will result in a 1,000,000 dollar prize. The problem is to prove that all non-trivial zeros of $\zeta(s)$ lie on the vertical line $x = 1/2$ in the complex $z = (x + iy)$-plane.

Ramanujan may have very well proved this himself had he not died at age 32 in 1920.