Sheaf Cohomology

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Math 518 - Homological Algebra

Fall 2007

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Chapter 1

Introduction

When studying a topological space, a mathematician invariably encounters a dichotomy between properties that are local and properties that are global. For example, consider smooth functions on smooth manifolds. A smooth function $f : M \to \mathbb{R}$ on a compact smooth manifold M can be expressed as a formula using local coordinates but cannot be expressed as a formula globally. Conversely, compatible smooth functions defined locally can be "glued" together to form a global smooth function. One can ask, given something defined locally on a topological space, when can it be extended to define something global? On the other hand, given something defined globally, what can one say about local restrictions?

Sheaves are designed to formalize this dichotomy. The simplest definition of sheaves is formulated in the language of categories and functors. Namely, given a topological space X, one can form a category $\mathfrak{Top}(X)$ whose objects are the open sets of X and whose morphisms are inclusion maps. A sheaf of abelian groups is a contravariant functor $\mathscr{F} : \mathfrak{Top}(X) \to \mathfrak{Ab}$ that satisfies an extra axiom that we will call the "gluing axiom." A similar construction yields sheaves of commutative rings, R-modules, etc. Examples of sheaves include smooth functions on smooth manifolds, holomorphic or meromorphic functions on complex manifolds, and sections of vector bundles over manifolds.

It turns out that the collection of all sheaves with values in a specified abelian category $\mathfrak{C}$ over a topological space X forms an abelian category $\mathfrak{C}(X)$. If $\mathfrak{C}$ has enough injectives, so does $\mathfrak{C}(X)$. The image of the entire space X under a sheaf $\mathscr{F} \in \mathfrak{C}$, sometimes denoted $\Gamma(X, \mathscr{F})$, defines an additive left-exact functor from $\mathfrak{C}(X)$ to $\mathfrak{C}$. This functor, called the global section functor, can be used to define sheaf cohomology. Specifically, the sheaf cohomology groups of a sheaf $\mathscr{F}$ on a topological space X, denoted $H^n(X, \mathscr{F})$, are defined to be the right derived functors of the global section functor. Under fairly tame conditions, the sheaf cohomology groups coincide with the Čech cohomology groups, which we will define below. The Čech cohomology groups can be computed. For example, we will show below that for the sheaf $\mathscr{F}$ of locally constant functions on a smooth manifold, the Čech cohomology groups $\check{H}^n(X, \mathscr{F})$ coincide with the de Rham cohomology groups. The first half of this paper will lay the groundwork be defining all the necessary terms and proving necessary results leading up to sheaf cohomology and Čech cohomology.

The last chapter of this paper will turn to what the sheaf cohomology tells us about the underlying topological space. Since the global sheaf functor is left exact, the zeroth level sheaf cohomology is naturally isomorphic to the global sheaf functor itself. The first cohomology is more interesting. For simplicity, we will restrict our attention to compact Riemann surfaces. We will fix a sheaf of commutative rings (with identity) $\mathcal{O}$ and define invertible sheaves with respect to $\mathcal{O}$. The set of isomorphism classes of invertible sheaves form a group under the tensor product, called the Picard group. It turns out that for the sheaf $\mathcal{O}^\times$ of units of $\mathcal{O}$, the first sheaf cohomology $H^1(X, \mathcal{O}^\times)$ is naturally isomorphic to the Picard group. In the context of compact Riemann surfaces, the Picard group is also isomorphic to the divisor class group of X as well as the group of line bundles over X. This interpretation brings geometry together with number theory and illustrates the power of sheaf cohomology.

As a convention, in this paper all rings will be rings with identity.

Chapter 2

Sheaves and Presheaves

One of the earliest results in complex analysis was the discovery that if a collection of holomorphic functions agree on open subsets then there exists an analytic continuation which "glues together" the local data to produce a global function. Specifically, if $\{U_n\}$ is a finite collection of open subsets of $\mathbb{C}$ and $f_n : U_n \to \mathbb{C}$ are holomorphic functions which agree on overlaps, $f_n|_{U_n \cap U_m} = f_m|_{U_n \cap U_m}$ for all $U_n \cap U_m \neq \varnothing$, then there is a meromorphic function f such that $f|_{U_n} = f_n$ for all n. The modern definition of sheaves comes from Cartan's attempt to generalize the notion of "compatible" holomorphic functions to multiple variables ([Uen97] pg 69).

2.1 Definitions

Due to the strict nature of the compatibility conditions it is useful to weaken the axioms and define a presheaf before defining a sheaf.

Let X be a topological space and $\mathfrak{Top}(X)$ the category of open subsets of X; the morphisms in the category are just the inclusion maps between open subsets.

Definition 2.1. A **presheaf** $\mathscr{F}$ of abelian groups is a contravariant functor $\mathscr{F} : \mathfrak{Top}(X) \to \mathfrak{Ab}$. The classical presheaf axioms are a consequence of the categorical definition:

1. If $V \subseteq U$ are open sets then there must exist a map $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$,
2. $\rho_{UU} = \mathrm{id}_{\mathscr{F}(U)}$ is the identity,
3. If $W \subseteq V \subseteq U$ are open sets then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

A presheaf of rings, modules or with values in any category $\mathfrak{C}$ is defined similarly. Some authors include a presheaf axiom requiring $\mathscr{F}(\varnothing)$ to be the terminal object in the category $\mathfrak{C}$. We omit this axiom in our treatment, and simply note that it follows from the gluing axiom for a sheaf, defined below.

Elements of $\mathscr{F}(U)$ are called **sections**, sometimes denoted $\Gamma(U, \mathscr{F})$. Elements of $\Gamma(X, \mathscr{F})$ are called **global sections** . The maps ρ_{UV} are called **restrictions**. If $V \subseteq U$ and $s \in \mathscr{F}(U)$, then the restriction of s to the subset $\rho_{UV}(s)$ is frequently written $s|_V$.

Definition 2.2. Let $\mathscr{F}$ be a presheaf on X and $U \subseteq X$ an open subset. $\mathscr{F}$ is a **sheaf** if it has the following property: for every open cover $U = \bigcup_{\alpha \in I} U_\alpha$ and every collection of sections $s_\alpha \in \mathscr{F}(U_\alpha)$, $\alpha \in I$ that agree on the intersections $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$, there exists a unique $s \in \mathscr{F}(U)$ such that $s|_{U_\alpha} = s_\alpha$ for each α.

Throughout the paper we shall refer to this as the "gluing axiom" for a sheaf. If $\{U_\alpha\}$ is an open cover of X and $f|_{U_\alpha} = 0$ for all α then the gluing axiom implies that $f \equiv 0$ on all of X. [Sha94]

Example 2.3. Let X be a topological space and S a set with the discrete topology. For an open subset $U \subseteq X$ define $\mathscr{F}(U) = \{f : X \to S | f \text{ is continuous}\}$. If S has an abelian group structure, then $\mathscr{F}(U)$ is the section of locally constant functions on U. Any function $f \in \mathscr{F}(X)$ is constant on connected components of X, so the "gluing axioms" are trivially satisfied and $\mathscr{F}$ is a sheaf, called the **constant sheaf**.

Example 2.4. Let M be a smooth manifold. The functor $\mathscr{F}$ which takes open subsets U to the ring of smooth functions $C^\infty(U)$ is a sheaf of commutative rings on M. Suppose that $V \subset U \subset M$ are open subsets and $f \in C^\infty(U)$. Then the restriction map is simply $\rho_{UV}(f) = f|_V$. The remaining axioms of a presheaf are trivially satisfied. Now suppose that $\{V_i\}_{i \in I}$ form a locally finite open cover of U with corresponding smooth functions $f_i : V_i \to \mathbb{R}$ such that $f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$. Any open subset of a manifold is a smooth manifold, so we can take a partition of unity $\{\varphi_i\}$ subordinate to $\{V_i\}$ and define $f(x) = \sum_{i \in I} f_i(x)\varphi_i(x)$. $f(x)$ is a global section whose restriction of to each V_i is f_i, so $\mathscr{F}$ is a sheaf.

Example 2.5. Let X be a compact Riemann surface and let $\mathcal{M}(X)$ be the field of globally meromorphic functions on X (sometimes called the field of rational functions on X). The sheaf $\mathcal{O}_{X,\mathrm{alg}}$ of **regular functions** on X is defined by

$$\mathcal{O}_{X,\mathrm{alg}}(U) = \{f \in \mathcal{M}(X) \text{ such that } f|_U \text{ is holomorphic}\}$$

Example 2.6. If $\mathscr{F}$ is a sheaf on X then a **subsheaf** $\mathscr{F}'$ is a sheaf on X such that for every open subset $U \subseteq X$, $\mathscr{F}'(U) \subseteq \mathscr{F}(U)$ and $\mathscr{F}'$ is compatible with the restriction maps of $\mathscr{F}$.

Example 2.7. Let $Y \subset X$ be an open subset and $\iota : Y \hookrightarrow X$ the inclusion map. For a sheaf $\mathscr{F}$ on X, the **restriction sheaf** $\iota^{-1}\mathscr{F}$ is defined by

$$\iota^{-1}\mathscr{F}(V) = \mathscr{F}(V)$$

for every open $V \subset Y$. This is often written $\mathscr{F}|_Y$.

Example 2.8. Let $f : X \to Y$ be a continuous map of topological spaces and $\mathscr{F}$ a sheaf on X. The **direct image sheaf** $f_*\mathscr{F}$ on Y is defined by $(f_*\mathscr{F})(U) = \mathscr{F}(f^{-1}(U))$ for each open set $U \subseteq Y$.

For each $p \in X$ the collection $\mathscr{F}(U)$ for open neighborhoods $p \in U \subseteq X$ form a directed system under restriction.

Definition 2.9. Let $\mathscr{F}$ be a sheaf on X and $p \in X$. The **stalk** of $\mathscr{F}$ at p is defined to be the direct limit $\mathscr{F}_p = \varinjlim \mathscr{F}(U)$ of neighborhoods U containing p. A **germ** of $\mathscr{F}$ at p is an element $s_p \in \mathscr{F}_p$.

2.2 The Category of Sheaves

If $\mathfrak{C}$ is a category, the collection of sheaves with values in $\mathfrak{C}$ on X is denoted $\mathfrak{C}(X)$. This forms a category, also denoted by $\mathfrak{C}(X)$. A morphism $\varphi : \mathscr{F} \to \mathscr{G}$ in the category $\mathfrak{C}(X)$ is a collection of morphisms $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ in $\mathfrak{C}$ that are compatible with restriction. If $\varphi : \mathscr{F} \to \mathscr{G}$ then for every pair of open sets $V \subseteq U$ the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \\ \rho_{UV} \downarrow & & \downarrow \rho'_{UV} \\ \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V) \end{array}$$

Unsurprisingly, an isomorphism of sheaves is a morphism which has a two sided inverse.

If $\mathfrak{C}$ is an abelian category then $\mathfrak{C}(X)$ is also an abelian category. The zero object $\mathscr{Z}$ in $\mathfrak{C}(X)$ is defined by $\mathscr{Z}(U) = 0$ for every open $U \subset X$, where 0 denotes the zero object in $\mathfrak{C}$. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves in $\mathfrak{C}(X)$, we can define the direct sum of the sheaves $\mathscr{F} \oplus \mathscr{G}$ by taking the direct sum of the corresponding sections, $(\mathscr{F} \oplus \mathscr{G})(U) = \mathscr{F}(U) \oplus \mathscr{G}(U)$. Using the universal property of direct sums in the category $\mathfrak{C}$, is is easy to see that the direct sum of presheaves is a presheaf, and the direct sum of sheaves is another sheaf.

The image, kernel, and cokernel of a morphism of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$ are presheaves defined locally by Im (φ_U), $\ker(\varphi_U)$ and $\operatorname{coker}(\varphi_U)$. A morphism of sheaves is said to be injective if the kernel is the zero object $\mathfrak{C}(X)$. Similarly, a morphism of sheaves is surjective if its cokernel is zero.

The kernel of a sheaf is always a sheaf. However, the image and cokernel of of a sheaf is generally a presheaf that is not a sheaf. This is illustrated in the following example.

Example 2.10. Let $X = \mathbb{C} \setminus \{0\}$ be the punctured complex plane. Define $\mathcal{O}$ to be the sheaf which takes each open subset U to the additive group of holomorphic functions on U. Define $\mathcal{O}^\times$ to be the sheaf which takes each open subset of X to the multiplicative group of nonvanishing holomorphic functions on U; then the map $\varphi(f) = e^{2\pi i f}$ is a morphism of sheaves. The image Im (φ) *will not* be a sheaf. Let $U = \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ and $V = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, then $U \cup V = X$. Using the missing line as a branch cut, $\log(z)$ can be defined on U and V. If $\log_1(z)$ denotes the choice of log defined on U and $\log_2(z)$ denotes the branch of log on V, then $\varphi_U(\log_1)|_{\mathbb{C}\setminus\mathbb{R}} = \varphi_V(\log_2)|_{\mathbb{C}\setminus\mathbb{R}} = z|_{\mathbb{C}\setminus\mathbb{R}}$. However, $\varphi_U(\log_1)$ cannot be the restriction of a holomorphic function defined on all of X because $\log(z)$ is discontinuous on $\mathbb{C} \setminus \{0\}$. Therefore there is no way to "glue together" the image of the two functions to realize them as the restriction of a element of $\mathscr{F}(X)$.

The way to remedy this problem is to take the sheafication of a presheaf. This process in similar to the way that one completes a metric space.

Proposition 2.11. Let $\mathscr{F}$ be a presheaf on a space X with values in an abelian category $\mathfrak{C}$. Then there is a sheaf $\mathscr{F}' \in \mathfrak{C}(X)$ and an injective morphism $\theta : \mathscr{F} \to \mathscr{F}'$ such that for any $\mathscr{G} \in \mathfrak{C}(X)$ and morphism of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$, there exists a unique morphism of sheaves $\tilde{\varphi} : \mathscr{F}' \to \mathscr{G}$ such that $\varphi = \tilde{\varphi}\theta$. The pair $(\mathscr{F}', \theta)$ is unique up to unique isomorphism.

Sketch of Proof. Define $\mathscr{F}'$ by:

$$\mathscr{F}'(U) = \left\{ f : U \to \coprod_{p \in U} \mathscr{F}_p \right\}$$

where $f \in \mathscr{F}'(U)$ satisfies:

1. For each $p \in U$, $f(p) \in \mathscr{F}_p$.
2. For every point $p \in U$ there is a neighborhood V contained in U such that there exists $g \in \mathscr{F}(V)$ with the property that for all $q \in V$, the germs $g_q = f(q)$.

The map $\theta : \mathscr{F} \to \mathscr{F}'$ is given by $\theta(s) = \{p \mapsto s_p \in \mathscr{F}_p\}$ and is an injection.

For proof of the universal property, see Proposition 1.2 in Chapter II of [Har77]. □

Definition 2.12. The sheaf $\mathscr{F}'$ in Proposition 2.11 is called the **sheafication** of the presheaf $\mathscr{F}$.

The sheafication is sometimes called the sheaf associated to $\mathscr{F}$. If $\mathscr{F}$ is a sheaf, then $\mathscr{F}' \cong \mathscr{F}$.

The image and cokernel sheaves are the sheafication of the presheaves defined by Im (φ_U) and coker(φ_U). Viewing the kernels and images of morphisms as subsheaves, we are finally able to define an exact sequence in the category of sheaves. The sequence

$$\cdots \to \mathscr{F}^{i-1} \overset{\varphi^{i-1}}{\to} \mathscr{F}^{i} \overset{\varphi^{i}}{\to} \mathscr{F}^{i+1} \overset{\varphi^{i+1}}{\to} \cdots$$

is exact if $\ker \varphi^i = \operatorname{Im}\, \varphi^{i-1}$ for all i.

We leave to the reader the exercise of using the above constructions to check the axioms required for $\mathfrak{C}(X)$ to be an abelian category.

Note that if $\varphi : \mathscr{F} \to \mathscr{G}$ is an injective morphism of sheaves, and the category $\mathfrak{C}$ admits quotients, then the intuitive way to define the quotient $\mathscr{F}/\mathscr{G}$ is using the quotient in $\mathfrak{C}$: $(\mathscr{F}/\mathscr{G})(U) = \mathscr{F}(U)/\mathscr{G}(U)$. However, the quotient as defined is usually not a sheaf so it is necessary to take the sheafication of the presheaf defined by taking local quotients.

