A Discrete Nash Demand Game with Diagonal Punishment

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Introduction

We analyze the following two player game in which players simultaneously and independently choose an integer from the strategy space $S_1 = S_2 = \{\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6}\}$. For $i, j \in \{1,2\}$ with $i \neq j$ we define a payoff function $u_i : S_1 \times S_2 \to \mathbb{R}_{\geq 0}$ by

$$(\mathbf{m}_1,\mathbf{m}_2) \mapsto \begin{cases} m_i & \text{if } m_1+m_2<10, m_1 \neq m_2 \\ km_i & \text{if } m_1+m_2<10, m_1 = m_2 \\ 0 & \text{if } m_1+m_2 \geq 10 \end{cases}$$

for some fixed parameter $k \in [0,1]$. In particular, we note that payoffs are diminished if either the sum of their choices is too large or contracted if both players make the same choice. For concreteness, we'll focus on the case $k = 0.5$, in which case the normal form can represented by the figure that follows.

	3	**4**	**5**	**6**
3	1.5, 1.5	3, 4	3, 5	3, 6
4	4, 3	2, 2	4, 5	0, 0
5	5, 3	5, 4	0, 0	0, 0
6	6, 3	0, 0	0, 0	0, 0

Figure 1: Bimatrix Form for $k = 1/2$

Solution Concepts

First, we determine Nash equilibria. It's clear from the bimatrix in Figure 1 that the pure Nash equilibria lie on the off-diagonal: $(\mathbf{3},\mathbf{6}),(\mathbf{4},\mathbf{5}),(\mathbf{5},\mathbf{4}),(\mathbf{6},\mathbf{3})$. We would not expect, however, the equilibrium points $(\mathbf{3},\mathbf{6})$ or $(\mathbf{6},\mathbf{3})$ to be realized in this game since $\mathbf{6}$ is a high risk strategy. If, on the other hand, the game were played sequentially with rational players, then the extensive form would have perfect information and the unique backwards induction solution would yield the same payoff as $(\mathbf{6},\mathbf{3})$. What we do expect is a strong tendency to play $\mathbf{4}$, moderate tendencies to play $\mathbf{3}$ and $\mathbf{5}$, and a very weak tendency to play $\mathbf{6}$. It is therefore natural to extend the game to the mixed strategy spaces $\Delta S_1 = \Delta S_2$ and search for mixed strategies which somehow capture this expected behavior. Each $\sigma \in \Delta S_1$ is of the form

$$\sigma = x\mathbf{3} + y\mathbf{4} + t\mathbf{5} + v\mathbf{6}$$

for some $x,y,t,v \in [0,1]$ such that $x+y+t+v=1$. Thus viewing pure strategies as vertices allows us to regard ΔS_1 as a 3-simplex, i.e., a solid tetrahedron in $\mathbb{R}^3$ as seen in Figure 2. To find a non-pure symmetric Nash equilibrium (σ_N,σ_N) we use the standard technique of equating payoffs $u_2(\sigma_N,\mathbf{3}) = u_2(\sigma_N,\mathbf{4}) = u_2(\sigma_N,\mathbf{5}) = u_2(\sigma_N,\mathbf{6})$.

