MØLLER ENERGY AND WORMHOLES

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Abstract. Wormholes are theoretical objects of great intrigue in physics. When a wormhole model has been suggested, we can attempt to measure any associated energy. One method of doing this uses Møller energy. First, we define the Møller energy-momentum complex. Then we compute the Møller energy for a spherically symmetric spacetime with a diagonal metric. Lastly, we go through several examples of wormhole metrics and use the formula obtained to compute the energy in these cases.

1. Introduction

A wormhole is a region which connects two places within the same spacetime or connects two asymptotically flat spacetimes (parallel universes) via an Einstein-Rosen bridge (wormhole throat). The Kruskal extension of Schwarzschild geometry suggests the existence of the latter type of wormholes; in this case, however, the wormhole is not static and does not stay open. Rather, "the wormhole throat decreases and eventually pinches off in a singularity" as is written in [3]. For this reason, we say these wormholes are non-traversable.

Morris and Thorne studied the possibility of static, spherically symmetric wormholes, but they discovered that the Einstein equation implies such a geometry would violate the weak energy condition, meaning traversable wormholes will likely require the existence of so-called exotic matter (i.e., matter having negative energy). Even so, it is commented in [4] that "this condition can be violated quantum-mechanically, e.g. in the Casimir effect, or in alternative gravity theories." Also, in [7], it's proposed how a wormhole might be constructed with very particular matter sources which are nonetheless not exotic.

Given a possible wormhole metric, one would like to make energy measurements to see, for example, how such a wormhole could be supported or even constructed in the first place. How do we measure the energy in a wormhole? We need an appropriate notion of energy density, but, as noted in [6], the "problem of defining in an acceptable manner the energy-momentum density hasn't got a generally accepted answer yet." However, of the many different energy-momentum complexes, there is only one notable definition which does not require quasi-Cartesian coordinates. Indeed, the Møller energy-momentum complex, to be defined below, allows computation of energy densities independent of coordinates. We'll use this definition to compute the Møller energy of a physical system in general spherical coordinates, and then we'll specialize to the case of a spherically symmetric spacetime with a diagonal metric. In turn, we'll use this calculation in several examples of wormhole metrics. Throughout, the signature of every metric is $(-,+,+,+)$ and we take $c = G = \hbar = \varepsilon_0 4\pi = 1$ for convenience.

2. Møller Energy-Momentum Complex

Let $g_{\mu\nu}$ denote the metric coefficients of a four-dimensional spacetime with coordinates

$$\begin{aligned} x_0 &= t \\ x_1 &= r \\ x_2 &= \theta \\ x_3 &= \phi \end{aligned}$$

where t is the time component, r is the radial component, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$. Then, as in [1], [2], [6] or [8], we define the Møller energy-momentum complex by

$$M_\mu^\nu = \frac{1}{8\pi} \frac{\partial \chi_\mu^{\nu\sigma}}{\partial x^\sigma}$$

where

$$\chi_\mu^{\nu\sigma} = \sqrt{-g}\left(\frac{\partial g_{\mu\alpha}}{\partial x^\beta} - \frac{\partial g_{\mu\beta}}{\partial x^\alpha}\right) g^{\nu\alpha} g^{\beta\sigma}$$

is the superpotential with

$$g = \det(g_{\mu\nu}).$$

Note that the superpotential is anti-symmetric since the symmetry of the metric coefficients implies

$$\begin{aligned} \chi_\mu^{\sigma\nu} &= \sqrt{-g}\left(\frac{\partial g_{\mu\alpha}}{\partial x^\beta} - \frac{\partial g_{\mu\beta}}{\partial x^\alpha}\right) g^{\sigma\alpha} g^{\beta\nu} \\ &= -\sqrt{-g}\left(\frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\mu\alpha}}{\partial x^\beta}\right) g^{\nu\beta} g^{\alpha\sigma} \\ &= -\chi_\mu^{\nu\sigma}. \end{aligned}$$

Like the stress-energy tensor $T^{\mu\nu}$, the Møller energy-momentum complex has the following conservation laws

$$\frac{\partial M_\mu^\nu}{\partial x^\nu} = 0.$$

Also similar to the stress-energy tensor, the Møller energy density is given by M_0^0 while M_σ^0 represents the momentum density components for $\sigma = 1, 2, 3$. Thus the energy of the physical system within a ball B of radius $r = R$ is given by

$$P_0 = \iiint_B M_0^0 \, dx^1 dx^2 dx^3.$$

Letting S denote the boundary of B, we may apply the divergence theorem (noting that $\chi_0^{00} = 0$) to get

$$\begin{aligned} P_0 &= \frac{1}{8\pi} \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \sum_{\sigma=1}^{3} \frac{\partial \chi_0^{0\sigma}}{\partial x^\sigma} \, dx^1 dx^2 dx^3 \\ &= \frac{1}{8\pi} \iint_S \sum_{\sigma=1}^{3} \chi_0^{0\sigma} \eta_\sigma \, dS \end{aligned}$$

where (η_1, η_2, η_3) is the outward unit normal vector of dS in (r, θ, ϕ)-coordinates.

