Smooth Manifolds and Minkowski Spacetime

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3/2/11

Outline

Topology and Curves

Definition

to•pol•o•gy |tə'päləjē|
noun
the study of geometric properties and spatial relations unaffected by the continuous change of shape or size of figures.

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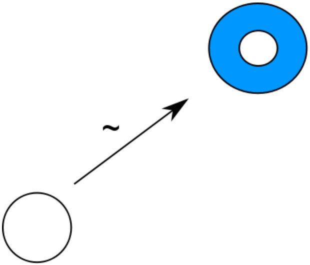
Remark

Actually, **topology** is the study of topological spaces (X, T) where X is a set and $T \supseteq \{\emptyset, X\}$ is a collection of subsets of X which is closed under arbitrary unions and finite intersections.

Types of Topological Equivalence

- Homotopy equivalent

  circle $\sim$ annulus



Types of Topological Equivalence

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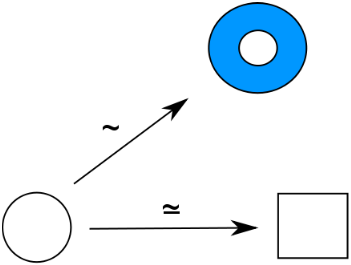
  circle $\sim$ annulus

- Homeomorphic

  circle $\simeq$ square



Types of Topological Equivalence

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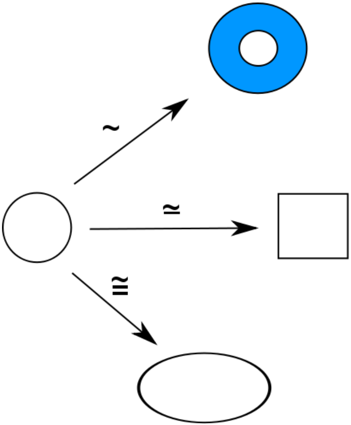
  circle $\sim$ annulus

- Homeomorphic

  circle $\simeq$ square

- Diffeomorphic

  circle $\cong$ ellipse



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- connected (all one piece)
- locally Euclidean (if we zoom in, things look flat)
- smooth (no sharp corners, so we can do calculus)

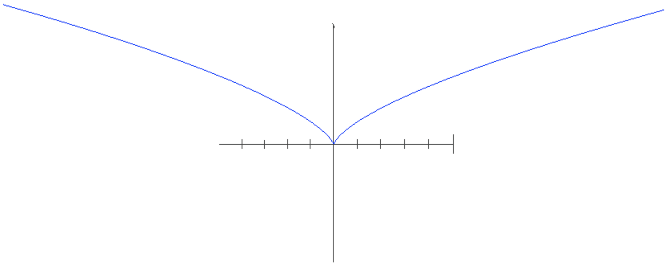


Figure: $y = x^{2/3}$ has a “cusp” at $(0, 0)$

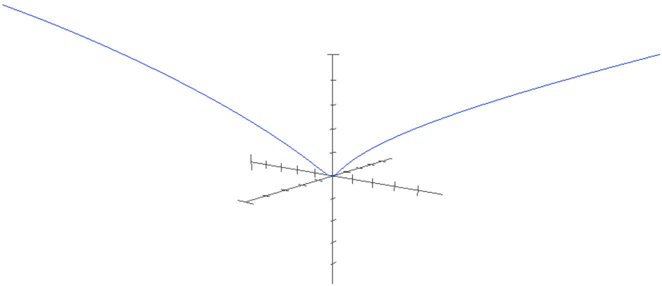
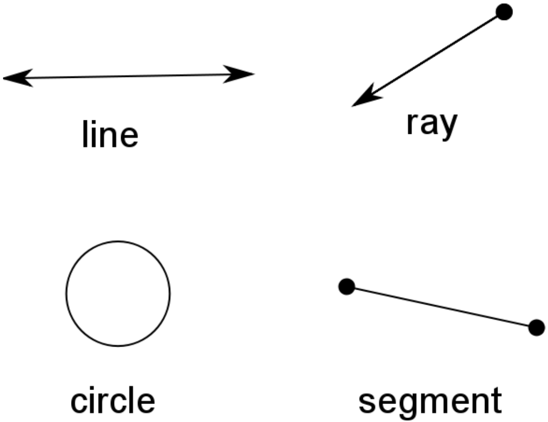
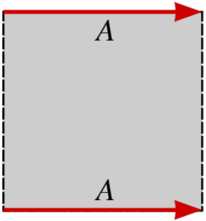


Figure: Maybe we're looking at a shadow of the twisted cubic

1-Dimensional Manifolds



Surfaces

Polygon	Picture	Boundary
		

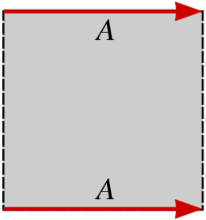
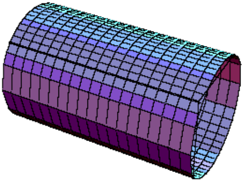
Polygon	Picture	Boundary
		