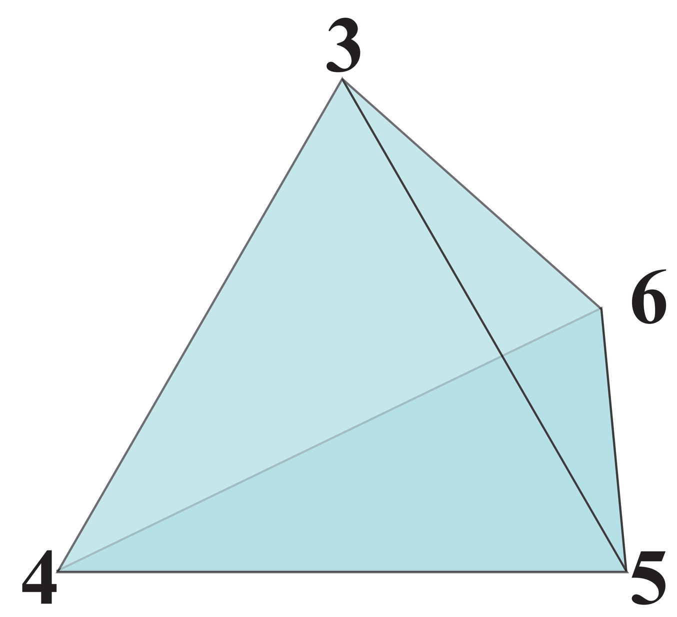


Figure 2: Space of Mixed Strategies

We get the linear system

$$\begin{aligned} 6x &= 1.5x + 3t + 3y + 3(1-x-y-t) \\ 6x &= 4x + 2y + 4t \\ 6x &= 5x + 5y, \end{aligned}$$

which has a unique solution via

$$\sigma_N = \frac{2}{5}\mathbf{3} + \frac{2}{25}\mathbf{4} + \frac{4}{25}\mathbf{5} + \frac{9}{25}\mathbf{6},$$

but the corresponding equilibrium point again does not agree with our intuition because there are relatively large weights placed on $\mathbf{3},\mathbf{6}$ compared to the relatively small weights placed on $\mathbf{4},\mathbf{5}$. Note that the payoff for this mixed Nash equilibrium is $u_1(\sigma_N,\sigma_N) = u_2(\sigma_N,\sigma_N) = 2.4$, but perhaps payoffs can be improved by assuming, in similar fashion, that players randomize choices according to the same mixed strategy (after all, there is no interaction between players). Specifically, we want to find σ_M such that

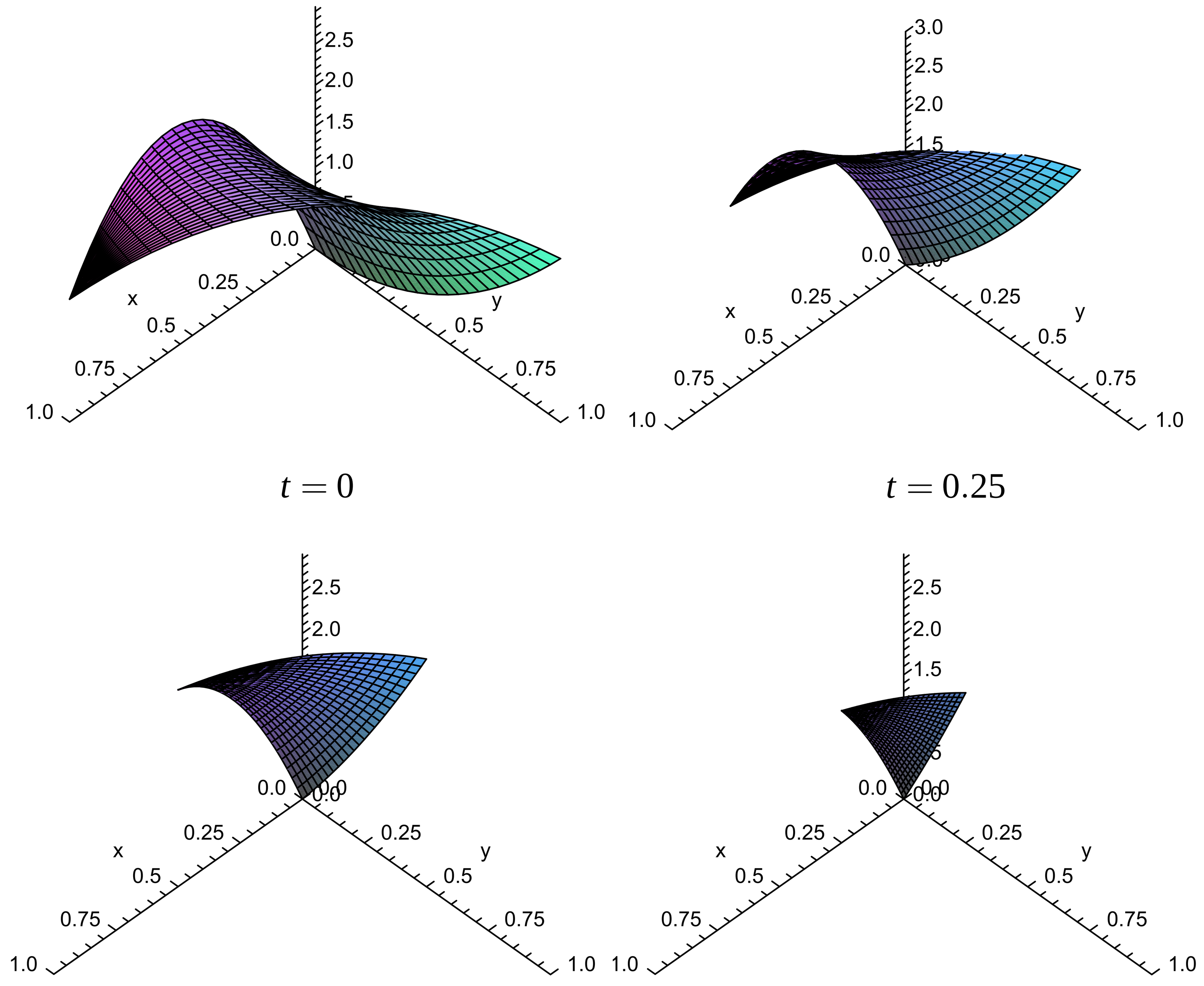
$$u_1(\sigma_M,\sigma_M) = \max\{u_1(\sigma,\sigma) : \sigma \in \Delta S_1\}.$$

