PHASE IV: WHY IT WORKS

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1. Background

Suppose we wanted to make a bank shot in a game on a pool table as seen in figure 1 below. Then we would need to know the angle at which to hit the desired ball onto the

Figure 1. Bank shots

cushion (that rubber attached to the railing). Notice from the picture of the bank shots on the pool table that the incoming angle of the ball to the cushion looks the same as the outgoing angle. Why should this be? It comes from the law deflection in physics, which states precisely what we observe in the pool table example: when a beam (say of light or sound) collides with any flat surface, then the incoming angle θ is equal to the outgoing angle of the reflected beam. This is illustrated in figure 2 below for two dimensions (i.e., things are happening in an xy-plane). So what happens when a beam is reflected off of a curved

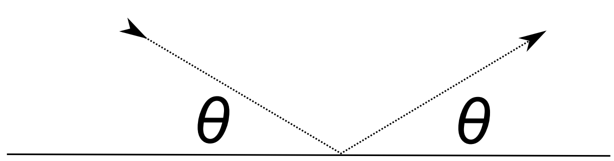

Figure 2. Beam reflected on a straight line

surface? Well, if the surface is smooth enough then the magnifying the surface around the point of contact will reveal that the surface is approximately flat in that small area. In this way, if we know the incoming angle to this flat surface, then we should be able to find the reflected beam using the deflection law for flat surfaces. In two dimensions (think xy-plane), we recall that when we continually zoom in on a function which has a derivative at a point, the function looks more and more like a line. The line which approximates the curve at a given point is the tangent line, which has slope equal to the derivative evaluated at that point. This is illustrated in figure 3 below. It turns out that if our incoming beam is vertical,

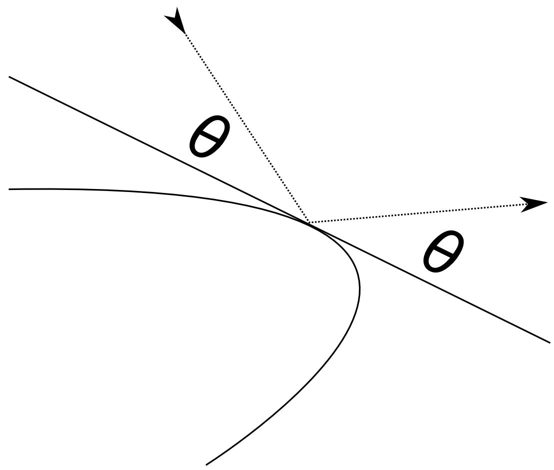

FIGURE 3. Beam reflected on a curved line

then the slope of the reflected beam is the average of the slopes of the tangent and normal lines:

$$\boxed{\text{slope of reflected beam } = \frac{1}{2}\left(m - \frac{1}{m}\right)} \tag{1.1}$$

Below is an illustration which shows a vertical beam and its reflected beam along with the tangent and normal lines to the curve.