Table: Finite Cylinder C

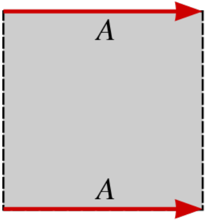
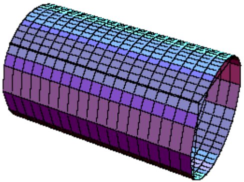
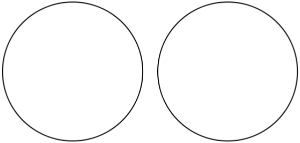
Polygon	Picture	Boundary
		

Table: Finite Cylinder C

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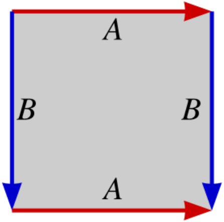		

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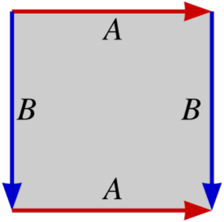
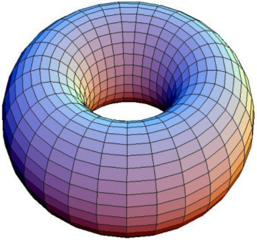
		

Table: Torus $\mathbb{T}^2$

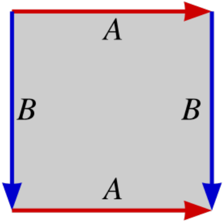
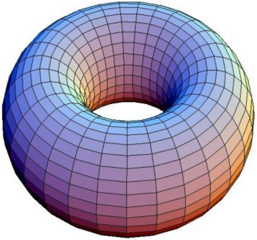
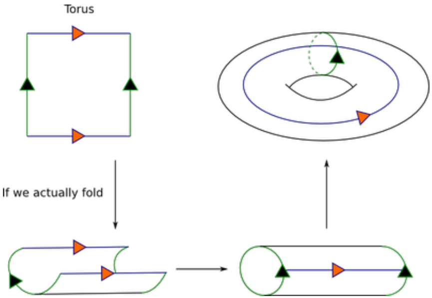
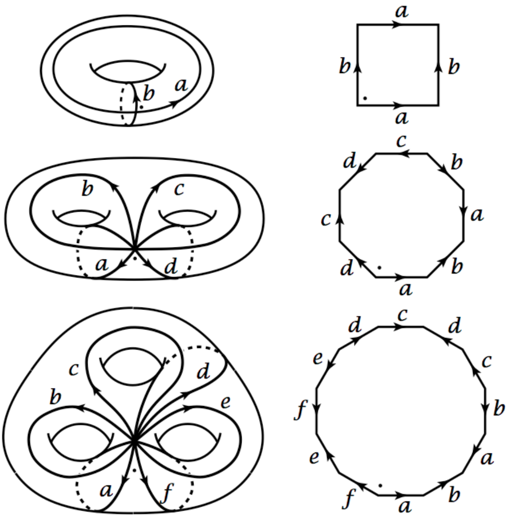
Polygon	Picture	Boundary
		$\emptyset$

Table: Torus $\mathbb{T}^2$

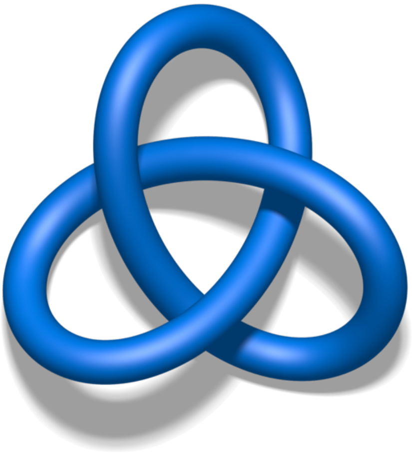
$C \rightsquigarrow \mathbb{T}^2$



Other Polygons Produce Multi-Holed Tori



A Knotted Torus is Still a Torus



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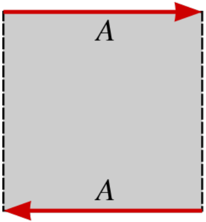		

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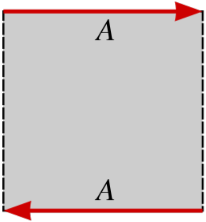	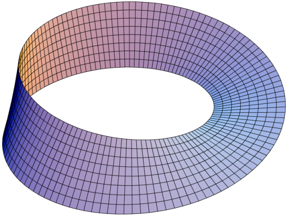	