Suppose that $\varphi : \mathscr{F} \to \mathscr{G}$ are sheaves on X and $p \in X$. Since the stalk at a point p is the direct limit of the sections of neighborhoods of p, φ induces a morphism on stalks by the universal mapping property of direct limits. The stalks of a sheaf completely characterize the sheaf in a way that the sections alone do not. For instance, a surjective morphism of sheaves induces a surjection on stalks, but may not be surjective on sections.

Proposition 2.13. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves, then φ is an isomorphism if and only if the induced maps $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ is an isomorphism for all $p \in X$.

Proof. If φ is an isomorphism then it induces isomorphisms on the sections. The stalks are direct limits of sections, so $\mathscr{F}$ clearly induces an isomorphism on stalks.

Now suppose that $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ is an isomorphism for all $p \in X$. First we show that φ is injective. Let $s \in \ker \varphi_U \subseteq \mathscr{F}(U)$. Then for every $p \in U$ the germ at that point satisfies $\varphi(s)_p = 0$. By hypothesis, the induced maps on the stalks are isomorphisms, so $s_p = 0$ for each $p \in U$. Since this is true in neighborhoods of every point the gluing axiom implies that $s = 0$. Therefore the maps of sections are injective.

Surjectivity is accomplished by building a preimage from pullbacks of local neighbhorhoods and then gluing them together. Let $t \in \mathscr{G}(X)$ be a section and for each $p \in X$ define $t_p \in \mathscr{G}_p$ to be the germ of t at p. φ_p is surjective, so there exists an $s_p \in \mathscr{F}_p$ such that $\varphi_p(s_p) = t_p$. The stalks are direct limits, so s_p can be represented by a section $s(p)$ on a neighborhood V_p of p. $\varphi(s(p))$ and $\varphi_p(s_p)$ are both elements of $\mathscr{G}(V_p)$ and share the same germ at p. Without loss of generality we can assume that $\varphi(s(p)) = t|_{V_p}$ in $\mathscr{G}(V_p)$ by choosing V_p to be smaller if necessary. X can be covered by the open neighborhoods V_p and for each V_p there is such a section $s(p) \in \mathscr{F}(V_p)$. For all $p, q \in X$

$$\varphi_{(V_p \cap V_q)}(s(p)|_{V_p \cap V_q}) = \varphi_{(V_p \cap V_q)}(s(q)|_{V_p \cap V_q}) \in \mathscr{F}(V_p \cap V_q)$$

and since we're already shown injectivity $s(p)|_{V_p \cap V_q} = s(q)|_{V_p \cap V_q}$. Therefore they agree on overlaps and by the gluing axiom there exists an $s \in \mathscr{F}(X)$ such that $\mathscr{F}|_{V_p} = s(p)$ for every point p. $\varphi(s), t \in \mathscr{G}(X)$ and for every p they agree on the neighborhoods $\varphi(s)|_{V_p} = t|_{V_p}$, so by the gluing axiom again $\varphi(s) - t \equiv 0$ on all of X and $\varphi(s) = t$. □

Remark 2.14. A consequence of the proof is that a collection of injective morphisms of stalks induces an injective morphism of sheaves.

Proposition 2.15. Let X be a topological space. Let $\mathfrak{C}$ be an abelian category with enough injectives. Suppose further that $\mathfrak{C}$ is closed under arbitrary direct sums and products. Then $\mathfrak{C}(X)$ has enough injectives.

Proof. The proof follows the proof of Proposition 2.2 in Chapter III of [Har77].

Let $\mathscr{F} \in \mathfrak{C}(X)$. Since $\mathfrak{C}$ is an abelian category with arbitrary direct sums, direct limits exist in $\mathfrak{C}$. Therefore, for each point $p \in X$ the stalk $\mathscr{F}_p$ is an object in the category $\mathfrak{C}$. Since $\mathfrak{C}$ has enough injectives, for every object A in $\mathfrak{C}$ there exists an injective object I in $\mathfrak{C}$ together with an injective morphism $A \hookrightarrow I$. Accordingly, for every $p \in X$ let $\mathscr{I}_p$ be an injective object in $\mathfrak{C}$ such that there exists an injective morphism $\varphi_p : \mathscr{F}_p \hookrightarrow I_p$.

For a point $p \in X$ the map $\iota : \{p\} \hookrightarrow X$ is a continuous map of topological spaces. It is clear that $\mathscr{I}_p$ defines a sheaf over $\{p\}$ so for each point p we can form the direct image sheaf $\iota_*(\mathscr{I}_p)$:

$$\iota_*(\mathscr{I}_p)(U) = \begin{cases} \varnothing & \text{if } p \notin U \\ \mathscr{I}_p & \text{if } p \in U \end{cases}$$

Define a sheaf $\mathscr{I}$ with values in $\mathfrak{C}$ by

$$\mathscr{I}(U) = \prod_{p\in U} \iota_*(\mathscr{I}_p).$$

I claim that there exists an injection $\varphi : \mathscr{F} \hookrightarrow \mathscr{I}$ and that $\mathscr{I}$ is an injective object in $\mathfrak{C}(X)$.

First we show the existence of an injection φ. Let $\mathscr{G}$ be a sheaf with values in $\mathfrak{C}$, then

$$\mathrm{Hom}_{\mathfrak{C}}(\mathscr{G}, \mathscr{I}) = \mathrm{Hom}_{\mathfrak{C}}\left(\mathscr{G}, \prod_{p\in X} \iota_*(\mathscr{I}_p)\right) \cong \prod_{p\in X} \mathrm{Hom}_{\mathfrak{C}}(\mathscr{G}, \iota_*(\mathscr{I}_p))$$

By definition of the direct image sheaf, $\mathrm{Hom}_{\mathfrak{C}}(\mathscr{G}, \iota_*(\mathscr{I}_p)) \cong \mathrm{Hom}_{\mathfrak{C}}(\mathscr{G}_p, \mathscr{I}_p)$. Therefore $\mathrm{Hom}_{\mathfrak{C}}(\mathscr{G}, \mathscr{I}) \cong \prod_{p\in X} \mathrm{Hom}_{\mathfrak{C}}(\mathscr{G}_p, \mathscr{I}_p)$. When $\mathscr{G} = \mathscr{F}$, the injections on the stalks $\phi_p : \mathscr{F}_p \hookrightarrow \mathscr{I}_p$ induce an injection $\varphi : \mathscr{F} \hookrightarrow \mathscr{I}$ by Remark 2.14. The establishes the first claim.

By definition, $\mathscr{I}$ is an injective object in $\mathfrak{C}(X)$ if and only if $\mathrm{Hom}_{\mathfrak{C}(X)}(-, \mathscr{I})$ is an exact functor. A morphism of sheaves is entirely determined by its action on the stalks, so if $F_p : \mathscr{G} \to \mathscr{G}_p$ are the family of functors that take the sheaf $\mathscr{G}$ to its stalk at p and $G_p : \mathscr{G}_p \mapsto \mathrm{Hom}_{\mathfrak{C}}(-, \mathscr{I}_p)$ are the local Hom functors, then $\mathrm{Hom}_{\mathfrak{C}(X)}(-, \mathscr{I}) \cong \prod_{p\in X} G_p \circ F_p$. F_p is always exact, and since for every $p \in X$ $\mathscr{I}_p$ is an injective object in $\mathfrak{C}$, so is G_p. The composition of exact functors if an exact functor, therefore $\mathrm{Hom}_{\mathfrak{C}}(-, \mathscr{I})$ is exact and $\mathscr{I}$ is an injective object in $\mathfrak{C}(X)$.

Therefore for every sheaf $\mathscr{F}$ in $\mathfrak{C}(X)$, there is an injective object $\mathscr{I}$ in $\mathfrak{C}(X)$ together with an injective morphism of sheaves $\varphi : \mathscr{F} \hookrightarrow \mathscr{I}$. Thus $\mathfrak{C}(X)$ has enough injectives. □

2.3 Sheaf Cohomology

Let $\mathscr{F}$ be a sheaf on X with values in a category $\mathfrak{C}$. Recall that $\Gamma(U, \mathscr{F}) = \mathscr{F}(U)$ where $U \subseteq X$ is an open subset . Then $\Gamma(X, -) : \mathfrak{C}(X) \to \mathfrak{C}$ is a functor which takes a sheaf on X to its global section, suitably called the **global section functor** on X.

Proposition 2.16. If $\mathfrak{C}$ is an abelian category then the global section functor $\Gamma(X, -)$: $\mathfrak{C}(X) \to \mathfrak{C}$ is additive and left exact.

Sketch of Proof. Let $\mathscr{F}, \mathscr{G}$ be sheaves on X and let $\varphi, \psi : \mathscr{F} \to \mathscr{G}$ be morphisms of sheaves. For an arbitrary $s \in \mathscr{F}(U)$ we have $(\varphi+\psi)(U)(s) = \varphi(U)(s)+\psi(U)(s)$. Applying the global section functor, $\Gamma(\varphi+\psi) = \Gamma\varphi+\Gamma\psi$. The remaining axioms for the definition of an additive functor are left to the reader.

Let $0 \to \mathscr{F}' \xrightarrow{\mu} \mathscr{F} \xrightarrow{\epsilon} \mathscr{F}''$ be a left exact sequence of sheaves on X, then we claim that $0 \to \Gamma(X, \mathscr{F}') \xrightarrow{\mu_*} \Gamma(X, \mathscr{F}) \xrightarrow{\epsilon_*} \Gamma(X, \mathscr{F}'')$ is exact. The induced sequence is exact at $\Gamma(X, \mathscr{F})$ since $\epsilon_*\mu_*(s) = (\epsilon_X \circ \mu_X)(s) = 0$ and $s \in \ker \epsilon_* \iff \epsilon_X(s) = 0 \iff s \in \mathrm{Im}\, \mu_X \iff s \in \mathrm{Im}\, \mu_*$. Suppose that $\mu_*(s) = 0$, then for every open neighborhood U the restriction

$\mu_*(s)|_U = 0$. But for each point the induced map on stalks $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$ is an injection, so $s = 0$. □

We can finally define sheaf cohomology on an arbitrary space. We specialize to the category $\mathfrak{C} = \mathfrak{Ab}$ of abelian groups.

Definition 2.17. Let X be a topological space. The n**th cohomology functor** $H^n(X, -) = R_n\Gamma(X, -)$ is defined to be the nth right derived functor of $\Gamma(X, -)$. If $\mathscr{F}$ is a sheaf on X then $H^n(X, \mathscr{F})$ is the n**th cohomology group of** $\mathscr{F}$.

Remark 2.18. Since $\Gamma(X, -)$ is left exact $H^0 \cong \Gamma(X, -)$ are naturally equivalent functors.

Chapter 3

Čech Cohomology of Manifolds and Presheaves

In this chapter, we'll define the Čech cohomology for an open cover of a manifold and then more generally for an arbitrary presheaf on a topological space. In both cases, special hypotheses about the cover or scheme ensure that the Čech cohomology is, in fact, something more familiar such as de Rham or sheaf cohmology. This allows one to perform computations. Specifically, when we fix a presheaf and vary the space we'll get objects resembling singular cohomology with coefficients or when we fix the space and vary the sheaf we get objects resembling sheaf cohomology.

3.1 The Čech-de Rham Complex

First we setup the machinery needed to define Čech cohomology for an open cover of a manifold. We will then prove results about this cohomology on covers by introducing a double complex with exact rows and turning this into a single complex.

Let M be a smooth manifold and $\mathcal{U} = (U_n)_{n\in\mathbb{N}_0}$ be an open cover of M. For $p \in \mathbb{N}_0$ define

$$N^p := \{(n_0, \ldots, n_p) \in \mathbb{N}_0^{p+1} | n_0 < \ldots < n_p\},$$

and for

$$\sigma = (n_0, \ldots, n_{p+1}) \in N^{p+1} \text{ and } k \in \{0, \ldots, p+1\}$$

define

$$\widehat{\sigma}(k) := (n_0, \ldots, n_{k-1}, n_{k+1}, \ldots, n_{p+1}) \in N^p \text{ and } \sigma(k) := n_k;$$

also, define the notation

$$n \in \sigma$$

to mean that there is some $j \in \{0, \ldots, p+1\}$ with

$$n = n_j.$$

Now let Ω^q denote smooth q-forms on a manifold and consider the inclusions

$$U_\sigma := \bigcap_{n\in\sigma} U_n \subseteq \bigcap_{n\in\widehat{\sigma}(k)} U_n =: U_{\widehat{\sigma}(k)}.$$

These inclusions induce restriction maps

$$\Omega^q(U_{\widehat{\sigma}(k)}) \to \Omega^q(U_\sigma)$$

given by

$$\omega \mapsto \omega|_{U_\sigma}.$$

So we're fixing q and dealing with all possible intersections and the induced maps on smooth q-forms obtained above.

Now we bundle together the smooth q-forms for all $p+1$-fold intersections (again with q still fixed). Define

$$C^p(\mathcal{U}, \Omega^q) := \prod_{\tau\in N^p} \Omega^q(U_\tau)$$

and define

$$\delta^p : C^p(\mathcal{U}, \Omega^q) \to C^{p+1}(\mathcal{U}, \Omega^q)$$

by

$$(\omega_\tau)_{\tau\in N^p} \mapsto \left(\sum_{k=0}^{p+1} (-1)^k \omega_{\widehat{\sigma}(k)}|_{U_\sigma}\right)_{\sigma\in N^{p+1}}$$

with each

$$\omega_\tau \in \Omega^q(U_\tau).$$

The δ maps here look quite similar to the boundary maps encountered for simplicial homology.

It's useful to have a convention for the index -1. Define

$$C^{-1}(\mathcal{U}, \Omega^q) := \Omega^q(M)$$

and define

$$\delta^{-1} : C^{-1}(\mathcal{U}, \Omega^q) \to C^0(\mathcal{U}, \Omega^q)$$

by

$$\omega \mapsto (\omega|_{U_n})_{n\in\mathbb{N}_0}.$$

Theorem 3.1. *The $\mathbb{Z}$-graded $\mathbb{R}$-module $\overline{C} = (C^p(\mathcal{U}, \Omega^q))_{p\in\mathbb{Z}}$ along with the map $\overline{\delta} = (\delta^p)_{p\in\mathbb{Z}}$ form a cochain complex with trivial cohomology where we've extended the indexing by zero (i.e., we define $C^p(\mathcal{U}, \Omega^q) = 0$ whenever $p < -1$).*

Proof. Since the restriction of a sum is the sum of the restrictions it's clear that $\overline{\delta} : \overline{C} \to \overline{C}$ is an endomorphism of degree $+1$. Also, for each $\sigma \in N^{p+2}$ we have that the σ-coordinate of

$$\delta^{p+1}\delta^p((\omega_\tau)_{\tau\in N^p})$$

is

$$\sum_{0\le j<k\le p+2}(-1)^j(-1)^k\omega_{\widehat{\sigma}(j,k)}|_{U_\sigma}+\sum_{0\le k<j\le p+2}(-1)^{j-1}(-1)^k\omega_{\widehat{\sigma}(j,k)}|_{U_\sigma}=0$$

where when $j<k$ we have

$$\widehat{\sigma}(j,k)=\widehat{\widehat{\sigma}(k)}(j)$$

and when $k>j$ we have

$$\widehat{\sigma}(j,k)=\widehat{\widehat{\sigma}(k)}(j-1).$$

Since

$$\delta^0((\omega|_{U_n})_{n\in\mathbb{N}_0})=((\omega|_{U_n})|_{U_{(m,n)}}-(\omega|_{U_m})|_{U_{(m,n)}})_{m<n}=(\omega|_{U_{(m,n)}}-\omega|_{U_{(m,n)}})_{m<n}=0,$$

$$\delta^0\delta^{-1}=0.$$

Thus $\overline{C}$ is a cochain complex.