3. Møller Energy of a Spherically Symmetric Spacetime

Now we compute the energy P_0 for a general spherically symmetric diagonal metric, which will later be used to find P_0 for various wormhole metrics. Each such metric under consideration is of the form

$$ds^2 = -A(r,t)^2dt^2 + B(r,t)^2dr^2 + C(r,t)^2r^2(d\theta^2 + \sin^2(\theta)d\phi),$$

with $A, B, C \geq 0$, so we have

$$(g_{\mu,\nu}) = \mathrm{diag}(-A^2, B^2, C^2r^2, C^2r^2\sin^2(\theta))$$

and

$$(g^{\mu\nu}) = (g_{\mu\nu})^{-1} = \mathrm{diag}(-A^{-2}, B^{-2}, C^{-2}r^{-2}, C^{-2}r^{-2}\csc^2(\theta)).$$

Hence

$$\sqrt{-g} = \sqrt{A^2B^2C^4r^4\sin^2(\theta)} = ABC^2r^2\sin(\theta)$$

and

$$\begin{aligned}
\chi_0^{01} &= \sqrt{-g}\left(\frac{\partial g_{0\alpha}}{\partial x^\beta} - \frac{\partial g_{0\beta}}{\partial x^\alpha}\right)g^{0\alpha}g^{\beta 1} \\
&= \sqrt{-g}\left(\frac{\partial g_{00}}{\partial x^1} - \frac{\partial g_{01}}{\partial x^0}\right)g^{00}g^{11} \\
&= \sqrt{-g}\left(\frac{\partial(-A^2)}{\partial r} - 0\right)(-A^{-2})B^{-2} \\
&= ABC^2r^2\sin(\theta)2A\frac{\partial A}{\partial r}A^{-2}B^{-2} \\
&= \frac{2C^2r^2\sin(\theta)}{B}\cdot\frac{\partial A}{\partial r}
\end{aligned}$$

while if $i = 2, 3$ then

$$\begin{aligned}
\chi_0^{0i} &= \sqrt{-g}\left(\frac{\partial g_{0\alpha}}{\partial x^\beta} - \frac{\partial g_{0\beta}}{\partial x^\alpha}\right)g^{0\alpha}g^{\beta i} \\
&= \sqrt{-g}\left(\frac{\partial g_{00}}{\partial x^i} - \frac{\partial g_{0i}}{\partial x^0}\right)g^{00}g^{ii} \\
&= \sqrt{-g}\,(0-0)\,g^{00}g^{ii} \\
&= 0.
\end{aligned}$$

Therefore, by the above, the energy becomes

$$
\begin{aligned}
P_0 &= \frac{C(R,t)^2 R^2}{4\pi B(R,t)} \cdot \frac{\partial A}{\partial r}(R,t) \int_0^{2\pi}\int_0^{\pi} \sin(\theta)\, d\theta d\phi \\
&= \frac{C(R,t)^2 R^2}{B(R,t)} \cdot \frac{\partial A}{\partial r}(R,t).
\end{aligned}
$$

4. Examples of Wormhole Metrics

In this section we'll explore several specific wormhole geometries, find a couple of embeddings which display wormhole throats, and compute the associated Møller energy.

- **Hartle's Toy Wormhole.** Consider the metric (as found in [3])

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2(\theta) d\phi^2)$$

where $b > 0$ is a constant. This represents a static spacetime which can be embedded as a surface in three space. To visualize this geometry, we use the method of [3] as follows. Fix a time $t = t_0$ and the equatorial plane $\theta = \pi/2$. The resulting slice has the form

$$d\Sigma^2 = dr^2 + (b^2 + r^2) d\phi^2.$$

On the other hand, if (ρ, ψ, z) are cylindrical coordinates in $\mathbb{R}^3$ with

$$
\begin{aligned}
x &= \rho\cos(\psi) \\
y &= \rho\sin(\psi),
\end{aligned}
$$

then the metric for flat space is

$$
\begin{aligned}
dS^2 &= dx^2 + dy^2 + dz^2 \\
&= (\cos(\psi) d\rho - \rho\sin(\psi) d\psi)^2 + (\sin(\psi) d\rho + \rho\cos(\psi) d\psi)^2 + dz^2 \\
&= d\rho^2 + \rho^2 d\psi^2 + dz^2.
\end{aligned}
$$

