

MATH 5B PROBLEM SET #1

Problem #1: A methane molecule consists of four hydrogen atoms bonded to a central carbon atom. Electrostatic forces require that the hydrogen atoms be as far apart from each other as possible (and all the bond-lengths are the same) so they arrange themselves in space as the vertices of a regular tetrahedron with the carbon atom in the center.

- Find the bond angle for a methane molecule.
- Find the angle between two adjacent edges of the tetrahedron.
- Find the area of a face of the tetrahedron (assume edges have unit length).

Hint: You can find four vertices of a cube that form a regular tetrahedron. Another way to assign coordinates to the vertices of a tetrahedron is to view it as sitting in \mathbb{R}^4 . Then, the vertices are just the unit vectors along the four coordinate axes.

Problem #2: Let T be a tetrahedron (not necessarily regular). The *face vector* for a face F of T is the outward pointing normal vector to F with length equal to the area of F .

- Show that the sum of all the face vectors of T is zero.
- Use part (a) to show that the sum of the face vectors for any polytope is zero.

Hint: As a warm-up, you could show analogous results in the plane for triangles and polygons and their “side vectors.”

Problem #3 Consider a triangle in the plane with vertices P_1 , P_2 , and P_3 and let P be a fourth point. Let v_i be the vector from P_i to P . Find the condition (in terms of v_1 , v_2 , v_3 and the dot product) that P lies on the altitude of the triangle through P_1 .

Problem #4: Orthogonal Matrices. A matrix A is *orthogonal* if it satisfies $A^T A = I$ (equivalently $AA^T = I$ or $A^T = A^{-1}$).

- Show that if A is an orthogonal matrix then the column vectors of A are all orthogonal and unit length. Geometrically speaking, what does this tell you about A ? Convince yourself that there are only two types of orthogonal matrices: rotations and reflections. What is the determinant of an orthogonal matrix?
- In 2-dimensions, rotation matrices are of the form

$$A_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Convince yourself that this is the case by thinking about what a rotation matrix should do to the standard basis. Verify that A_θ is indeed orthogonal. Also show that $A_{-\theta} = A_\theta^{-1}$ and explain why this makes sense geometrically.

- Use addition formulas for sine and cosine to show that $A_{\theta_1} A_{\theta_2} = A_{\theta_1 + \theta_2}$. What does this tell you about rotation matrices?
- There are four 2×2 orthogonal matrices with first entry $a_{11} = \frac{1}{\sqrt{2}}$. Find them. Think about what they do to the plane by drawing what they do to the letter F.