Next, we show that the cohomology of $\overline{C}$ is trivial. It's clear that $\ker(\delta^{-1})=0$ since a q-form ω will vanish if it vanishes on each set in a cover. Hence $H^{-1}(\overline{C})=0$. Now suppose $p\ge 0$ and $(\omega_\tau)_{\tau\in N^p}\in\ker(\delta^p)$. Let $(\psi_n:M\to\mathbb{R})_{n\in\mathbb{N}_0}$ be a smooth partition of unity subordinate to the cover $\mathcal{U}$. Set

$$x=\left(\sum_{n\in\mathbb{N}_0}\psi_n\omega_{(n,\sigma)}\right)_{\sigma\in N^{p-1}}$$

with appropriate conventions taken when $p=0$. One can check that, in fact,

$$\delta^{p-1}(x)=(\omega_\tau)_{\tau\in N^p},$$

so the complex is exact as claimed. □

Above, we fixed q and defined cochain complexes with the boundary maps δ. Now we note that for each $p\ge 0$, the exterior derivative

$$d:C^p(\mathcal{U},\Omega^q)\to C^p(\mathcal{U},\Omega^{q+1})$$

acting component-wise gives rise to the following double complex

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
C^0(\mathcal{U},\Omega^1) & \xrightarrow{\delta} & C^1(\mathcal{U},\Omega^1) & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
C^0(\mathcal{U},\Omega^0) & \xrightarrow{\delta} & C^1(\mathcal{U},\Omega^0) & \xrightarrow{\delta} & \cdots
\end{array}$$

This is called the Čech-de Rham complex. We will construct a single complex. For $n\in\mathbb{N}_0$ consider the direct sum along a diagonal

$$K^n:=\bigoplus_{0\le p\le n}C^p(\mathcal{U},\Omega^{n-p})$$

and define

$$D^n : K^n \to K^{n+1}$$

by setting

$$C^p(\mathcal{U}, \Omega^{n+1-p})\text{-component of } D^n = \begin{cases} d & \text{when } p = 0 \\ \delta + (-1)^p d & \text{when } 0 < p < n+1 \\ \delta & \text{when } p = n+1. \end{cases}$$

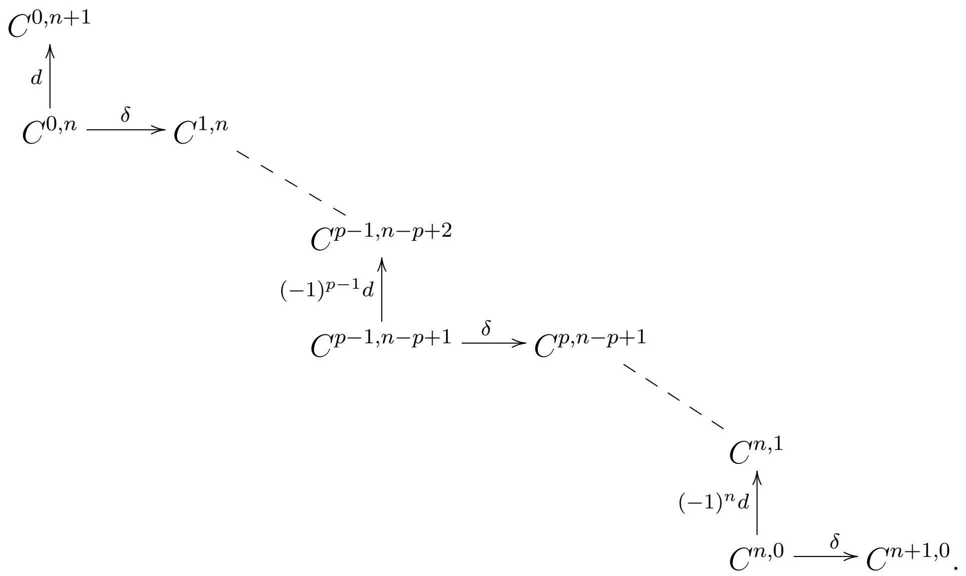
We illustrate this map below for the $n = 1$ case where we're temporarily taking $C^{a,b} = C^a(\mathcal{U}, \Omega^b)$ to unclutter notation:

$$D^1 : C^{0,1} \oplus C^{1,0} \to C^{0,2} \oplus C^{1,1} \oplus C^{2,0}$$

is given by

$$(x, y) \mapsto (d(x), \delta(x) - d(y), \delta(y)).$$

The daunting diagram for an arbitrary n appears as follows:



Theorem 3.2. The $\mathbb{Z}$-graded $\mathbb{R}$-module $\overline{K} = (K^n)_{n\in\mathbb{Z}}$ along with the map $\overline{D} = (D^n)_{n\in\mathbb{Z}}$ form a cochain complex where we've extended the indexing by zero as before. Moreover, the maps δ^{-1} defined for each q induce a cochain map

$$\overline{\delta^{-1}} : (\Omega^q(M))_{q\in\mathbb{Z}} \to \overline{K}$$

which in turn induces isomorphisms on cohomology:

$$\delta_*^{-1} : H^n_{dR}(M) \to H^n(\overline{K}).$$

Proof. First, it's clear that $\overline{D}$ is homomorphism of degree $+1$. Also, the linear operator d commutes with restriction and d acts pointwise on tuples, so d commutes with δ, giving

$$\delta(\delta + (-1)^p d) + (-1)^{p+1} d(\delta + (-1)^p d) = (-1)^p \delta d + (-1)^{p+1} d\delta = 0.$$

It follows that $\overline{D}^2 = 0$, so $(\overline{K}, \overline{D})$ is a cochain complex.

Next, we consider the composition

$$\Omega^q(M) \stackrel{\delta^{-1}}{\to} C^0(\mathcal{U}, \Omega^q) \hookrightarrow K^q$$

which we also denote δ^{-1}. Since

$$(\delta^0 + d)\delta^{-1} = \delta^0\delta^{-1} + d\delta^{-1} = \delta^{-1}d,$$

$\overline{\delta^{-1}}$ is a cochain map.

Thus there are induced maps δ^{-1}_* on cohomology. We'll show these maps are surjective and injective. Suppose $\phi = (\phi_p)_{0\le p\le n} \in \ker(D : K^n \to K^{n+1})$ with $\phi_p \in C^p(\mathcal{U}, \Omega^{n-p})$ for all p. Then, in particular, we have

$$\delta(\phi_n) = 0.$$

The rows of the Čech-de Rham complex are exact by (3.1), so

$$\phi_n = \delta(\psi_{n-1})$$

for some $\psi_{n-1} \in C^{n-1}(\mathcal{U}, \Omega^0)$. Set $\psi = (\psi_k)_{0\le k\le n-1} \in K^{n-1}$ where if $0 \le k \le n-2$ we have $\psi_k = 0 \in C^k(\mathcal{U}, \Omega^{n-1-k})$. Then

$$\phi - D(\psi) = (\phi_k - D(\psi)_k)_{0\le k\le n}$$

is cohomologous with ϕ and

$$\phi_n - D(\psi)_n = \phi_n - \delta(\psi_{n-1}) = \phi_n - \phi_n = 0.$$

Moreover,

$$\begin{aligned} & \phi_{n-1} - D(\psi)_{n-1} = \phi_{n-1} - (\delta(\psi_{n-2}) + (-1)^{n-1}d(\psi_{n-1})) \\ = \; & \phi_{n-1} - \delta(0) - (-1)^{n-1}d(\psi_{n-1}) = \phi_{n-1} + (-1)^n d(\psi_{n-1}), \end{aligned}$$

so

$$\begin{aligned} & \delta(\phi_{n-1} - D(\psi)_{n-1}) = \delta(\phi_{n-1}) + (-1)^n\delta(d(\psi_{n-1})) \\ = \; & \delta(\phi_{n-1}) + (-1)^n d(\delta(\psi_{n-1})) = \delta(\phi_{n-1}) + (-1)^n d(\phi_n) \\ = \; & D(\phi)_n = 0. \end{aligned}$$

Therefore by replacing ϕ with $\phi - D(\psi)$ we may assume $\phi_n = 0$ and $\delta(\phi_{n-1}) = 0$. Playing the same game with the next row up, we may assume $\phi_n = \phi_{n-1} = 0$ and $\delta(\phi_{n-3}) = 0$. Continue. Hence without loss of generality $\phi_k = 0$ whenever $0 < k \le n$ with $\delta^0(\phi_0) = 0 = d(\phi_0)$, so again by exactness of the rows we get

$$\delta^{-1}(\eta) = \phi_0$$

for some closed form $\eta \in \Omega^n(M)$, whence

$$\delta^{-1}_*([\eta]) = \phi.$$

This proves surjectivity.

Now suppose $\delta_*^{-1}([\eta]) = D(\phi)$ for some closed $\eta \in \Omega^n(M)$ and $\phi = (\phi_k)_{0\le k\le n-1} \in K^{n-1}$. Then

$$\delta^{-1}(\eta) = d(\phi_0),$$

so $\eta|_{U_k}$ is exact for all $k \in \mathbb{N}_0$, whence η is exact. This can be seen by using a partition of unity argument. This proves injectivity. □

3.2 Čech Cohomology of Manifolds

For each $k \in \mathbb{N}_0$ define

$$C^k(\mathcal{U}, \mathbb{R}) := \ker(d : C^k(\mathcal{U}, \Omega^0) \to C^k(\mathcal{U}, \Omega^1)).$$

Then $C^k(\mathcal{U}, \mathbb{R})$ is the set of locally constant functions on the $(k+1)$-tuple intersections. Letting δ denote the restriction of the δ maps in the previous section to $\ker(d)$, we have that

$$\cdots \to 0 \to 0 \to C^0(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathbb{R}) \xrightarrow{\delta} \cdots$$

is a cochain complex. We define the nth Čech cohomology group of the cover $\mathcal{U}$, denoted $H^n(\mathcal{U}, \mathbb{R})$, to be the nth cohomology group of this complex.

Definition 3.3. An open cover $\mathcal{U}$ of a manifold M is called a **good cover** if all finite nonempty intersections from $\mathcal{U}$ are contractible.

Theorem 3.4. If $\mathcal{U}$ is a good cover of M, then for each n we have

$$H^n(\mathcal{U}, \mathbb{R}) \cong H^n_{dR}(M).$$

Sketch of Proof. We know $H^q_{dR}(U_\sigma) = 0$ for all $q \ge 1$ and all $\sigma \in \bigcup_{p\ge 0} N^p$ since $\mathcal{U}$ is a good cover, so the columns of the Čech-de Rham complex augmented with the row of Čech cohomology groups by inclusion maps are exact. We may view the Čech-de Rham complex again as a single complex with the roles of δ and d interchanged, in which case it's not hard to see this has the same cohomology as the single complex of the preceding section. Likewise, using arguments similar to those in (3.2) (i.e., inductively removing components using exactness of columns), we may conclude these cohomology groups are the Čech cohomology groups. □

This theorem is useful for computations and applications since every smooth manifold has a good cover. A suitable choice can allow one to apply a generalized Mayer-Vietoris method using the Čech-de Rham complex. For example, one can compute the cohomology of the spheres $\mathbb{S}^1, \mathbb{S}^2$ in this way. Also, (3.2) may be used in proving the Künneth formula:

$$H^\bullet_{dR}(M \times N) \cong H^\bullet_{dR}(M) \otimes_{\mathbb{R}} H^\bullet_{dR}(N)$$

where M, N are smooth manifolds and N has finite dimensional cohomology.

A couple of handy and immediate consequences of (3.2) are contained in the following corollary.

Corollary 3.5. The Čech cohomology $H^\bullet(\mathcal{U}, \mathbb{R})$ is the same for all good covers $\mathcal{U}$ of M, and if M has a finite good cover (e.g., when M is compact), then the de Rham cohomology $H^\bullet_{dR}(M)$ is finite-dimensional.

3.3 Čech Cohomology of Presheaves

In the last section we defined the Čech cohomology groups $H^\bullet(\mathcal{U}, \mathbb{R})$ corresponding to an open cover $\mathcal{U}$ of a smooth manifold M. Now we'll show how to generalize this construction for an arbitrary presheaf $\mathscr{F}$ and an open cover $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of a topological space X where I is a well-ordered set. We've already dealt with two presheaves thus far, namely, Ω^q and the constant presheaf denoted $\mathbb{R}$ (which assigns to an open set U the locally constant functions $U \to \mathbb{R}$).

For each $p \in \mathbb{N}_0$ take

$$C^p(\mathcal{U}, \mathscr{F}) := \prod_{\sigma \in I^p} \mathscr{F}(U_\sigma)$$

where I^p now denotes the strictly increasing $p+1$ tuples of elements in the index set I and U_σ is as before with the additional convention that if $\omega \in C^p(\mathcal{U}, \mathscr{F})$ and σ is an arbitrary $(p+1)$-tuple with entries in I, then

$$\omega_\sigma = \begin{cases} 0 & \text{if } \sigma \text{ has repeated entries} \\ -\omega_\tau & \text{if swapping two entries in } \sigma \text{ gives } \tau \end{cases}$$

Define

$$\delta^p : C^p(\mathcal{U}, \mathscr{F}) \to C^{p+1}(\mathcal{U}, \mathscr{F})$$

by setting the $\sigma \in I^{p+1}$ component equal to

$$\sum_{k=0}^{p+1} (-1)^k \mathscr{F}(\iota_k^\sigma)$$

where

$$\mathscr{F}(\iota_k^\sigma) : \mathscr{F}(U_{\widehat{\sigma}(k)}) \to \mathscr{F}(U_\sigma)$$

is induced by the inclusion

$$\iota_k^\sigma : U_\sigma \hookrightarrow U_{\widehat{\sigma}(k)}.$$

Note that this definition of δ agrees with the one above regarding Ω^q as a presheaf where $\mathscr{F}(\iota_k^\sigma)$ now takes the place of restriction. We have the following analogue for the first part of (3.1) whose proof is similar.

Theorem 3.6. The $\mathbb{Z}$-graded $\mathbb{R}$-module $\overline{C}_{\mathcal{U}} = (C^p(\mathcal{U}, \mathscr{F}))_{p \in \mathbb{Z}}$ along with the map $\overline{\delta} = (\delta^p)_{p \in \mathbb{Z}}$ form a cochain complex where we've extended the indexing by zero.

Definition 3.7. We call the cohomology groups $H^\bullet(\mathcal{U}, \mathscr{F})$ of this complex the **Čech cohomology groups** of the cover $\mathcal{U}$ with values in $\mathscr{F}$.

Definition 3.8. Let $\mathcal{V} = (V_\beta)_{\beta\in J}$ and $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ be open covers of a topological space X where I, J are well-ordered sets. Then we say $\mathcal{V}$ is a **refinement** of $\mathcal{U}$ and write $\mathcal{U} \leq \mathcal{V}$ if there is a map $\phi : J \to I$ with $V_\beta \subseteq U_{\phi(\beta)}$ for all $\beta \in J$. Such a map ϕ is called a **refinement map**.

Suppose $\mathcal{V} = (V_\beta)_{\beta\in J}$ and $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ are open covers of X such that $\mathcal{U} \leq \mathcal{V}$ with refinement map ϕ. For each $\sigma = (s_0, \dots, s_p) \in J^p$ consider

$$\phi(\sigma) = (\phi(s_0), \dots, \phi(s_p));$$

we have inclusions

$$i^\sigma : V_\sigma \hookrightarrow U_{\phi(\sigma)}.$$

These inclusions induce maps

$$\phi^p : C^p(\mathcal{U}, \mathscr{F}) \to C^p(\mathcal{V}, \mathscr{F})$$

defined by taking the $\sigma \in J^p$ component of $\phi^p(\omega)$ to be

$$\mathscr{F}(i^\sigma)(\omega_{\phi(\sigma)}).$$

Lemma 3.9. (i) The map $\overline{\phi} : \overline{C}_{\mathcal{U}} \to \overline{C}_{\mathcal{V}}$ is a cochain map.

(ii) If $\psi : J \to I$ is another refinement map, then $\overline{\phi}$ and $\overline{\psi}$ are homotopic.

(iii) For each n the collection $(H^n(\mathcal{U}, \mathscr{F}))_{\mathcal{U}\in S}$ can be made into a directed system of groups where S is the directed set of well-ordered covers $\mathcal{U}$ of M with the relation $\leq$ given by refinement.

