

MATH 5B PROBLEM SET #3

Problem #1: A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is *homogeneous of degree d* if for each $\mathbf{x} = (x_1, \dots, x_n)$ and for every $k \in \mathbb{R}$, we have

$$f(kx_1, kx_2, \dots, kx_n) = k^d f(x_1, x_2, \dots, x_n)$$

- (a) Which of the following is homogeneous?
- $f(x, y) = x^3 + xy^2 - 7y^3$
 - $g(x, y, z) = x^2yz + 2xy^3 - yz^5$
 - $h(x, y) = e^{x/y}$
 - $i(x, y, z) = \frac{x^3 + x^2z - 2yz^2}{xyz - y^3}$
- (b) If $f(x_1, \dots, x_n)$ is a polynomial, explain (with words) what it means for f to be homogeneous of degree d .
- (c) Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is homogeneous of degree d . Prove *Euler's formula*:

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \dots + x_n \frac{\partial f}{\partial x_n} = df$$

Hint: Start with the equation that defines what it means for f to be homogeneous and differentiate.

Problem #2: Let S be the surface in \mathbb{R}^3 defined by the equation $x^4 - 3x^2y^2 + 2y^2z^2 - z^4 = -1$. The tangent plane to S at the point $(1, 1, 1)$ intersects the x -axis at point P . Find the coordinates of P .

Problem #3: Consider a triangle inscribed in the unit circle in the plane, with one vertex at $(1, 0)$ and the two other vertices given by (counterclockwise) polar angles θ_1 and θ_2 in that order, so $0 \leq \theta_1 \leq \theta_2 \leq 2\pi$.

- (a) Find a function $A(\theta_1, \theta_2)$ for the area of the triangle in terms of the angles θ_1 and θ_2 . Sketch the domain of A in the $\theta_1\theta_2$ -plane.
- (b) Find the critical points of A in the interior of the domain.
- (c) Find any boundary critical points of A . What are the minimum and maximum values for A ?
- (d) Use the second derivative test to confirm the nature of the critical points you found in part (b).

Problem #4: Find the dimensions of the largest rectangular box that can be inscribed in the ellipsoid $x^2 + 2y^2 + 4z^2 = 12$. Assume the faces of the box are parallel to the coordinate planes.

Problem #5: Consider the function $f(x_1, \dots, x_n) = e^{-x_1^2 - x_2^2 - \dots - x_n^2}$. Find the unique critical point of f and use the second derivative test to determine its type. Use level sets to graph f for $n = 2$.