Since the original spacetime is spherically symmetric, the embedding should be axis-symmetric, so we assume that z, ρ are functions of r only and that $\psi = \phi$. Using this in the above we get

$$
\begin{aligned}
dS^2 &= \left(\frac{d\rho}{dr} dr\right)^2 + \rho^2 d\phi^2 + \left(\frac{dz}{dr} dr\right)^2 \\
&= \left[\left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + \rho^2 d\phi^2.
\end{aligned}
$$

Thus in order for $d\Sigma^2 = dS^2$, we need

$$\rho^2 = r^2 + b^2$$
$$1 = \left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2.$$

The first equation implies

$$\left(\frac{d\rho}{dr}\right)^2 = \frac{r^2}{r^2+b^2},$$

so the second equation implies

$$\left(\frac{dz}{dr}\right)^2 = 1 - \left(\frac{d\rho}{dr}\right)^2 = \frac{b^2}{r^2+b^2},$$

whence choosing $z = 0$ when $r = 0$ we find

$$z = \pm \int \sqrt{\frac{b^2}{r^2+b^2}}\, dr = \pm b \sinh^{-1}(r/b) = \pm b \ln(\sqrt{(\rho/b)^2 - 1} + \rho/b).$$

Therefore, taking $b = 1$ for simplicity, the embedding in $\mathbb{R}^3$ is the surface of revolution about the z-axis of the curve

$$z = \pm \ln(\sqrt{x^2 - 1} + x).$$

The resulting picture is seen below.

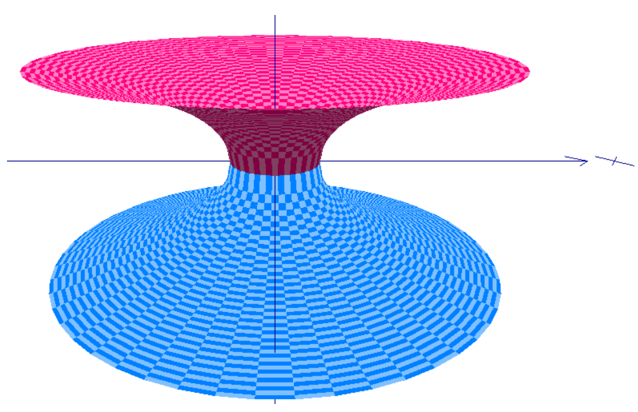


FIGURE 1. Embedding of Toy Wormhole

Now we use the formula for P_0 at the end of the last section to compute the energy. Here $A(r,t) = 1$, so $\partial A/\partial r = 0$, giving $P_0 = 0$.

- **Conformal Wormhole.** Consider the metric (as found in [1])

$$ds^2 = \Omega(t)\left[-dt^2 + \left(1 - \frac{b(r)}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)\right]$$

where $\Omega(t) > 0$ is the conformal factor and $b(r) < r$ is the shape function. As mentioned in [1], this metric "was considered by Kar (1994) in order to find out if within classical general relativity a class of nonstable... wormholes [which do not violate energy conditions] could exist. It was found that evolving geometry can support a wormhole." To visualize this geometry we mimic the method above by fixing $t = t_0$ and $\theta = \pi/2$, so cylindrical coordinates (ρ, ψ, z) give the conditions

$$\begin{aligned} \rho^2 &= \Omega(t_0)r^2 \\ \frac{\Omega(t_0)}{1 - b(r)/r} &= \left(\frac{d\rho}{dr}\right)^2 + \left(\frac{dz}{dr}\right)^2. \end{aligned}$$