Proof. (i) Let $\omega = (\omega_\sigma)_{\sigma\in I^p} \in C^p(\mathcal{U}, \mathscr{F})$ with $\omega_\sigma \in \mathscr{F}(U_\sigma)$. Then

$$\begin{aligned}
\delta^p \phi^p \omega &= \delta^p (\mathscr{F}(i^\tau)\omega_{\phi(\tau)})_{\tau\in J^p} \\
&= \left(\sum_{k=0}^{p+1} (-1)^k \mathscr{F}(\iota_k^\sigma)\mathscr{F}(i^{\widehat{\sigma}(k)})\omega_{\phi(\widehat{\sigma}(k))}\right)_{\sigma\in J^{p+1}} \\
&= \left(\sum_{k=0}^{p+1} (-1)^k \mathscr{F}(i^\sigma)\mathscr{F}(\iota_k^{\phi(\sigma)})\omega_{\widehat{\phi(\sigma)}(k)}\right)_{\sigma\in J^{p+1}} \\
&= \phi^{p+1}\left(\sum_{k=0}^{p+1} (-1)^k \mathscr{F}(\iota_k^\tau)\omega_{\widehat{\tau}(k)}\right)_{\tau\in I^p} \\
&= \phi^{p+1}\delta^p\omega
\end{aligned}$$

as required.

(ii) Define $\Sigma_q : C^q(\mathcal{U}, \mathscr{F}) \to C^{q-1}(\mathcal{V}, \mathscr{F})$ by

$$\Sigma_q \omega = \left(\sum_{k=0}^{q-1} (-1)^k \mathscr{F}(\widetilde{i}_k^\beta)\omega_{\widetilde{\beta}(k)}\right)_{\beta\in J^{q-1}}$$

where

$$\widetilde{\beta}(k) = (\phi(\beta_0), \ldots, \phi(\beta_k), \psi(\beta_k), \ldots \psi(\beta_{q-1}))$$

and

$$\widetilde{i}_k^\beta : V_\beta \hookrightarrow U_{\widetilde{\beta}(k)}$$

are inclusion maps. It's a mildly painful computation to show that indeed

$$\overline{\psi} - \overline{\phi} = \overline{\delta\Sigma} + \overline{\Sigma\delta}.$$

(iii) The relation $\leq$ on well-ordered covers is clearly reflexive (since the identity map on the index set is a refinement map) and transitive (since the composition of refinement maps is a refinement map). If $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ and $\mathcal{V} = (V_\beta)_{\beta \in J}$ are arbitrary open covers of X with well-ordered sets I, J, then $I \times J$ is a well-ordered set with the lexicographic ordering and $\mathcal{W} = (U_\alpha \cap V_\beta)_{(\alpha,\beta) \in I \times J}$ is an open cover; moreover, $\mathcal{W}$ is a refinement of both $\mathcal{U}$ and $\mathcal{V}$ since, for example, the projection $I \times J \twoheadrightarrow I$ is a refinement map. Thus $\mathcal{W} \geq \mathcal{U}, \mathcal{V}$, so S is a directed set. If $\mathcal{U} \leq \mathcal{V}$ in S, then there is a unique induced map on cohomology since any two refinement maps are homotopic. Therefore we have a directed system as claimed since the induced map of a composition is the composition of induced maps and the induced map of an identity map is an identity map. □

Definition 3.10. We now define the nth **Čech cohomology group** of X with values in $\mathscr{F}$ as the direct limit

$$\check{H}^n(X, \mathscr{F}) = \varinjlim_{\mathcal{U}} H^n(\mathcal{U}, \mathscr{F})$$

where the limit is taken over the directed set of well-ordered open covers $\mathcal{U}$.

Theorem 3.11. We have $\check{H}^n(M, \mathbb{R}) \cong H^n_{dR}(M)$ for all n where $\mathbb{R}$ is the constant presheaf with group $\mathbb{R}$ on M.

Sketch of Proof. Since every open cover on a manifold has a refinement which is a good cover, we may use only good covers in the direct limit defining $\check{H}^n(M, \mathbb{R})$, but then because the refinement is compatible with the isomorphisms in (3.4) we have the desired result. □

3.4 A Note on Sheaf Cohomology

In a far more general setting, namely, that of schemes with nice properties, Čech cohomology serves the role of computational advantage. In particular, when the scheme and the cover are nice enough, the Čech and sheaf cohomology are the same. The following is Theorem 4.5 in Chapter III of [Har77].

Theorem 3.12. Suppose $\mathcal{U}$ is an affine open cover of a Noetherian, separated scheme X and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then the Čech cohomology of the cover $\mathcal{U}$ with values in $\mathscr{F}$ and the sheaf cohomology of X with values in $\mathscr{F}$ are isomorphic.

With such a scheme taking the place of a manifold, we have a vast generalization of theorem (3.11).

We can relax the conditions on X and $\mathscr{F}$ considerably if we're only interested in H^1. The following theorem is taken from part (c) of Exercise 4.4 in Chapter III of [Har77].

Theorem 3.13. Let X be an arbitrary topological space and $\mathscr{F}$ be a sheaf of abelian groups on X. Then

$$\check{H}^1(X, \mathscr{F}) \cong H^1(X, \mathscr{F}).$$

Chapter 4

The Interpretation of $H^1(X, \mathcal{O}^\times)$

We will now interpret $H^1(X, \mathscr{F}^\times)$ where $\mathscr{F}$ is a sheaf of commutative rings and $\mathscr{F}^\times(U)$ is the group of units $(\mathscr{F}(U))^\times$ for every open $U \subset X$. To keep things simple, we will assume X is a compact Riemann surface and $\mathscr{F}$ is the sheaf of regular functions $\mathcal{O}_{X,\text{alg}}$. We will denote this sheaf by $\mathcal{O}$. First we will define invertible sheaves over $\mathcal{O}$. It turns out the isomorphism classes of invertible sheaves over $\mathcal{O}$ form a group, called the Picard group. The Picard group is naturally isomorphic to the first sheaf cohomology $H^1(X, \mathcal{O}^\times)$. Moreover, on a Riemann surface, this group is isomorphic to the divisor class group as well as the group of line bundles. All of these isomorphisms are natural. This illustrates the power of sheaf cohomology. It is a versatile tool that on the one hand generalizes de Rham cohomology and on the other hand provides a bridge between number theory and geometry.

4.1 Invertible Sheaves

We will study the invertible sheaves on a compact Riemann surface X. Let $\mathcal{M}(X)$ be the meromorphic functions $f : X \to \mathbb{C}$. For each open subset U of X, define

$$\mathcal{O}_{X,\text{alg}}(U) = \{f \in \mathcal{M}(X) |\ f|_U \text{ is holomorphic}\}.$$

This is the sheaf of **regular** functions on X. For simplicity, we will denote this sheaf by $\mathcal{O}$. The sheaf $\mathcal{O}$ is a sheaf of commutative rings. An invertible sheaf is a locally free rank one sheaf of $\mathcal{O}$-modules. Our first task is to understand what this statement means.

Before formally defining an invertible sheaf, we must define a sheaf of $\mathcal{O}$-modules.

Definition 4.1. A sheaf $\mathscr{F}$ on X is a **sheaf of $\mathcal{O}$-modules** if

1. for every open set $U \subset X$, $\mathscr{F}(U)$ is an $\mathcal{O}(U)$-module; and
2. whenever $V \subset U$, the restriction map $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$ is $\mathcal{O}$-linear in the sense that if $r \in \mathcal{O}(U)$ and $f \in \mathscr{F}(U)$ then $\rho_{UV}(r \cdot f) = \rho_{UV}(r) \cdot \rho_{UV}(f)$.

Example 4.2. Let $\mathcal{V}$ be a complex vector bundle over a compact Riemann surface X. Define $\mathscr{F}$ by

$$\mathscr{F}(U) = \left\{\text{sections } \sigma \text{ of } \mathcal{V} \middle|\ \sigma \text{ is meromorphic on } \mathcal{V} \text{ and } \sigma\big|_U \text{ is holomorphic}\right\}.$$

Then $\mathscr{F}$ is a sheaf of $\mathcal{O}$-modules on X called the sheaf of **regular sections** of $\mathcal{V}$.

A sheaf map $\phi : \mathscr{F} \to \mathscr{G}$ between two $\mathcal{O}$-modules is simply a sheaf map such that for every U, $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is a homomorphism of $\mathcal{O}(U)$-modules.

We can now state the definition of an invertible sheaf. Recall that if $\mathscr{F}$ is a sheaf on X, and $U \subset X$ is an open set, there is a restricted sheaf $\mathscr{F}|_U$ on U defined by setting $\mathscr{F}|_U(V) = \mathscr{F}(V)$ for any open set $V \subset U$.

Definition 4.3. Let X be a topological space, and let $\mathscr{F}$ be a sheaf of $\mathcal{O}$-modules. We say $\mathscr{F}$ is **invertible** if for every $p \in X$ there is an open neighborhood U of p, such that $\mathscr{F}|_U \cong \mathcal{O}|_U$ as sheaves of $\mathcal{O}|_U$-modules on the space U.

An isomorphism $\mathscr{F}|_U \to \mathcal{O}|_U$ is called a trivialization of $\mathscr{F}$ over U. An equivalent way of defining an invertible sheaf is to require that there is an open cover $\{U_i\}$ of X such that for each i, $\mathscr{F}|_{U_i} \cong \mathcal{O}|_{U_i}$ as sheaves of $\mathcal{O}|_{U_i}$-modules on U_i. Note that we will shortly see the reason such sheaves are called "invertible."

It is sometimes convenient to express the invertibility of a sheaf $\mathscr{F}$ in terms of generators for the modules $\mathscr{F}(V)$. Suppose that U is an open subset of X on which $\mathscr{F}|_U \cong \mathcal{O}|_U$ as sheaves of $\mathcal{O}|_U$-modules. Then there are isomorphisms $\phi_V : \mathcal{O}(V) \to \mathscr{F}(V)$ for all open $V \subset U$. Moreover each map ϕ_V is a map of $\mathcal{O}(V)$-modules, and these isomorphisms commute with the restriction maps. In particular there is an isomorphism $\phi_U : \mathcal{O}(U) \to \mathscr{F}(U)$ on the entire open subset U. Let $f_U \in \mathscr{F}(U)$ be the image of $1_U \in \mathcal{O}(U)$. Then f_U is a generator for the free module $\mathscr{F}(U)$ over $\mathcal{O}(U)$. If we define $f_V = \rho_{UV}(f_U)$ for every open set $V \subset U$, then f_V is also a generator of the free module $\mathscr{F}(V)$, since $f_V = \rho_{UV}(f_U) = \rho_{UV}(\phi_U(1)) = \phi_V(\rho_{UV}(1)) = \phi_V(1)$. Thus the element f_U is not only a generator for $\mathscr{F}(U)$, but it restricts to generators f_V for $\mathscr{F}(V)$ for every open $V \subset U$.

It is easy to express the fact that $\mathscr{F}(V)$ is free of rank one over $\mathcal{O}(V)$ using a generator f_V. We simply require that f_V generates $\mathscr{F}(V)$ over $\mathcal{O}(V)$, and that it has a trivial annihilator in $\mathcal{O}(V)$ - that is, if $r \in \mathcal{O}(V)$ and $r \cdot f_V = 0$ then $r = 0$. This proves the following:

Lemma 4.4. Let X be a compact Riemann surface, and let $\mathscr{F}$ be a sheaf of $\mathcal{O}$-modules. Then $\mathscr{F}$ is invertible if and only if for every $p \in X$ there is an open neighborhood U of p and a section $f_U \in \mathscr{F}(U)$ such that for all $V \subset U$, the restricted section $f_V = \rho_{UV}(f_U)$ generates the module $\mathscr{F}(V)$ over $\mathcal{O}(V)$, and has a trivial annihilator.

Such an element f_U will be called a **local generator** for the invertible sheaf $\mathscr{F}$ at the point p. Hence we may loosely say that a sheaf $\mathscr{F}$ is invertible if it has a local generator at every point of X.

Example 4.5. Let X be a Riemann surface. Then the sheaf Ω^1 of 1-forms on X is invertible. A local generator for Ω^1 in a neighborhood of a point p is the 1-form dz, where z is a local complex coordinate for X centered at p.

Let $\mathcal{U} = \{U_i\}$ be an open cover of X such that each $\mathscr{F}(U_i)$ has a local generator f_i. If $U_i \cap U_j \neq \varnothing$, then f_i and f_j both restrict to a local generator of $\mathscr{F}(U_i \cap U_j)$, hence f_i and f_j

only differ by a multiple of a unit in $\mathcal{O}(U_i \cap U_j)$ on $U_i \cap U_j$. Suppose $f_i = t_{ij} f_j$ on $U_i \cap U_j$, where $t_{ij} \in (\mathcal{O}(U_i \cap U_j))^\times$. Then $f_j = (t_{ij})^{-1} f_i = t_{ji} f_i$. On U_i, since $U_i \cap U_i = U_i$, t_{ii} is the identity. If $U_i \cap U_j \cap U_k \neq \varnothing$, then $f_i = t_{ij} f_j$ and $f_j = t_{jk} f_k$, hence $f_i = t_{ij} t_{jk} f_k$ on $U_i \cap U_j \cap U_j$. On the other hand, $f_i = f_{ik} f_k$. Since f_i, f_j and f_k are local generators, we get the following conditions that the t_{ij}'s satisfy:

1. t_{ii} is the identity;
2. $t_{ji} = t_{ij}^{-1}$; and
3. $t_{ik} = t_{ij} t_{jk}$, whenever $U_i \cap U_j \cap U_k \neq \varnothing$.

These are called the **cocycle conditions**. We will see why below.

Finally, we turn to putting a group structure on the set of isomorphism classes of invertible sheaves over $(X, \mathcal{O})$. This requires us to generalize two constructions in the category of R-modules. If M and N are two free modules of rank one over a ring R, then the tensor product $M \otimes_R N$ and the dual module $M^* := \operatorname{Hom}_R(M, R)$ are also free modules of rank one over R.

One's first instinct in defining $\mathscr{F} \otimes_{\mathcal{O}} \mathscr{G}$ and $\mathscr{F}^*$ is to set

$$\mathscr{F} \otimes_{\mathcal{O}} \mathscr{G}(U) = \mathscr{F}(U) \otimes_{\mathcal{O}(U)} \mathscr{G}(U)$$

$$\mathscr{F}^*(U) = \operatorname{Hom}_{\mathcal{O}(U)}(\mathscr{F}(U), \mathcal{O}(U))$$

for all open sets U. This does produce presheaves of $\mathcal{O}$-modules, but not in general sheaves of $\mathcal{O}$-modules. We need to use Proposition 2.11 to sheafify the above presheaves to get the correct sheaves.

Definition 4.6. Let $\mathscr{F}$ and $\mathscr{G}$ be two invertible sheaves of $\mathcal{O}$-modules on X. Then we define the **tensor product** $\mathscr{F} \otimes_{\mathcal{O}} \mathscr{G}$ to be the sheaf associated to the presheaf defined by $U \mapsto \mathscr{F}(U) \otimes_{\mathcal{O}(U)} \mathscr{G}(U)$, and define the **dual sheaf** $\mathscr{F}^*$ to be the sheaf associated to the presheaf defined by $U \mapsto \operatorname{Hom}_{\mathcal{O}(U)}(\mathscr{F}(U), \mathcal{O}(U))$.

The above definitions do not give easy descriptions of the sheaves. Accordingly, we describe more concrete definitions. Let $\{U_i\}$ be the collection of all open subsets of X on which both $\mathscr{F}$ and $\mathcal{G}$ may be trivialized. This forms an open covering of X. For any open subset U of X, define

$$\mathscr{F} \otimes_{\mathcal{O}} \mathscr{G}(U) = \{(s_i) \in \prod_i \mathscr{F}(U \cap U_i) \otimes_{\mathcal{O}(U \cap U_i)} \mathscr{G}(U \cap U_i) \mid s_i|_{U \cap U_i \cap U_j} = s_j|_{U \cap U_i \cap U_j} \text{ for all } i, j\}$$

$$\mathscr{F}^*(U) = \{(s_i) \in \prod_i \operatorname{Hom}_{\mathcal{O}(U \cap U_i)}(\mathscr{F}(U \cap U_i), \mathcal{O}(U \cap U_i)) \mid s_i|_{U \cap U_i \cap U_j} = s_j|_{U \cap U_i \cap U_j} \text{ for all } i, j\}$$

Then $\mathscr{F} \otimes_{\mathcal{O}} \mathscr{G}$ and $\mathscr{F}^*$ are the required sheaves.