Table: Möbius Strip M

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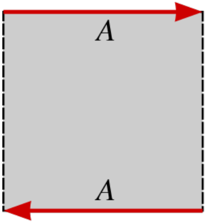	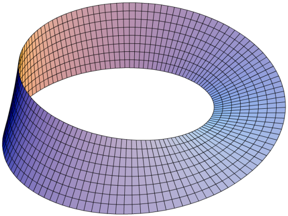	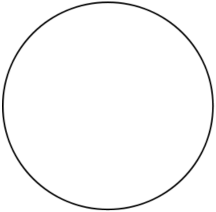

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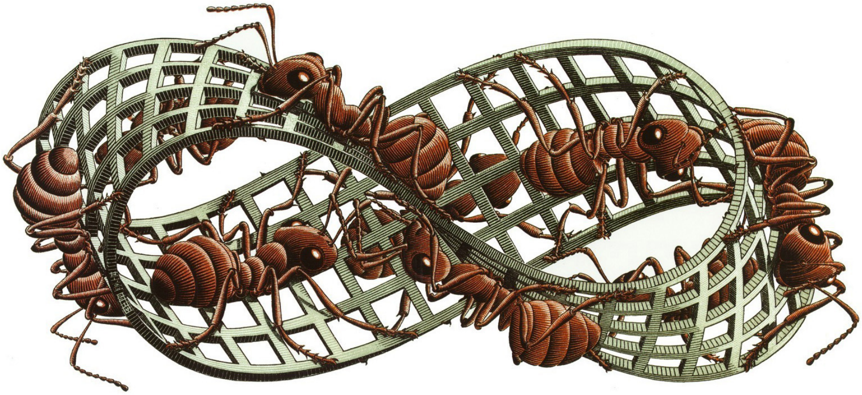
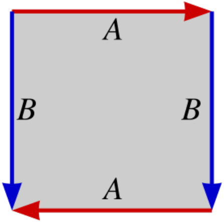


Figure: M. C. Escher, “Möbius Strip II”

Polygon	Picture	Boundary
		

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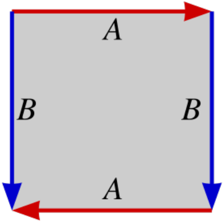	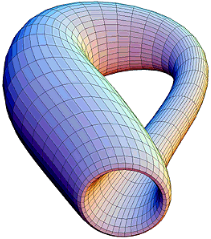	

Table: Klein Bottle K

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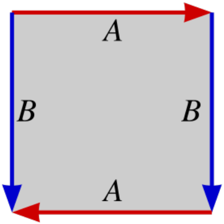	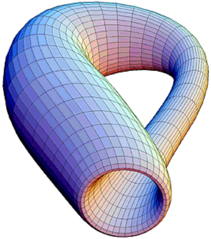	$\emptyset$

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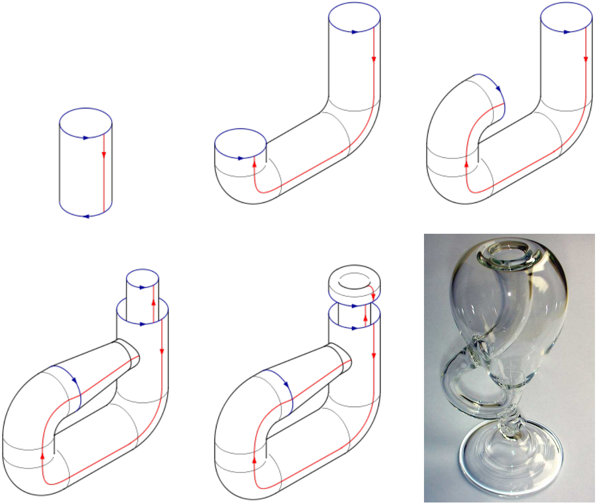

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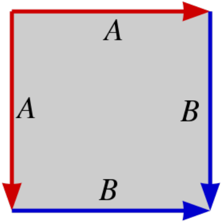		

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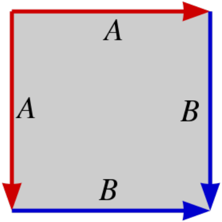	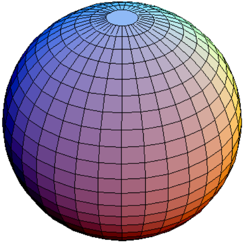	

Table: Sphere $\mathbb{S}^2$

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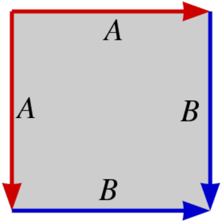	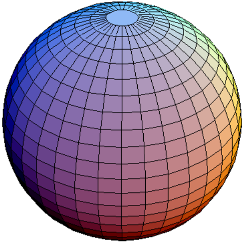	$\emptyset$