These imply

$$\frac{dz}{dr} = \pm\sqrt{\Omega(t_0)\frac{b(r)}{r - b(r)}},$$

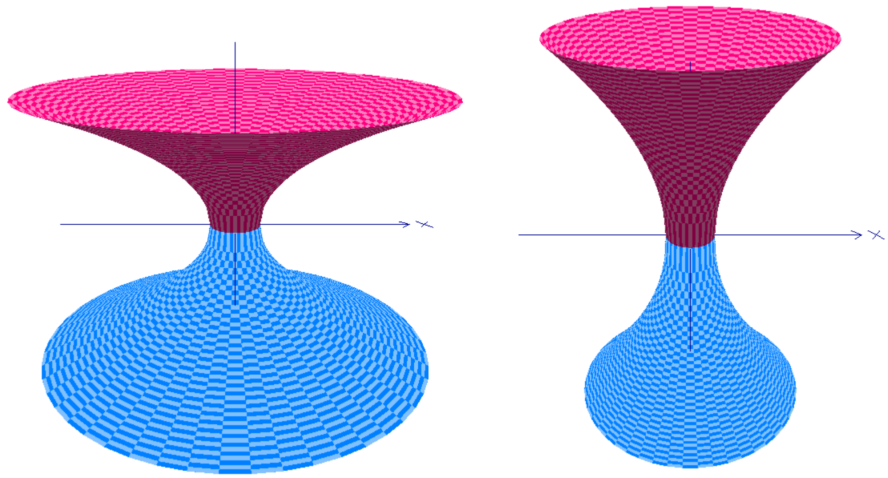
so if $b(r) = b$ is constant, then choosing $z = 0$ when $r = 0$ gives

$$z = \pm 2\sqrt{\Omega(t_0)b(r - b)}.$$

Therefore taking $b = 1$ for simplicity, the embedding in $\mathbb{R}^3$ is the surface of revolution about the z-axis of the curve

$$z = \pm 2\sqrt{\Omega(t_0)(x - 1)},$$

which is a family of hyperboloids when we allow t to vary. The resulting pictures for $\Omega(t) = (t + 1)^2$ at $t = 0, t = 1$ are seen below.



$t = 0$ $t = 1$

FIGURE 2. Embeddings of Conformal Wormhole with $b(r) = 1, \Omega(t) = (t+1)^2$

As with the toy wormhole, the energy here is zero since $A(r,t) = \sqrt{\Omega(t)}$ does not depend on r.

- **Charged Wormhole.** Consider the metric (c.f. [8], which has an abundance of errors)

$$ds^2 = -\left(1 + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{b(r)}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)$$

where Q is the charge and again $b(r)$ is the shape function. Note that if $b(r) \equiv 0$, then this spacetime becomes a Reissner-Nordström black hole with no mass. An embedding here for special choices of $b(r), Q$ could be obtained with symbolic integration, but one finds this to be unenlightening. The Møller energy, however, is nontrivial; specifically, we have

$$\begin{aligned}
A(R,t)^2 &= 1 + \frac{Q^2}{R^2} \\
\frac{\partial A}{\partial r}(R,t) &= \frac{1}{2}\left(1 + \frac{Q^2}{R^2}\right)^{-1/2} \frac{-2Q^2}{R^3} \\
B(R,t) &= \left(1 - \frac{b(R)}{R} + \frac{Q^2}{R^2}\right)^{-1/2} \\
C(R,t)^2 &= 1,
\end{aligned}$$

so

$$\begin{aligned} P_0 &= \frac{1 \cdot R^2}{\left(1 - \frac{b(R)}{R} + \frac{Q^2}{R^2}\right)^{-1/2}} \left(1 + \frac{Q^2}{R^2}\right)^{-1/2} \frac{-Q^2}{R^3} \\ &= \frac{-Q^2}{R} \sqrt{\frac{R^2 - b(R)R + Q^2}{R^2 + Q^2}}. \end{aligned}$$

- **Inflating Wormhole.** Consider the metric (as found in [1])

$$ds^2 = -A(r)^2 dt^2 + e^{2\chi t} \left[\left(1 - \frac{b(r)}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2(\theta) d\phi^2) \right]$$

where $\Lambda = 3\chi^2$ is the cosmological constant. As pointed out in [1], "Roman (1993) [explored] the possibility that inflation might provide a natural mechanism for the enlargement of such wormholes to macroscopic size... It was shown that the throat and the proper length of the wormhole inflate." If $b(r) = b$ is a constant, then this spacetime has the same embedding as the conformal wormhole with the conformal factor equal to an exponential function. The energy here, however, does not vanish as it did for the conformal wormhole since we allow A to depend on r; specifically, we have

$$\begin{aligned} A(R,t)^2 &= A(R)^2 \\ \frac{\partial A}{\partial r}(R,t) &= \frac{dA}{dr}(R) \\ B(R,t) &= e^{\chi t} \left(1 - \frac{b(R)}{R}\right)^{-1/2} \\ C(R,t)^2 &= e^{2\chi t}, \end{aligned}$$

so

$$\begin{aligned} P_0 &= \frac{e^{2\chi t} \cdot R^2}{e^{\chi t} \left(1 - \frac{b(R)}{R}\right)^{-1/2}} \cdot \frac{dA}{dr}(R) \\ &= e^{\chi t} \frac{dA}{dr}(R) \sqrt{R - b(R)}. \end{aligned}$$

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