Remark 4.7. In the definitions above, we used every open set U_i on which both $\mathscr{F}$ and $\mathscr{G}$ could be trivialized. This is not necessary. If we use any collection of such U_i's which cover X, we will obtain sheaves isomorphic to the those defined above.

Note that if $\mathscr{F}$ and $\mathscr{G}$ both trivialize on an open set U, then so does $\mathscr{F} \otimes_{\mathcal{O}} \mathscr{G}$. Moreover a local generator is induced by the tensor product of the local generators for $\mathscr{F}$ and $\mathscr{G}$.

Lemma 4.8. Let $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ be three invertible sheaves. Then:

1. $\mathcal{O} \otimes_{\mathcal{O}} \mathscr{F} \cong \mathscr{F}$;
2. $\mathscr{F} \otimes_{\mathcal{O}} \mathscr{G} \cong \mathscr{G} \otimes_{\mathcal{O}} \mathscr{F}$;
3. $(\mathscr{F} \otimes_{\mathcal{O}} \mathscr{G}) \otimes_{\mathcal{O}} \mathscr{H} \cong \mathscr{F} \otimes_{\mathcal{O}} (\mathscr{G} \otimes_{\mathcal{O}} \mathscr{H})$; and
4. $\mathscr{F} \otimes_{\mathcal{O}} \mathscr{F}^* \cong \mathcal{O}$.

Sketch of Proof. We only prove the last isomorphism $\mathscr{F} \otimes_{\mathcal{O}} \mathscr{F}^* \cong \mathcal{O}$ and leave the others to the reader. Let $\{U_i\}$ be an open cover on which $\mathscr{F}$ may be trivialized. Let f_i be a local generator of $\mathscr{F}(U_i)$. For each i, define a morphism of $\mathcal{O}(U_i)$-modules $f_i^* : \mathscr{F}(U_i) \to \mathcal{O}(U_i)$ by setting $f_i^*(f_i) = 1$. Then f_i^* is a local generator of $\mathscr{F}^*(U_i)$. Thus $f_i \otimes f_i^*$ is a local generator of $\mathscr{F} \otimes \mathscr{F}^*(U_i)$. We define homomorphisms of $\mathcal{O}(U_i)$-modules $\phi_{U_i} : \mathscr{F} \otimes \mathscr{F}^*(U_i) \to \mathcal{O}(U_i)$ by setting $\phi_{U_i}(f_i \otimes f_i^*) = 1$. Then it is easy to see that each ϕ_{U_i} is an isomorphism and these maps can be extended to a sheaf isomorphism $\mathscr{F} \otimes \mathscr{F}^* \cong \mathcal{O}$. □

This lemma shows that the set of isomorphism classes of invertible sheaves forms an abelian group, where $\otimes_{\mathcal{O}}$ gives a group multiplication, the class of $\mathcal{O}$ gives the identity and the class of the dual sheaves gives the inverse.

Definition 4.9. The group of isomorphism classes of invertible sheaves on a topological space X is called the **Picard group** and denoted $\mathrm{Pic}(X)$.

4.2 The Picard Group and $H^1(X, \mathcal{O}^\times)$

Given the sheaf $\mathcal{O}$ of commutative rings, we define the sheaf $\mathcal{O}^\times$ by $\mathcal{O}^\times(U) = (\mathcal{O}(U))^\times$, the group of units of $\mathcal{O}(U)$. A function $f \in \mathcal{O}(U)$ is a unit if and only if f does not vanish on U. Therefore, if $f \in \mathcal{O}^\times(U)$ and $V \subset U$, $\rho_{UV}(f)$ doesn't vanish on V and hence $\rho_{UV}(f) \in \mathcal{O}^\times(V)$. Therefore $\mathcal{O}^\times$ defines a presheaf of abelian groups on X. If U is covered by open sets V_i and for each i we have $f_i \in \mathcal{O}^\times(V_i)$ such that $f_i = f_j$ on $V_i \cap V_j$, since $\mathcal{O}$ is a sheaf there is a function $f \in \mathcal{O}(U)$ such that $\rho_{UV_i}(f) = f_i$. Since f_i is non-vanishing on each V_i it follows that f is non-vanishing on U and hence $f \in \mathcal{O}^\times(U)$. Therefore $\mathcal{O}^\times$ satisfies the gluing axiom and defines a sheaf of abelian groups on X. Since the category of abelian groups is an abelian category with enough injectives that is closed under arbitrary direct sums and products, we can define the sheaf cohomology groups $H^n(X, \mathcal{O}^\times)$. We are specifically interested in interpreting $H^1(X, \mathcal{O}^\times)$. Since the group operation in $\mathcal{O}^\times(U)$ is multiplication of functions, we will write the group operation in our cohomology groups as multiplication for the remainder of this chapter.

Now we prove the main theorem of this chapter, that $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}^\times)$. We will do so by proving $\mathrm{Pic}(X) \cong \check{H}^1(X, \mathcal{O}^\times)$, which will suffice by Theorem 3.13 above.

We define a map from $\mathrm{Pic}(X)$ to $\check{H}^1(X, \mathcal{O}^\times)$. Let $\mathscr{F} \in \mathrm{Pic}(X)$. Then there is an open cover $\mathcal{U} = \{U_i\}$ of X such that $\mathscr{F}$ trivializes over each U_i. For each i, let f_i be a local generator of $\mathscr{F}$ over U_i. For each pair i, j there exists $t_{ij} \in \mathcal{O}^\times(U_i \cap U_j)$ such that $f_i = t_{ij} f_j$ in $\mathscr{F}(U_i \cap U_j)$. Since $t_{ij} \in \mathcal{O}^\times(U_i \cap U_j)$, $t_{ij} \neq 0$ on $U_i \cap U_j$. The functions $\{t_{ij}\}$ will be referred to below as the **transition functions** of $\mathscr{F}$ with respect to the open cover $\mathcal{U}$.

Lemma 4.10. The tuple (t_{ij}) defined above is a cocycle in $\check{H}^1(\mathcal{U}, \mathcal{O}^\times)$.

Proof. The tuple $(t_{ij})_{i<j}$ is a cochain in $\check{C}^1(\mathcal{U}, \mathcal{O}^\times)$ by the first two cocycle conditions. By the third cocycle condition, for every i, j, k then $t_{ij} t_{jk} t_{ki}$ is the identity on $U_i \cap U_j \cap U_k$. Since $t_{ki} = t_{ik}^{-1}$, $t_{ij} t_{ik}^{-1} t_{jk} = t_{ij} t_{jk} t_{ik}^{-1} = 1$. On the other hand:

$$\delta t_{ij} = t_{ij} t_{ik}^{-1} t_{jk}.$$

Thus $\delta t_{ij} = 1$ and therefore $(t_{ij})_{i<j}$ is a cocycle for the sheaf $\mathcal{O}^\times$ with respect to $\mathcal{U}$. Thus $(t_{ij})_{i<j}$ represents a class in $\check{H}^1(\mathcal{U}, \mathcal{O}^\times)$. Since the direct limit is taken over refinements of the open covers, and $\mathscr{F}$ trivializes over any refinement of $\mathcal{U}$, $(t_{ij})_{i<j}$ defines an element of $\check{H}^1(X, \mathcal{O}^\times)$. □

Now define $H : \mathrm{Pic}(X) \to \check{H}^1(X, \mathcal{O}^\times)$ by $H(\mathscr{F}) = (t_{ij})$ where (t_{ij}) is as defined above.

Lemma 4.11. The map H is well defined.

Proof. We have to show that H is independent of the choice of local generators, the choice of open cover and choice of representative of the isomorphism class of $\mathscr{F}$.

Starting with the generators, suppose $\{g_i\}$ are another collection of local generators of $\mathscr{F}$ with respect to the open cover $\mathcal{U}$. Then for every i there is a nowhere zero regular function s_i such that $g_i = s_i f_i$. The tuple (s_i) forms a cochain in C^0. Since $s_i \neq 0$ on U_i for every i, $f_j = s_j^{-1} g_j$. Thus on $U_i \cap U_j$ we have

$$g_i = s_i f_i = s_i t_{ij} f_j = s_i t_{ij} s_j^{-1} g_j.$$

Observe that $s_i s_j^{-1}\big|_{U_i \cap U_j} = ds_i$ and therefore is a coboundary. Thus, (t_{ij}) and $(s_i t_{ij} s_j^{-1})$ define the same cohomology class in $\check{H}^1(\mathcal{U}, \mathcal{O}^\times)$, and it follows that they define the same class in the limit $\check{H}^1(X, \mathcal{O}^\times)$.

Now we show that H is independent of our choice of open cover. As a consequence of part (iii) of Lemma 3.9, any two open covers have a common refinement. Accordingly it suffices to show that $H(\mathscr{F})$ is invariant under refinement. Suppose $\mathcal{V} = \{V_j\}$ is a refinement of $\mathcal{U}$ with refinement map $r : J \to I$. Then for each j, $V_j \subset U_{r(j)}$. As a consequence, for each j we can choose $f_{r(j)}$ as a local generator with respect to the cover $\mathcal{V}$. Let (t'_{ij}) be the co-cycle computed using $\mathcal{V}$. It follows that $(t'_{ij}) = H(r)(t_{ij})$ where $H(r) : \check{H}^1(\mathcal{U}, \mathcal{O}^\times) \to \check{H}^1(\mathcal{V}, \mathcal{O}^\times)$ is the map obtained via refinement. Since the direct limit is taken over the directed system $(H^1(\mathcal{W}, \mathcal{O}^\times))_{\mathcal{W} \in \mathcal{S}}$ (using the notation from Lemma 3.9), and $H(r)$ is one of the maps making this collection into a directed system, it follows that $(t'_{ij}) = (t_{ij})$ in $\check{H}^1(X, \mathcal{O}^\times)$ and H is independent of the choice of open cover.

Finally, suppose $\mathscr{F} \cong \mathscr{F}'$ are isomorphic invertible sheaves. Then both $\mathscr{F}$ and $\mathscr{F}'$ trivialize over the open cover $\mathcal{U} = \{U_i\}_{i \in I}$, and have respective local generators $\{f_i\}$ and $\{f_i\}'$

with respective transition functions $\{t_{ij}\}$ and $\{t'_{ij}\}$. Since $\mathscr{F} \cong \mathscr{F}'$, for each $i \in I$ there is an isomorphism $\varphi_{U_i} : \mathscr{F}(U_i) \to \mathscr{F}'(U_i)$ such that $\varphi_{U_i}(f_i) = f'_i$. Then on $U_i \cap U_j$:

$$t_{ij} f'_i = t_{ij}\varphi_{U_i}(f_i) = \varphi_{U_i}(t_{ij} f_i) = \varphi_{U_i}(f_j) = f'_j = t'_{ij} f'_i.$$

It follows that $t_{ij} = t'_{ij}$ and hence $H(\mathscr{F}) = H(\mathscr{F}')$. Thus H is independent of the choice of representative for $\mathscr{F} \in \mathrm{Pic}(X)$.

Therefore $H : \mathrm{Pic}(X) \to \check{H}^1(X, \mathcal{O}^\times)$ is a well defined map. □

Next I will show that H is bijective by defining an explicit inverse.

Lemma 4.12. Suppose $\mathcal{U} = \{U_i\}$ is an open cover of X with a well-ordered indexing set I. Let $(t_{ij}) \in \check{H}^1(\mathcal{U}, \mathcal{O}^\times)$. Then there exists a corresponding invertible sheaf $\mathscr{F}$ that trivializes over $\mathcal{U}$ whose transition functions with respect to $\mathcal{U}$ are given by (t_{ij}). The sheaf $\mathscr{F}$ is unique up to isomorphism.

Proof. First we prove existence. Define a sheaf $\mathscr{E}$ as follows. For an open set $V \subset X$,

$$\mathscr{E}(V) = \bigcup_{i \in I} \left(\mathcal{O}(V \cap U_i) \times \{i\}\right).$$

This is the precisely the disjoint union, but it is convenient to be explicit. It is clear that $\mathscr{E}$ is a sheaf of abelian groups since $\mathcal{O}$ satisfies the gluing axioms.

If $(f, i) \in \mathscr{E}(V)$, $f \in \mathcal{M}(X)$ and is holomorphic on $V \cap U_i$. For $h \in \mathcal{O}(V)$, $hf \in \mathcal{M}(X)$ and is holomorphic on $V \cap U_i$ and thus $(hf, i) \in \mathscr{E}(V)$. It is clear that this multiplication is compatible with the restriction maps. We therefore define a $\mathcal{O}(V)$-module structure on $\mathscr{E}(V)$ by function multiplication. This makes the sheaf $\mathscr{E}$ into a sheaf of $\mathcal{O}$-modules.

Now we define an equivalence relation on $\mathscr{E}(V)$ for all open sets $V \subset X$. The components of the cocycle (t_{ij}) only exist for $i < j$. Accordingly, we extend the collection t_{ij} by defining $t_{ji} = 1/t_{ij}$ when $i < j$ and defining $t_{ii} = 1$. We will say that $(f_i, i) \sim (f_j, j) \in \mathscr{E}(V)$ if $f_i = t_{ij} f_j$ on $V \cap U_i \cap U_j$. By our conventions for t_{ii} and t_{ji} with $i < j$, this relation is reflexive and symmetric. Since the tuple (t_{ij}) is a cocycle in $\check{H}^1(X, \mathcal{O}^\times)$, the relation is transitive and defines an equivalence relation.

Suppose $h \in \mathcal{O}(V)$ and $(f_i, i) \sim (f_j, j)$ in $\mathscr{E}(V)$. Then $f_i = t_{ij} f_j$ on $V \cap U_i \cap U_j$ and therefore

$$hf_i = h(t_{ij} f_j) = t_{ij}(hf_j)$$

on $V \cap U_i \cap U_j$. Thus $hf_i \sim hf_j$. Therefore the $\mathcal{O}$-module action is well defined on equivalence classes.

Define a sheaf $\mathscr{F}$ by $\mathscr{F}(V) = \mathscr{E}(V)/\sim$. It is clear that the sheaf structure of $\mathscr{E}$ descends to $\mathscr{F}$, and since the $\mathcal{O}$-module action is well defined on equivalence classes in $\mathscr{E}$, $\mathscr{F}$ is a sheaf of $\mathcal{O}$-modules. I claim that $\mathscr{F}$ is an invertible sheaf that trivializes over $\mathcal{U}$ with transition functions t_{ij} with respect to $\mathcal{U}$.

For every i, let 1_{U_i} be a generator of $\mathcal{O}(U_i)$. For an open subset $V \subset U_i$, set $f_V = \rho_{U_i V}(1_{U_i})$. I claim that the class $[f_V]$ generates $\mathscr{F}(V)$ as an $\mathcal{O}(V)$-module and the annihilator of f_V is trivial.

Suppose $[g] \in \mathscr{F}(V)$. Since $V \subset U_i$, the set of representatives for $[g]$ in $\mathscr{E}$ is $\{t_{ij}g\}_{j \in I}$. Each of these representatives defines a function $t_{ij}g \in \mathcal{O}(V \cap U_j)$. By definition of the equivalence relation, for $j, k \in I$, $t_{ij}g = t_{ik}g$ on $V \cap U_j \cap U_k$. Since $\{V \cap U_j\}_{j \in I}$ covers V, by the gluing axiom there is some $\hat{h} \in \mathcal{O}(V)$ such that $\rho_{V(V \cap U_j)}\hat{h} = t_{ij}g$. Since f_V generates $\mathcal{O}(V)$, there is some $h \in \mathcal{O}(V)$ such that $\tilde{h} = hf_V$. By construction, $[g] = [\tilde{h}]$ and therefore by definition of the $\mathcal{O}(V)$-module structure on $\mathscr{F}(V)$, $[g] = h[f_V]$. Thus $[f_V]$ generates $\mathscr{F}(V)$.

Now suppose α annihilates f_V. Thus $\alpha[f_V] = [\alpha f_V] = 0$. Therefore $t_{ij}\alpha f_V = 0$ for every $j \in I$. In particular, $t_{ii}\alpha f_V = 0$. But t_{ii} is the identity on U_i and $V \subset U_i$. Hence $\alpha f_V = 0$. Since $f_V = \rho_{U_i V}(1_{U_i}) \neq 0$ and $\mathcal{O}(V)$ has no zero divisors, it follows that $\alpha = 0$. Therefore the annihilator of f_V is trivial.