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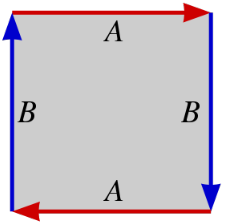		

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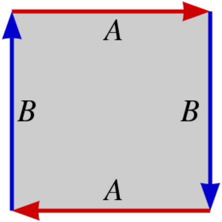
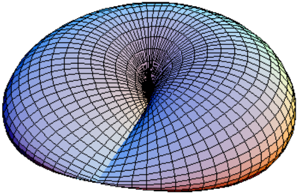
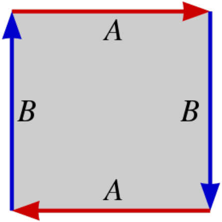
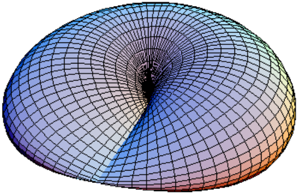
		

Table: Real Projective Plane $\mathbb{RP}^2$

Polygon	Picture	Boundary
		$\emptyset$

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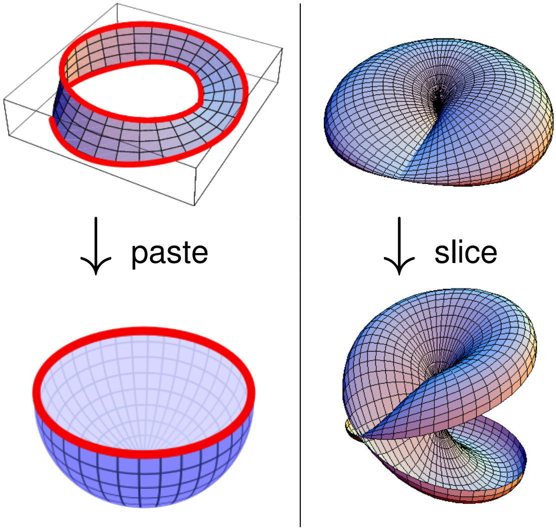
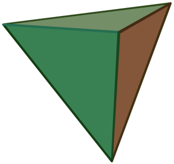
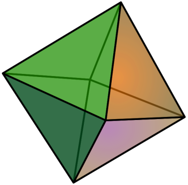

paste
slice

Spaces and the Schwarzschild Metric

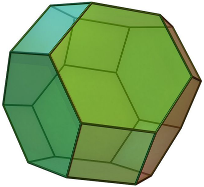
We can also take polyhedra (instead of polygons) and attach faces (instead of sides) to get 3-dimensional manifolds



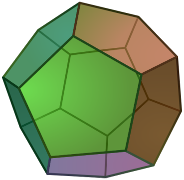
tetrahedron



octahedron



truncated oct.



dodecahedron

Table: Some Polyhedra

Minkowski Spacetime

We can indicate geometries with "line elements"

$$ds = \text{an infinitesimal element of length}$$

This allows us to do calculus: tangents, areas

Flat Spaces

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Flat Spaces

$$ds^2 = dx^2$$ linear

Minkowski Spacetime

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Flat Spaces

$ds^2 = dx^2$	linear
$ds^2 = dx^2 + dy^2$	planar

Minkowski Spacetime

We can indicate geometries with “line elements”

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Flat Spaces

$ds^2 = dx^2$	linear
$ds^2 = dx^2 + dy^2$	planar
$ds^2 = dx^2 + dy^2 + dz^2$	spatial

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Flat Spaces

$ds^2 = dx^2$	linear
$ds^2 = dx^2 + dy^2$	planar
$ds^2 = dx^2 + dy^2 + dz^2$	spatial
$ds^2 = dx^2 + dy^2 + dz^2 - dt^2$	spatial-temporal

$$= -dt^2 + dr^2 + r^2(d\theta^2 + sin^2(\theta)d\varphi^2)$$

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Einstein did this in 1915 via 10 “field equations.”

In the case of a non-charged, non-rotating, uniformly dense, spherical body of mass *M* we get...

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2(\theta)d\varphi^2)$$

Embedding diagrams help us visualize this geometry

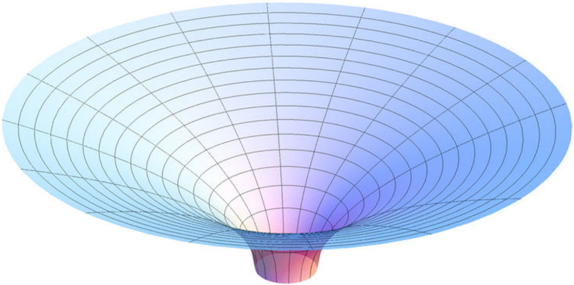

Figure: Flamm's Paraboloid

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Figure: An artist's interpretation of a black hole