To do so, we apply the method of Lagrange multipliers to the function $f(x,y,t,v) := u_1(\sigma,\sigma)$ (viewed as map from the tetrahedron to $\mathbb{R}$) with the constraint $g(x,y,t,v) := x+y+t+v-1 = 0$ by first setting $\nabla f = \lambda \nabla g$ and then checking for solutions on the interior of the tetrahedron, the four faces, and the six edges (the four vertices correspond to pure strategies). Using the computer software Maple, we plot f as an evolving surface in $\mathbb{R}^3$ changing with time t as seen in Figure 3, and we find that

$$\sigma_M = \frac{11}{41}\mathbf{3} + \frac{37}{82}\mathbf{4} + \frac{23}{82}\mathbf{5} \approx (0.27)\mathbf{3} + (0.45)\mathbf{4} + (0.28)\mathbf{5}$$

with payoff $u_1(\sigma_M,\sigma_M) = 509/164 \approx 3.103658537$. This distribution seems to be a much more reasonable snapshot of the probabilities we anticipate players to select strategies with. Note, however, that (σ_M,σ_M) is <u>not</u> a Nash equilibrium since

$$u_1(\mathbf{5},\sigma_M) = 295/82 \approx 3.598 > u_1(\sigma_M,\sigma_M).$$



$t=0$ | $t=0.25$ | $t=0.5$ | $t=0.75$

Figure 3: Level Surfaces of f

Hence if a player forms a belief like σ_M, then we expect a greater trend to choose $\mathbf{5}$, especially if the game is played only once. Thus the balance point

$$\frac{\sigma_M + \mathbf{5}}{2} \approx (0.13)\mathbf{3} + (0.23)\mathbf{4} + (0.64)\mathbf{5}$$

might provide a reasonable solution concept if we expect players to split between a conservative and risky mindset.

The Experiment and Conclusions

The team ran an experiment in which 27 people played the above game with $k = 1/2$ and $k = 1$. We found the following distributions:

for $k = 1/2$: 4 chose $\mathbf{3}$, 5 chose $\mathbf{4}$, 17 chose $\mathbf{5}$, 1 chose $\mathbf{6}$
for $k = 1$: 1 chose $\mathbf{3}$, 14 chose $\mathbf{4}$, 12 chose $\mathbf{5}$, 0 chose $\mathbf{6}$

We conclude

- Nash equilibria do not comprise good solution concepts for this game.
- $(\sigma_M + \mathbf{5})/2$ appears to provide an accurate model for players' tendencies: compare the actual distribution $\approx (0.15, 0.19, 0.62, 0.04)$ for $k = 1/2$.
- The strategy $\mathbf{3}$ (resp. $\mathbf{5}$) would probably have been selected more (resp. less) often if subjects had played the game many times. In particular, σ_M may be a better model for tendencies in a repeated game.

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