Thus, $[f_V]$ generates $\mathscr{F}(V)$ as an $\mathcal{O}(V)$-module and has trivial annihilator. It follows by Lemma 4.4 that $\mathscr{F}$ is an invertible sheaf. It is clear from the construction that $\mathscr{F}$ trivializes over $\mathcal{U}$ and has transition functions t_{ij} with respect to $\mathcal{U}$. This establishes existence.

Now suppose $\mathscr{G}$ is another invertible sheaf that trivializes over $\mathcal{U}$ and has transition functions $\{t_{ij}\}$ with respect to $\mathcal{U}$. Since $\mathscr{F}$ and $\mathscr{G}$ trivialize over $\mathcal{U}$, for each i we obtain an isomorphism $\mathscr{F}(U_i) \cong \mathscr{G}(U_i)$ by sending a generator to a generator. Since $\mathscr{F}$ and $\mathscr{G}$ have the same transition functions with respect to the open cover $\mathcal{U}$, the isomorphisms are compatible with restriction. It follows that for any $p \in X$, taking the direct limit over open sets containing p we obtain an isomorphism of the stalks $\mathscr{F}_p \cong \mathscr{G}_p$. Therefore by Proposition 2.13, $\mathscr{F} \cong \mathscr{G}$ as sheaves. □

Accordingly, for each open cover $\mathcal{U} \in S$ we get a map $K_{\mathcal{U}} : \check{H}^1(\mathcal{U}, \mathcal{O}^\times) \to \mathrm{Pic}(X)$.

Lemma 4.13. The maps $K_{\mathcal{U}}$ induce a unique map $K : \check{H}^1(X, \mathcal{O}^\times) \to \mathrm{Pic}(X)$ such that $K\iota_{\mathcal{U}} = K_{\mathcal{U}}$.

Proof. Since the maps $K_{\mathcal{U}}$ are invariant under refinement, for any refinement $\mathcal{V}$ of $\mathcal{U}$ with refinement map r, $K_{\mathcal{U}}H(r) = K_{\mathcal{V}}$. Therefore the desired result follows by the universal property of direct limits. □

Lemma 4.14. The maps H and K are inverses and H is a bijection.

Proof. Suppose $\tau = (t_{ij}) \in \check{H}^1(X, \mathcal{O}^\times)$. Then for any open cover $\mathcal{U}$ over which $K(\tau)$ trivializes, and any local generators f_i for $K(\tau)$, by definition $f_j = t_{ij}f_j$ in $U_i \cap U_j$ and hence $HK(\tau) = \tau$. In the other direction, let $\mathscr{G} = KH(\mathscr{F})$. Then $H(\mathscr{G}) = H(\mathscr{F})$. This implies that there is an open cover over which both $\mathscr{F}$ and $\mathscr{G}$ trivialize and have the same transition functions. Hence by uniqueness up to isomorphism in Lemma 4.12, $\mathscr{G} \cong \mathscr{F}$, so $KH(\mathscr{F}) \cong \mathscr{F}$ and define the same element of $\mathrm{Pic}(X)$. Therefore H and K are inverses and it follows that H is bijective. □

We now have that H is a bijection from $\text{Pic}(X)$ onto $\check{H}^1(X, \mathcal{O}^\times)$. The last step is showing that H is a group homomorphism.

Lemma 4.15. The map H is a group homomorphism.

Proof. Suppose $\mathscr{F}, \mathscr{G} \in \text{Pic}(X)$. Let $\mathcal{U} = \{U_i\}$ be a covering of X over which both $\mathscr{F}$ and $\mathscr{G}$ trivialize. Let f_i be local generators for $\mathscr{F}$ and g_i be local generators of $\mathscr{G}$. Then $f_i \otimes g_i$ are local generators for $\mathscr{F} \otimes_{\mathcal{O}} \mathscr{G}$. For any $U_i \cap U_j$, there is $r_{ij}, s_{ij} \in \mathcal{O}^\times(U_i \cap U_j)$ such that $f_j = r_{ij} f_i$ and $g_j = s_{ij} g_i$. Then:

$$f_j \otimes g_j = (r_{ij} f_i) \otimes (s_{ij} g_i) = (r_{ij} s_{ij}) f_i \otimes g_i.$$

Since $r_{ij}, s_{ij} \in \mathcal{O}^\times(U_i \cap U_j)$ then $r_{ij} s_{ij} \in \mathcal{O}^\times(U_i \cap U_j)$. Therefore:

$$H(\mathscr{F} \otimes_{\mathcal{O}} \mathscr{G}) = (r_{ij} s_{ij}) = (r_{ij})(s_{ij}) = H(\mathscr{F}) H(\mathscr{G}).$$

Thus H is a group homomorphism. □

Finally, our theorem has been reduced to a consequence of the above lemmas.

Theorem 4.16. $\text{Pic}(X) \cong H^1(X, \mathcal{O}^\times)$.

Proof. By Lemmas 4.11, 4.14 and 4.15 the map $H : \text{Pic}(X) \to \check{H}^1(X, \mathcal{O}^\times)$ is a well-defined bijective group homomorphism and therefore $\text{Pic}(X) \cong \check{H}^1(X, \mathcal{O}^\times)$. By Theorem 3.13, $\check{H}^1(X, \mathcal{O}^\times) \cong H^1(X, \mathcal{O}^\times)$ completing the proof. □

Note that the only reason we needed X to be a Riemann surface was to have a concrete description of the sheaf $\mathcal{O}^\times$. The arguments can be abstracted to show that this result holds for any "ringed space" $(X, \mathcal{O}_X)$ (where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings called the structure sheaf). Therefore, we can interpret the first sheaf cohomology to be the group of invertible sheaves. Note that if $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, we can use the isomorphism between the Picard group and the first sheaf cohomology to define an induced map $f^* : \text{Pic}(Y) \to \text{Pic}(X)$. This can be defined independent of the isomorphism. For the construction, see Lecture 5 in [Mum66], which uses the construction of the pullback of a sheaf of modules $f^*\mathscr{F}$ defined in Section 5 of Chapter II in [Har77]. It turns out the isomorphism $\text{Pic}(X) \cong H^1(X, \mathcal{O}^\times)$ is natural.

4.3 The Divisor Class Group

The divisor class group is an analogue for function fields of the ideal class group for rings of integers. As such, it has applications in number theory. On a Riemann surface, the divisor class group is isomorphic to the Picard group (sometimes the divisor class group is called the Picard group). We identify the connection in this section. We will begin by defining the divisor class group, and then we will identify the isomorphism with the Picard group. We will conclude with identifying a direct isomorphism between the divisor class group and $H^1(X, \mathcal{O}^\times)$.

Let X be a Riemann surface. Denote the set of all functions $X \to \mathbb{Z}$ by $\mathbb{Z}^X$. For any function $D : X \to \mathbb{Z}$, the **support** of D is the set of all $p \in X$ such that $D(p) \neq 0$.

Definition 4.17. A function $D : X \to \mathbb{Z}$ is called a **divisor** on X if its support is discrete. The set of all divisors on X from a group under pointwise addition denoted by $\mathrm{Div}(X)$.

A divisor D is typically denoted by

$$D = \sum_{p \in X} D(p) \cdot p.$$

Since X is a compact Riemann surface, the support of every divisor is finite. Therefore the group $\mathrm{Div}(X)$ for a compact Riemann surface X is precisely the free abelian group on the points of X. Now we define a subgroup of $\mathrm{Div}(X)$ called the principal divisors. Suppose f is a nonzero meromorphic function on X. Then we make the following definition.

Definition 4.18. For every point $p \in X$, the **order** of f at p is defined by

$$\mathrm{ord}_p(f) = \begin{cases} k & : \text{ if } p \text{ is a zero of } f \text{ of order } k \\ -k & : \text{ if } p \text{ is a pole of } f \text{ of order } k \\ 0 & : \text{ otherwise} \end{cases}.$$

Note that a nonzero meromorphic function $f : X \to \mathbb{C}$ defines a holomorphic function $f : X \to \mathbb{CP}^1$ (see Theorem 1.15 of [For81]). Accordingly, by the identity theorem, the zeros of f (and poles of f) form a discrete (and therefore finite) subset of X. See pages 6 - 8 of [For81] for details. Accordingly we can make the following definition.

Definition 4.19. The **divisor of f**, denoted by $\mathrm{div}(f)$, is the divisor defined by the order function:

$$\mathrm{div}(f) = \sum_{p \in X} \mathrm{ord}_p(f) \cdot p.$$

A divisor of this form is called a **principal divisor**.

The set of principal divisors is a subset of $\mathrm{Div}(X)$. In fact, it is a subgroup. The proofs of the following are left to the reader.

Proposition 4.20. Suppose f and g are nonzero meromorphic functions on X. Then for every $p \in X$:

1. $\mathrm{ord}_p(fg) = \mathrm{ord}_p(f) + \mathrm{ord}_p(g)$
2. $\mathrm{ord}_p(f/g) = \mathrm{ord}_p(f) - \mathrm{ord}_p(g)$
3. $\mathrm{ord}_p(1/f) = -\mathrm{ord}_p(f)$.
4. $\mathrm{ord}_p(f \pm g) \geq \min\{\mathrm{ord}_p(f), \mathrm{ord}_p(g)\}$

Corollary 4.21. Suppose f and g are nonzero meromorphic functions on X. Then for every $p \in X$:

1. $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$
2. $\mathrm{div}(f/g) = \mathrm{div}(f) - \mathrm{div}(g)$
3. $\mathrm{div}(1/f) = -\mathrm{div}(f)$.

Corollary 4.22. The set of principal divisors on X forms a subgroup of $\mathrm{Div}(X)$ denoted by $\mathrm{PDiv}(X)$.

We can now define the divisor class group.

Definition 4.23. For a Riemann Surface X, the divisor class group, which we will denote by $\mathrm{DCl}(X)$[1] is the quotient

$$\mathrm{DCl}(X) = \frac{\mathrm{Div}(X)}{\mathrm{PDiv}(X)}.$$

We will define a partial ordering on the divisors on X which will allow us to identify the relationship between $\mathrm{DCL}(X)$ and $\mathrm{Pic}(X)$. Given a divisor $D \in \mathrm{Div}(X)$, we say $D \geq 0$ if $D(p) \geq 0$ for all $p \in X$, and we say $D > 0$ if $D \geq 0$ and $D \neq 0$. Given two divisors $D_1, D_2 \in \mathrm{Div}(X)$, we say $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$, and $D_1 > D_2$ if $D_1 - D_2 > 0$. This defines a partial ordering on $\mathrm{Div}(X)$.

Given a divisor D on X, we use the partial ordering on $\mathrm{Div}(X)$ to construct the sheaf of meromorphic functions with poles bounded by D:

$$\mathcal{O}[D](U) = \left\{ f \in \mathcal{M}(X) \;\middle|\; \mathrm{div}(f) \geq -D \text{ on } \; U \right\}.$$

As a consequence of Proposition 4.20, for each U we have $\mathcal{O}[D](U)$ is an $\mathcal{O}^\times(U)$-module, the action being multiplication of functions. On a compact Riemann surface X, the sheaf $\mathcal{O}[D]$ is an invertible sheaf. For any $p \in X$, a local generator is given by $z^{-D(p)}$ where z is a meromorphic function on X with a simple zero at p. See Lemma 1.5 in Chapter XI of [Mir95] for the proof. If $[D_1] = [D_2]$ in $\mathrm{DCl}(X)$, there is some $f \in \mathcal{M}(X)$ such that $D_1 - D_2 = \mathrm{div}(f)$. Multiplication by f induces an isomorphism $\mathcal{O}[D_1] \cong \mathcal{O}[D_2]$ This gives us a well defined map

$$\mathcal{O}[-] : \mathrm{DCl}(X) \to \mathrm{Pic}(X).$$

We will show $\mathcal{O}[-]$ is a group isomorphism.

Lemma 4.24. The map

$$\mathcal{O}[-] : \mathrm{DCl}(X) \to \mathrm{Pic}(X)$$

is a group homomorphism.

Proof. It suffices to show that for any two divisors D_1 and D_2 on X,

$$\mathcal{O}[D_1 + D_2] \cong \mathcal{O}[D_1] \otimes_{\mathcal{O}} \mathcal{O}[D_2].$$

The local generator for $\mathcal{O}[D]$ for any divisor D is the map $f = z^{-D(p)}$, where in the appropriate neighborhood of p we have $-D = \mathrm{div}(f)$.

Let $\{U_i\}$ be an open covering on which both sheaves $\mathcal{O}[D_1]$ and $\mathcal{O}[D_2]$ trivialize. Let $f_i^{(1)}$ and $f_i^{(2)}$ be local generators for $\mathcal{O}[D_1]$ and $\mathcal{O}[D_2]$ respectively. Then for each $j = 1, 2$, on U_i we have $\mathrm{div}(f_i^{(j)}) = -D_j$. Hence by Proposition 4.21 it follows that $\mathrm{div}(f_i^{(1)} f_i^{(2)}) = -D_1 - D_2$ and thus $f_i^{(1)} f_i^{(2)}$ is a local generator for $\mathcal{O}[D_1 + D_2]$ on U_i.

[1]This notation is *not* standard.

By the first statement of Corollary 4.21, there is a natural bilinear map induced by multiplication from $\mathcal{O}[D_1] \times \mathcal{O}[D_2]$ to $\mathcal{O}[D_1 + D_2]$. By the universal properties of tensor products and sheafication, this descends to the tensor product inducing a map of sheaves

$$\mu : \mathcal{O}[D_1] \otimes_{\mathcal{O}} \mathcal{O}[D_2] \to \mathcal{O}[D_1 + D_2].$$

Since $\mathcal{O}[D_1] \otimes_{\mathcal{O}} \mathcal{O}[D_2]$ trivializes over $\{U_i\}$ with local generators $f_i^{(1)} \otimes f_i^{(2)}$, and $\mu(f_i^{(1)} \otimes f_i^{(2)}) = f_i^{(1)} f_i^{(2)}$, it follows that μ sends local generators to local generators. Hence μ is an isomorphism of sheaves and

$$\mathcal{O}[D_1 + D_2] \cong \mathcal{O}[D_1] \otimes_{\mathcal{O}} \mathcal{O}[D_2].$$

Therefore $\mathcal{O}[-]$ is a group homomorphism. □

We need a couple of lemmata before we can show that $\mathcal{O}[-]$ is bijective.

Lemma 4.25. On an open set $U \subset X$, for a meromorphic function $f \in \mathcal{M}(X)$, a meromorphic function $g \in \mathcal{M}(X)$ is an element of $\mathcal{O}[-\operatorname{div}(f)](U)$ if and only if there is $h \in \mathcal{M}(X)$ that is holomorphic on U such that $g = fh$ on U.

Proof. One direction is trivial. By Lemma 4.21, if $g = fh$ on U then $\operatorname{div}(g) = \operatorname{div}(h) + \operatorname{div}(f)$. It follows that $\operatorname{div}(g) - \operatorname{div}(f) = \operatorname{div}(h)$. Since h is holomorphic on U and therefore has no poles on U, $\operatorname{div}(h) \geq 0$ on U. Hence $\operatorname{div}(g) \geq \operatorname{div}(f)$ and $g \in \mathcal{O}[-\operatorname{div}(f)](U)$.

For the other direction, we suppose $g \in \mathcal{O}[-\operatorname{div}(f)](U)$. Then $\operatorname{div}(g) \geq \operatorname{div}(f)$ and thus $\operatorname{div}(g) - \operatorname{div}(f) \geq 0$. Tracing the definitions, this implies that for every $p \in U$, $\operatorname{ord}_p(g) - \operatorname{ord}_p(f) \geq 0$. Therefore, if p is a zero of f then p is a zero of g of higher order. Similarly if p is a pole of g then p is a pole of f of higher order. It follows by expanding the Laurent series for f and g (and noting that U is biholomorphic to an open subset of $\mathbb{C}$) that there is a holomorphic function h on U such that $g = fh$. This concludes the proof. □

Lemma 4.26. If $\mathcal{O}[D_1] = \mathcal{O}[D_2]$, then $D_1 = D_2$.