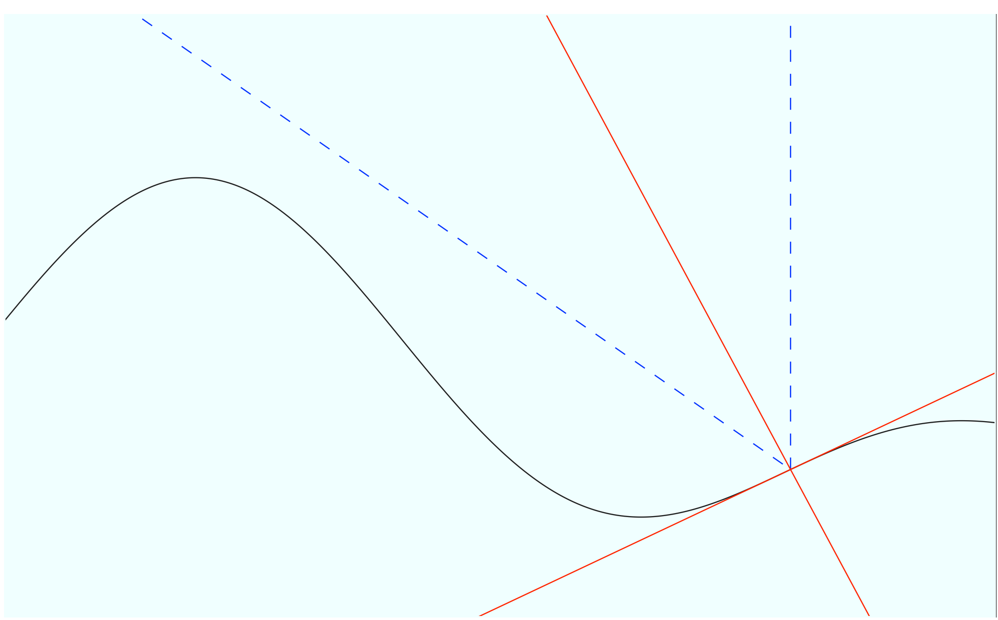

FIGURE 4. Vertical beam reflected showing tangent and normal lines

2. So What?

What does all of this have to do with your satellite dishes? Well, the parabola is a curve with the property that any incoming beams perpendicular to the directrix are reflected to the focus. Huh? OK, let's take it slow. Suppose we have the parabola $4y = x^2$. Then the focus is $(0, 1)$ and the derivative is

$$y'(x) = \left(\frac{x^2}{4}\right)' = \frac{2x}{4} = \frac{x}{2},$$

which is the slope of the tangent line at a point (x, y) on the parabola. Let's pick a point on the parabola, say $(1, 1/4)$. The slope of the tangent line at $(1, 1/4)$ is then $y'(1) = 1/2$, and the slope of the normal line at $(1, 1/4)$ is $-1/(1/2) = -2$. This means that if a vertical beam makes contact with the parabola at the point $(1, 1/4)$, then equation 1.1 tells us

$$\text{slope of reflected beam } = \frac{1}{2}\left(\frac{1}{2} - 2\right) = \frac{1}{2}(-1.5) = -0.75 \tag{2.1}$$

The point-slope formula then tells us that the equation of the reflected beam is

$$y = -0.75(x - 1) + 1/4 = -0.75x + .75 + 0.25 = -0.75x + 1 \tag{2.2}$$

Notice that the y-intercept is 1, which means the reflected beam passes through the focus $(0, 1)$. Indeed, any vertical beam will be reflected through the focus, as is illustrated in the graph below.

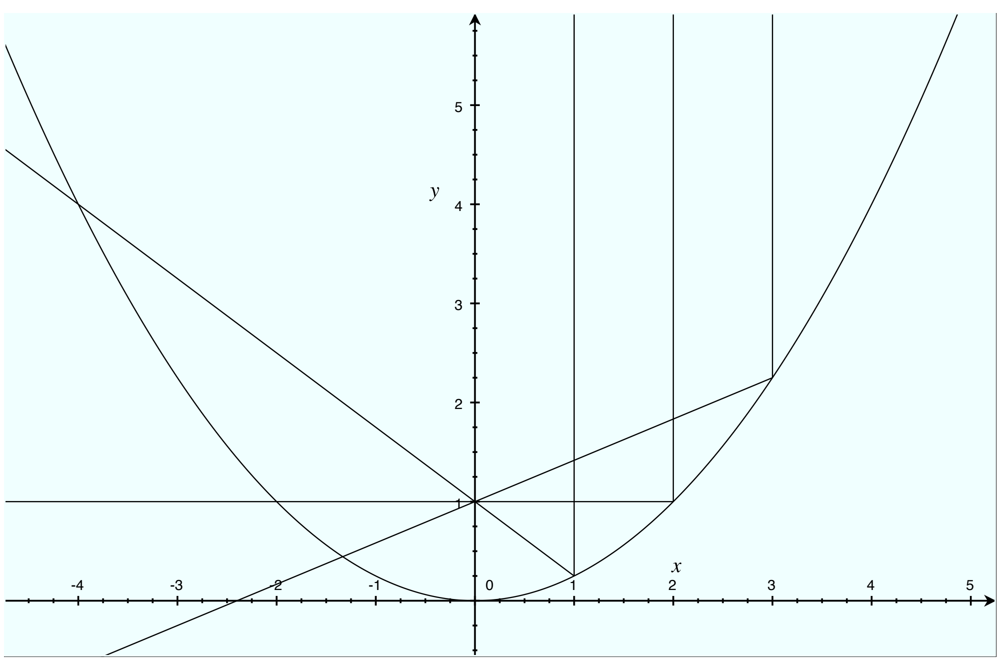

Figure 5. Graph of $y = x^2/4$ with vertical and refelcted beams

3. What you need to do

(1) Pick three distinct points on your parabola, i.e., the equation you used in constructing your dishes.
(2) For each point, find the slopes for the tangent and normal lines.
(3) Next, compute the equation of each reflected line and verify that they all pass through the focus.
(4) Test the accuracy of your model by dropping a small rubber ball straight into your satellite dish. If you have a nice shape, the ball should bounce into the focus regardless of where you drop it. Record what you notice. Explain why you think your shape was truly parabolic or not.
(5) Watch your grade improve significantly.