Proof. Let $p \in X$ be arbitrary and suppose $D_1(p) < D_2(p)$. There exists a meromorphic function z with order one at p. Then for an open neighborhood U of p, $z^{-D_2(p)} \in \mathcal{O}[D_2](U)$ but $z^{-D_2(p)} \notin \mathcal{O}[D_1](U)$. This contradicts the hypothesis that $\mathcal{O}[D_1] = \mathcal{O}[D_2]$. Hence for every $p \in X$, $D_1(p) \geq D_2(p)$. By symmetry $D_2(p) \geq D_1(p)$ for every $p \in X$. Thus $D_1(p) = D_2(p)$ for every $p \in X$ and hence $D_1 = D_2$. □

Proposition 4.27. The homomorphism $\mathcal{O}[-]$ is injective.

Proof. Suppose $[D] \in \ker(\mathcal{O}[-])$. Recall that the sheaf $\mathcal{O}$ is the identity element of $\operatorname{Pic}(X)$. Therefore $\mathcal{O}[D] \cong \mathcal{O}$. It follows that $\mathcal{O}[D]$ is globally trivial and has a global generator f. Therefore for every open set $U \subset X$, f generates the free rank-one $\mathcal{O}$-module $\mathcal{O}[D](U)$. Therefore $\mathcal{O}[D](U)$ is precisely the set of multiples of f by elements of $\mathcal{O}(U)$. Since $\mathcal{O}(U)$ by definition consists precisely of meromorphic functions on X that are holomorphic on U, it follows that

$$\mathcal{O}[D] = \left\{ fh \;\middle|\; h \in \mathcal{M}(X) \text{ such that } h \text{ is holomorphic on } U \right\}.$$

Therefore by Lemma 4.25,

$$\mathcal{O}[D] = \mathcal{O}[-\mathrm{div}(f)].$$

This is an equality, not a mere isomorphism. Hence by Lemma 4.26, $D = -\mathrm{div}(f)$ which, by the fourth part of Lemma 4.21 implies $D = \mathrm{div}(1/f)$. It follows that D is principal and thus $D \in \mathrm{PDiv}(X)$. Hence $[D] = 0$ in $\mathrm{DCl}(X)$. Therefore $\ker(\mathcal{O}[-])$ is trivial and $\mathcal{O}[-]$ is injective. □

Proposition 4.28. The homomorphism $\mathcal{O}[-]$ is surjective.

Proof. Let $\mathscr{F}$ be an invertible sheaf on X and let $\{U_i\}$ be an open cover of X over which $\mathscr{F}$ trivializes. Choose corresponding local generators $\{f_i\}$.

Choose an index, say $i = 0$ (where the symbol '0' stands for the minimal element of the well ordered index set I). Then for each i, there are nonzero meromorphic functions t_{i0} on U_i that are holomorphic on $U_i \cap U_0$ such that $f_i = t_{i0} f_0$. On U_0, $t_{00} = 1$. Hence for each $p \in U_0$, $\mathrm{ord}_p(t_{00}) = 0$. Note that for each i, on $U_i \setminus (U_i \cap U_0)$, t_{i0} may have zeros or poles.

Now we define a divisor be defining a function $D : X \to \mathbb{Z}$. Define D by

$$D(p) = -\mathrm{ord}_p(t_{i0}) \;\text{ if }\; p \in U_i.$$

This is well defined because of the cocycle condition. To wit, if $p \in U_i \cap U_j$, $t_{i0} = t_{ij} t_{j0}$ on $U_i \cap U_j$ and $\mathrm{ord}_p(t_{ij}) = 0$ on $U_i \cap U_j$. Hence by the first part of Lemma 4.20, $\mathrm{ord}_p(t_{i0}) = \mathrm{ord}_p(t_{j0})$. By definition, $\mathcal{O}[D]$ trivializes over $\{U_i\}$ and has local generators $\{t_{i0}\}$.

Finally, we show $\mathcal{O}[D] \cong \mathscr{F}$. For each i, since both sheaves trivialize on U_i, we can define an isomorphism $\mathcal{O}[D] \to \mathscr{F}$ by mapping $t_{i0} \mapsto f_i$. These maps are compatible with restrictions by the cocycle condition. It follows that this collection of isomorphisms induce an isomorphism $\mathcal{O}(D)_p \cong \mathscr{F}_p$ of stalks for each $p \in X$. Hence by Proposition 2.13, $\mathcal{O}[D] \cong \mathscr{F}$.

Therefore for every $\mathscr{F} \in \mathrm{Pic}(X)$, there is a divisor D on X such that $\mathscr{F} \cong \mathcal{O}[D]$ and $\mathcal{O}[-]$ is surjective. □

Theorem 4.29. The map $\mathcal{O}[-] : \mathrm{DCl}(X) \to \mathrm{Pic}(X)$ is an isomorphism of groups.

Proof. By Lemma 4.24, $\mathcal{O}[-]$ is a group homomorphism and by Propositions 4.27 and 4.28, $\mathcal{O}[-]$ is bijective. Hence $\mathcal{O}[-]$ is a group isomorphism. □

The isomorphism $\mathcal{O}[-]$ is natural, in the sense that given a holomorphic map of Riemann surfaces $f : X \to Y$, one can use precomposition to induce a group homomorphism $f^* : \mathrm{DCl}(Y) \to \mathrm{DCl}(X)$ making DCl into a contravariant functor. Then the isomorphism $\mathcal{O}[-]$ becomes a natural equivalence between the functors DCl and Pic.

As a consequence, it follows that for any Riemann surface X, $\mathrm{DCl}(X)$ is naturally isomorphic to $H^1(X, \mathcal{O}^\times)$. This can be accomplished directly using a long exact sequence on cohomology.

First we fix the following sheaves. Define the sheaf $\mathcal{M}^*$ to be the constant sheaf of meromorphic functions on X that are not identically zero. Define a sheaf $\mathcal{Div}$ by

$$\mathcal{Div}(U) = \{\text{divisors with finite support contained in } U\}$$

for each open set U. For each $f \in \mathcal{M}^*(X)$ and each open set $U \subset X$, we can define a divisor by

$$D_{f,U}(p) = \begin{cases} \operatorname{ord}_p(f) & : \ p \in U \\ 0 & : \ \text{otherwise} \end{cases}.$$

This gives us a sheaf map

$$\operatorname{div} : \mathcal{M}^* \to \mathcal{Div}.$$

It turns out div is surjective. See Lemma 3.1 in Chapter XI of [Mir95]. A meromorphic function f will satisfy $D_{f,U}(p) = 0$ for all $p \in U$ if and only if f has no poles or zeros on U, which in turn is true if and only if $f \in \mathcal{O}^\times(U)$. Hence the kernel of div is exactly the sheaf $\mathcal{O}^\times$. This gives us a short exact sequence of shaves:

$$0 \to \mathcal{O}^\times \to \mathcal{M}^* \overset{\text{div}}{\to} \mathcal{Div} \to 0.$$

This short exact sequence of sheaves induces a long exact sequence on cohomology. Recall that since the global section functor is left exact, there is a natural equivalence $H^0(X, \mathscr{F}) \cong \mathscr{F}(X)$ for any appropriate sheaf $\mathscr{F}$. Moreover, since $\mathcal{M}^*$ is a constant sheaf, $H^1(X, \mathcal{M}^*)$ is trivial (see Proposition 2.1 in Chapter X of [Mir95]). Therefore, if we denote the connecting homomorphism by Δ, the zeroth-level of the long-exact sequence of cohomology is given by

$$0 \to \mathcal{O}^\times(X) \to \mathcal{M}^*(X) \overset{\text{div}}{\to} \mathcal{Div}(X) \overset{\Delta}{\to} H^1(X, \mathcal{O}^\times) \to 0.$$

Since X is assumed to be a compact Riemann surface and $\mathcal{O}^\times(X)$ is the ring of global nonzero holomorphic functions, which are constant, $\mathcal{O}^\times(X) \cong \mathbb{C}^*$. The group $\mathcal{M}^*(X)$ is simply the multiplicative group of the rational function field $\mathcal{M}$, which is $\mathcal{M}(X) \setminus \{0\}$. Finally, by definition the group $\mathcal{Div}(X)$ is the group $\operatorname{Div}(X)$ of global divisors on X. Therefore we have the following exact sequence:

$$0 \to \mathbb{C}^* \to \mathcal{M}(X) \setminus \{0\} \overset{\text{div}}{\to} \operatorname{Div}(X) \overset{\Delta}{\to} H^1(X, \mathcal{O}^\times) \to 0.$$

Thus $\Delta : \operatorname{Div}(X) \to H^1(X, \mathcal{O}^\times)$ is the cokernel of the map div. The image of the global divisor map $\operatorname{div} : \mathcal{M}(X)\setminus\{0\} \to \operatorname{Div}(X)$ is $\operatorname{PDiv}(X)$. Therefore it follows that the connecting homomorphism induces a natural isomorphism

$$\Delta : \operatorname{DCl}(X) \to H^1(X, \mathcal{O}^\times).$$

This isomorphism is explicitly described on page 347 of [Mir95]. This is used to show that the isomorphisms $\mathcal{O}[-] : \operatorname{DCl}(X) \to \operatorname{Pic}(X)$ and $H : \operatorname{Pic}(X) \to H^1(X, \mathcal{O}^\times)$ satisfy

$$\Delta = H \circ \mathcal{O}[-].$$

See Proposition 3.7 in Chapter XI of [Mir95] for the proof.

4.4 Line Bundles

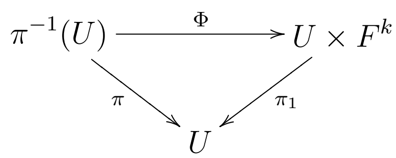
Now we look at the geometric side of the story. We will consider the collection of line bundles over a Riemann surface X. It turns out they form an abelian group that is also isomorphic to the Picard group. We will first show the group of line bundles is isomorphic to $H^1(X, \mathcal{O}^\times)$ and then consider the resulting isomorphism onto $\text{Pic}(X)$. We will complete the picture by determining the isomorphism with $\text{DCl}(X)$.

We remind the reader of the definition of an arbitrary rank k vector bundle over a smooth manifold M.

Definition 4.30. Let M be a smooth manifold. A vector bundle of rank k over M is a smooth manifold E together with a surjective smooth map $\pi : E \to M$ satisfying:

1. For each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ is endowed with the structure of a k-dimensional vector space over some fixed field F, and

2. For each $p \in M$, there exists a neighborhood U of p in M and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times F^k$ (called a **local trivialization** of E over U) , such that the following diagram commutes:



   (where π_1 is projection onto the first factor); and such that for each $q \in U$, the restriction of Φ to E_q is a linear isomorphism from E_q onto $\{q\} \times F^k \cong F^k$.

When the rank $k = 1$, the vector bundle is called a F-**line bundle**. For a Riemann surface X, we denote the set of isomorphism classes of complex line bundles by $\text{LB}(X)$.

Note that for a complex line bundle L over X, we can use a cover $\{U_i\}$ of complex charts to obtain the local trivialization for L. Then for each i, we call $z_i := \Phi_i^{-1}(p, 1)$ for $p \in U_i$ a **fiber coordinate** with respect to Φ_i. On overlaps, we have well-defined nonzero holomorphic functions, called the transition functions. To be precise, if $U_i \cap U_j \neq 0$, then there exists a holomorphic function t_{ij} on $U_i \cap U_j$ such that the local trivializations Φ_i and Φ_j satisfy

$$t_{ij} = \Phi_i \Phi_j^{-1} : \mathbb{C} \times (U_i \cap U_j) \to \mathbb{C} \times (U_i \times U_j).$$

The maps t_{ij} are called the **transition functions** . They satisfy three particular conditions:

1. $t_{ii} = 1$ on U_i;

2. $t_{ji} = 1/t_{ij}$ on $U_i \cap U_j$; and

3. $t_{ki} t_{ij} t_{jk} = 1$ on $U_i \cap U_j \cap U_k$.

These conditions are called the **cocycle conditions** .

Proposition 4.31. Let X be a Riemann surface, $\{U_i\}$ an open cover of X, and for each pair of indices i, j, suppose that t_{ij} is a nowhere zero holomorphic function on $U_i \cap U_j$, such that the collection $\{t_{ij}\}$ satisfies the cocycle conditions. Then there is a line bundle L, unique up to isomorphism, with local trivializations over the cover $\{U_i\}$ and the collection $\{t_{ij}\}$ as its transition functions. For any fiber coordinate z_i on $\pi^{-1}(U_i)$ we have

$$z_i = t_{ij} z_j$$

on $U_i \cap U_j$.

Sketch of Proof. Let $\tilde{L}$ be the disjoint union

$$\tilde{L} = \coprod_i (\mathbb{C} \times U_i).$$

Define a relation $\sim$ on $\tilde{L}$ by declaring $(s, p) \in \mathbb{C} \times U_j$ to be related to $(t_{ij}s, p) \in \mathbb{C} \times U_i$ whenever $p \in U_i \cap U_j$. The cocycle conditions for the transition functions imply that $\sim$ is an equivalence relation. Let

$$L = \tilde{L}/\sim .$$

Then L is the desired line bundle, where $\pi : L \to X$ is given by $\pi([s, p]) = p$ (which is clearly well defined and holomorphic). The composition

$$\mathbb{C} \times U_i \hookrightarrow \tilde{L} \to L$$

is injective, and the local trivializations are given by the inverse of this composition on $\pi^{-1}(U_i)$. See Theorem 29.7 of [For81] for the details. Alternatively, one may at this point invoke the vector bundle construction lemma, which is Lemma 5.5 of [Lee03]. □

The point of Proposition 4.31 is that a line bundle is defined uniquely (up to isomorphism) by the local data provided by its transition functions. This is the fact we will exploit to show $\mathrm{LB}(X) \cong \mathrm{Pic}(X)$.

First we will show that there is a bijective correspondence between $\mathrm{LB}(X)$ and $H^1(X, \mathcal{O}^\times)$. Given a line bundle L trivialized over an open cover $\mathcal{U} = \{U_i\}$ of X, define $H'(L) = (t_{ij})$, where the collection $\{t_{ij}\}$ are the transition functions. As with Lemma 4.10, the cocycle conditions imply that $\{t_{ij}\}$ define a cocycle class in $\check{H}^1(\mathcal{U}, \mathcal{O}^\times)$ and hence in the direct limit $\check{H}^1(X, \mathcal{O}^\times)$. Using a nearly identical proof as Lemma 4.11, this gives a well defined map $H' \to \check{H}^1(X, \mathcal{O}^\times)$.

Proposition 4.32. The map

$$H' : \mathrm{LB}(X) \to \check{H}^1(X, \mathcal{O}^\times)$$

is a bijection.

Proof. Since the cocycle conditions are precisely the conditions for (t_{ij}) to be a cocycle class in $\check{H}^1(X, \mathcal{O}^\times)$, Proposition 4.31 implies that H' is surjective.

It remains to show injectivity. Suppose L^1, L^2 are line bundles over X and $H'(L^1) = H'(L^2)$.

We assume we have a cover over which L^1 and L^2 both trivialize. Since $H'(L_1) = H'(L_2)$, we may pass to a possibly finer cover such that the cocycles forming $H'(L_1)$ and $H'(L_2)$ differ by a coboundary. Let $\mathcal{U} = \{U_i\}$ denote this cover. Let $\left\{t_{ij}^{(l)}\right\}$ denote the transition functions for the line bundles L_l, with $l = 1, 2$ and let $\Phi_i^{(l)}$ denote the maps for the local trivialization over U_i for L_l, with $l = 1, 2$. Since the cocycles forming $H'(L_1)$ and $H'(L_2)$ over $\mathcal{U}$ differ by a coboundary, there are holomorphic nowhere zero functions s_i on U_i for each i such that $t_{ij}^{(1)} s_i/s_j = t_{ij}^{(2)}$ on $U_i \cap U_j$ for every i and j.

For each i, we can construct a line bundle automorphism $S_i : \mathbb{C} \times U_i \to \mathbb{C} \times U_i$ given by $S_i(z_i, p) = (s_i z_i, p)$. This gives us a local trivialization $\Phi_i^{(1a)} : \pi^{-1}(U_i) \to \mathbb{C} \times U_i$ defined by $\Phi_i^{(1a)} = S_i \circ \Phi_i^{(1)}$. Since the maps $\Phi_i^{(1a)}$ are compatible with the maps $\Phi_i^{(1)}$ (in the sense that $\Phi_i^{(1a)} \circ (\Phi_i^{(1)})^{-1}$ is a biholomorphism from $\mathbb{C} \times U_i \to \mathbb{C} \times U_i$), it follows that the maps $\Phi_i^{(1a)}$ define a compatible local trivialization for L_1. The transition maps become exactly

$$t_{ij}^{(1a)} = t_{ij}^{(1)} s_i/s_j = t_{ij}^{(2)}.$$

Hence $L_1 \cong L_2$ as line bundles by the uniqueness statement of Proposition 4.31. □

Define a new map

$$H'' : \mathrm{LB}(X) \to H^1(X, \mathcal{O}^\times)$$

by $H''(L) = 1/H'(L)$. The map H'' is also a bijection. Since $\check{H}^1(X, \mathcal{O}^\times) \cong H^1(X, \mathcal{O}^\times)$ we may assume that H'' is a bijection between $\mathrm{LB}(X)$ and $H^1(X, \mathcal{O}^\times)$. As a consequence we have a bijective correspondence between $\mathrm{LB}(X)$ and $\mathrm{Pic}(X)$. We will determine this bijection explicitly. This relationship comes via the notion of a section of a line bundle.

Definition 4.33. Let $\pi : L \to X$ be a line bundle over a compact Riemann surface X and let $U \subset X$ be an open subset. A **regular section of L over U** is a function $s : U \to L$ such that

1. for every $p \in U$, $s(p)$ lies in the fiber of L over p and

$$\pi \circ s = \mathrm{Id}_U$$

and

2. for every local trivialization $\Phi_V : \pi^{-1}(V) \to \mathbb{C} \times V$ for L, the composition

$$\mathrm{pr}_1 \circ \Phi_V \circ s\big|_{U \cap V} : U \cap V \to \mathbb{C}$$

is a holomorphic function on $U \cap V$.

Denote the set of regular sections of L over U by $\mathcal{O}\{L\}(U)$. This defines a sheaf $\mathcal{O}\{L\}$ for any line bundle L over X. The sheaf $\mathcal{O}\{L\}$ is an invertible sheaf. See Proposition 2.14 in Chapter XI of [Mir95] for the proof.

We have now defined a function

$$\mathcal{O}\{-\} : \mathrm{LB}(X) \to \mathrm{Pic}(X).$$

We will show that this is a bijection, and is related to the map H'' defined above.

Proposition 4.34. For a compact Riemann surface X, the map $\mathcal{O}\{-\} : \mathrm{LB}(X) \to \mathrm{Pic}(X)$ is a bijection. Moreover, the composition

$$\mathrm{LB}(X) \xrightarrow{\mathcal{O}\{-\}} \mathrm{Pic}(X) \xrightarrow{H} H^1(X, \mathcal{O}^\times)$$

is the bijection H''.

Proof. Only the last statement requires proof, the former statement is an immediate consequence. Suppose L is a line bundle over X. Let $\mathcal{U} = \{U_i\}$ be an open cover of X over which L trivializes. Denote the fiber coordinate with respect to Φ_i by z_i. Let $\{t_{ij}\}$ be the transition functions. Then $H'(L) = (t_{ij})$, so by the second cocycle condition, $H''(L) = (t_{ji})$.

The invertible sheaf $\mathcal{O}\{L\}$ also trivializes over $\mathcal{U}$. For each i, a local generator for $\mathcal{O}\{L\}(U_i)$ is given by the section s_i where $s_i(z_i) \equiv 1$ on U_i. We obtain the cocycle representing $H(\mathcal{O}\{L\})$ by writing the local generator for U_i as a multiple of the local generator for U_j. When $z_i = 1$ (which defines s_i), we have

$$z_j = t_{ji} z_i = t_{ji}$$

and as a consequence

$$s_i = t_{ji} s_j.$$

Thus the cocycle (t'_{ij}) representing $H(\mathcal{O}\{L\})$ satisfies

$$t'_{ij} = t_{ji}.$$

Therefore:

$$H(\mathcal{O}\{L\}) = (t'_{ij}) = (t_{ji}) = H''(L)$$

completing the proof. □

We now have compatible bijections between $\mathrm{LB}(X)$ and $H^1(X, \mathcal{O}^\times)$ and between $\mathrm{LB}(X)$ and $\mathrm{Pic}(X)$. We can use these bijections to induce a group structure on $\mathrm{LB}(X)$. The group structure on $\mathrm{LB}(X)$ is very similar to the group structure for invertible sheaves. The operation is the tensor product, the identity is the "trivial line bundle" $X \times \mathbb{C}$ and the inverse of a line bundle is its dual. It also follows, tautologically, that the maps H'' and $\mathcal{O}\{-\}$ are natural group isomorphisms.

Before reflecting on our results, we should wrap this discussion up with a description of the direct relationship between line bundles and divisors. In the interest of brevity, we will not prove any of the assertions that follow but merely reference where they can be obtained.

In order to define a map from $\mathrm{LB}(X)$ to $\mathrm{DCl}(X)$, we must start with a couple of definitions.

Definition 4.35. Let $\pi : L \to X$ be a line bundle over a compact Riemann surface X. A **rational section of L** is map $s : X \to L$ that is a regular section on an open set $U \subset X$ whose complement is finite such that for every local trivialization $\Phi_V : \pi^{-1}(V) \to \mathbb{C} \times V$ the composition

$$\mathrm{pr}_1 \circ \Phi_V \circ s\big|_V : V \to \mathbb{C}$$

is meromorphic.

As with meromorphic functions, given a rational section s of a line bundle L on X, we can define the order of s at a given point.

Definition 4.36. Let $\pi : L \to X$ be a line bundle over a compact Riemann surface and let $s : X \to L$ be a rational section. The **order of s at a point** $p \in X$, denoted by $\operatorname{ord}_p(s)$, is the order of the rational function $f = \operatorname{pr}_1 \circ \Phi_U \circ s$ where $\Phi_U : \pi^{-1}(U) \to \mathbb{C} \times U$ is a local trivialization of L in a neighborhood of p.

This is clearly well defined, since for another choice of local trivialization $\Psi_V : \pi^{-1}(V) \to \mathbb{C} \times V$ on another neighborhood V of p the resulting rational function f' is related to f by $f' = tf$ where t is the transition function. Since t is a nonvanishing holomorphic function on $U \cap V$, it follows that

$$\operatorname{ord}_p(f') = \operatorname{ord}_p(tf) = \operatorname{ord}_p(t) + \operatorname{ord}_p(f) = \operatorname{ord}_p(f).$$

All but finitely many $p \in X$ will have $\operatorname{ord}_p(s) = 0$ and therefore we can define a divisor by

$$\operatorname{div}(s) = \sum_{p \in X} \operatorname{ord}_p(s) \cdot p.$$

Finally, given any two rational sections s_1 and s_2 of a line bundle L, the transition functions give an equivalence $\operatorname{div}(s_1) \sim \operatorname{div}(s_2)$ modulo the subgroup of principal divisors. Hence in $\operatorname{DCl}(X)$, $[\operatorname{div}(s_1)] = [\operatorname{div}(s_2)]$. See Proposition 2.23 in Chapter XI of [Mir95] for the proof. Hence we have a well defined map

$$\operatorname{div} : \operatorname{LB}(X) \to \operatorname{DCl}(X).$$

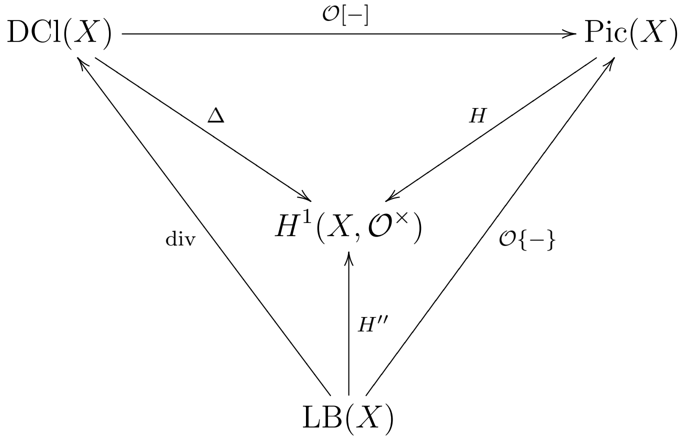
Finally, this map is a group isomorphism and we have

$$\Delta \circ \operatorname{div} = H''.$$

For the proof, see Proposition 3.11 of Chapter XI in [Mir95].

4.5 What Does This Mean?

We have now defined a series of natural isomorphisms allowing for several interpretations of the first sheaf cohomology group $H^1(X, \mathcal{O}^\times)$ for a compact Riemann surface. The isomorphisms can be organized into a commutative tetrahedron:



We can interpret the first cohomology group $H^1(X, \mathcal{O}^\times)$ as the Picard group, the group of invertible sheaves over the ringed space $(X, \mathcal{O})$. This group has a number-theoretic interpretation as the divisor class group of X and has a geometric interpretation as the group of line bundles over X. In such a way, the sheaf cohomology brings together number theory and geometry. In some literature, the divisor class group is called the Picard group, while sometimes the group of line bundles is called the Picard group.

As an example, one can show that for the Riemann sphere $\mathbb{CP}^1$, $H^1(\mathbb{CP}^1, \mathcal{O}^\times) \cong \mathbb{Z}$ (see the computation with a sheaf of differential forms in Example 4.0.3, Chapter III of [Har77] and its connection with $H^1(X, \mathcal{O}^\times)$ in Exercise 1.8, Chapter V of [Har77]). The generator for the group of line bundles is the tautological line bundle that assigns to each point $p \in \mathbb{CP}^1$ the complex line in $\mathbb{C}^2$ represented by p.

The equivalences above can be generalized ringed spaces $(X, \mathcal{O}_X)$. In particular, we can use this equivalence to **define** divisors or line bundles in a more abstract context.

The natural equivalences between $H^1(X, \mathcal{O}^\times)$, the group of invertible sheaves and the group of line bundles are manifestations of a deeper principle about sheaf cohomology.

Invertible sheaves are "locally trivial" in the sense that there is an open cover $\mathcal{U}$ of X over which the sheaf restricted to $U_i \in \mathcal{U}$ is isomorphic to the "trivial sheaf" restricted to U_i, where the "trivial sheaf" is the structure sheaf in the definition of a ringed space $(X, \mathcal{O})$. The automorphisms of a ring as a module over itself are given by multiplication by a unit. For an open set $U \subset X$, the units of $\mathcal{O}(U)$ are precisely $\mathcal{O}^\times(U)$. Hence on a ringed space $(X, \mathcal{O})$, one can define on X the sheaf of automorphisms of $\mathcal{O}$ which is isomorphic to the sheaf $\mathcal{O}^\times$.

Line bundles are "locally trivial" in the sense that there is an open cover $\mathcal{U}$ of X where the restriction to a subset $U_i \in \mathcal{U}$ is isomorphic to the "trivial bundle" $X \times \mathbb{C}$ restricted to U_i. A homomorphism of a trivial line bundle must have the form

$$\alpha(z, p) = (f(p) \cdot z, p)$$

where f is a holomorphic function. For this homomorphism to be an automorphism of the trivial line bundle, f must be nonvanishing. It follows that if we form a sheaf of automorphisms of the trivial line bundle, the sheaf will be isomorphic to the sheaf $\mathcal{O}^\times$.

In both of these cases, we had a collection of trivializations which naturally gave a 1-cocycle for the sheaf $\mathcal{O}^\times$ for a particular covering $\mathcal{U}$ of X. Changing the trivializations changed the cocycle by a coboundary. Passing to a finer open cover in the cohomology amounts to passing to a finer collection of trivializations. The limit group $H^1(X, \mathcal{O}^\times)$ then classifies isomorphism classes of both invertible sheaves and line bundles.

More generally, the first sheaf cohomology group can be used to classify "locally trivial" objects. This is accomplished as follows. Suppose X is a topological space, $F : \mathfrak{Top} \to \mathfrak{C}$ is a functor and $T = F(X)$ is "trivial" in some sense which in particular requires the ability to construct a sheaf of automorphisms $\mathrm{Aut}(T)$ on X. Moreover, suppose we have a set $\mathcal{X}$ of isomorphism classes of "locally trivial objects" in $\mathfrak{C}$. Such an object is "locally isomorphic" to T in the sense that we can define a restriction $S\big|_U$ for any open set $U \subset X$ and that there is an open cover $\mathcal{U} = \{U_i\}$ such that for every i, $S\big|_{U_i} \cong T\big|_{U_i} := F(U_i)$ in $\mathfrak{C}$. Then $\mathcal{X}$ is in bijective correspondence with $H^1(X, \mathrm{Aut}(T))$. For the proof, see Proposition 4.3 in Chapter XI of [Mir95]. This principle can be applied, for example, to vector bundles of rank k over a complex manifold (see Exercise 5.18 in Chapter II of [Har77]) or to $\mathbb{P}^n$ bundles over a scheme (see Exercise 7.10 in Chapter II of [Har77]).

Chapter 5

Concluding Remarks

Sheaf cohomology is a powerful generalization of de Rham cohomology. The first sheaf cohomology group unites number theory and geometry. This allows one to consider number theoretic or geometric notions in contexts where they would otherwise not be obvious. More generally, the first sheaf cohomology group can be used to classify properly defined "locally trivial" objects in some category.

This paper is a starting point for the investigation of sheaves and sheaf cohomology. One can proceed in several different directions for further study. There are several vanishing theorems that describe conditions under which $H^n(X, \mathscr{F}) = 0$ for $n > 0$. For example, there is a vanishing theorem for flasque sheaves. A flasque sheaf is a sheaf whose restriction maps are surjective. It follows that if there is some $n > 0$ such that $H^n(X, \mathscr{F}) \neq 0$ there is a cohomological obstruction to the ability to extend sections of $\mathscr{F}$. See Section 2, Chapter III of [Har77]. One can also describe Ext groups on sheaves. This leads to Serre duality, which says roughly that if X is a scheme with dimension n (using some appropriate notion of dimension), for certain types of sheaves $\mathscr{F}$ over X there is an isomorphism between the i^{th} Ext group of $\mathscr{F}$ (with respect to some "dualizing" sheaf ω) and the dual of $H^{n-i}(X, \mathscr{F})$ for every $0 \leq i \leq n$. See Sections 6 and 7 of Chapter III in [Har77] for the details. One can also proceed to add more structure to the Picard group. There is a canonical way in which the Picard group can be turned into a scheme. See [Mum66] for the construction and its applications.

All of this abstraction has brought together seemingly unrelated fields of mathematics and has broken many barriers. Sheaves, sheaf cohomology, and schemes have been used to solve formerly intractable problems, such as classification theorems for algebraic curves. The interested reader who would like to read more on the general theory should consult [Har77] or [Sha94]. If the reader is willing to specialize to Riemann surfaces and algebraic curves, [Mir95] is also a useful reference.

Bibliography

[BT82] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology.* Springer, 1982.

[EH00] David Eisenbud and Joe Harris. *The Geometry of Schemes.* Springer-Verlag, 2000.

[For81] Otto Forster. *Lectures on Riemann Surfaces.* Springer-Verlag, 1981.

[Har77] Robin Hartshorne. *Algebraic Geometry.* Springer, 1977.

[HS97] P.J. Hilton and U. Stammbach. *A Course in Homological Algebra.* Springer, second edition, 1997.

[Lee03] John M. Lee. *Introduction to Smooth Manifolds.* Springer, 2003.

[Mir95] Rick Miranda. *Algebraic Curves and Riemann Surfaces.* American Mathematical Society, 1995.

[Mum66] David Mumford. *Lectures on Curves on an Algebraic Surface.* Princeton University Press, 1966.

[Sha94] Igor R. Shafarevich. *Basic Algebraic Geometry*, volume 2. Springer-Verlag, second edition, 1994.

[Uen97] Kenji Ueno. *Algebraic Geometry 1: From Algebraic Varieties to Schemes.* Iwanami Shoten, 1997. Translated by Goro Kato.

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