How non-crossing partitions occur in free probability theory

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This is one of the survey talks at the workshop on "Braid groups, clusters, and free probability" at the American Institute of Mathematics, January 10-14, 2005. The goal of the talk is to give a quick and elementary introduction to some combinatorial aspects of free probability, addressed to an audience who is not familiar with free probability but is acquainted to lattices of non-crossing partitions. The (attempted) line of approach is to select a cross-section of relevant material, and increase its rate of absorption by working through a set of exercises (the exercises are embedded in the lecture, and are straightforward applications of the material presented). While this is hopefully an efficient approach, it is quite far from being comprehensive; for a more detailed treatment of the subject, one can check the references [3, 5, 8, 9] listed at the end of the lecture. (Among these references, the one which is closest to the spirit of the present lecture is [3] – but, of course, there is a lot of material from [3] that could not be even mentioned here.)

The plan of the lecture is as follows. The first section reviews a minimal amount of general terminology, and then the Sections 2 and 3 present some basic tools used in the combinatorics of free probability – the non-crossing cumulants and the R-transform. The main Section 4 is devoted to explaining how convolution in lattices of non-crossing partitions is intimately related to the multiplication of free random variables. This connection can in principle be used both ways, but (to my knowledge) its applications up to present have been focused on the direction of doing computations with free random variables via the combinatorics of $NC(n)$. We give some illustrative examples of how this is done, and we explain how the S-transform of Voiculescu (a device which is also used in connection to the multiplication of free random variables) fits in this picture.

1. Basic free probabilistic terminology

The framework used in free probability is the one of a *non-commutative probability space*. We will work with a version of this concept which goes under the name of "$*$-probability space".

1.1 Definition. By a **$*$-probability space** we will understand a system $(\mathcal{A}, \varphi)$ where:

• $\mathcal{A}$ is a unital algebra over $\mathbb{C}$.

• $\mathcal{A}$ is a endowed with a $*$-operation (an antilinear map $\mathcal{A} \ni a \mapsto a^* \in \mathcal{A}$ such that $(a^*)^* = a$ and such that $(ab)^* = b^* a^*$, for $a, b \in \mathcal{A}$).

• $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional which is *normalized* by the condition that $\varphi(1_{\mathcal{A}}) = 1$ (where $1_{\mathcal{A}}$ is the unit of $\mathcal{A}$), and is *strictly positive definite* in the sense that $\varphi(a^* a) \geq 0$ for all $a \in \mathcal{A}$, with equality holding if and only if $a = 0$.

1.2 Remarks *(and disclaimers).*

(a) A more suggestive name for an $(\mathcal{A}, \varphi)$ as above would probably be "generalized algebra of integrable functions" – the elements of $\mathcal{A}$ are viewed as some sort of integrable functions, and applying φ to an $a \in \mathcal{A}$ is viewed like an operation of taking an integral. This comment is just made to explain the terminology, we will not encounter the functional analysis side of free probability in the present lecture.

(b) There are various alternative "flavours" for the definition of a non-commutative probability space. For instance one sometimes requires that φ is a **trace** – i.e. that $\varphi(ab) = \varphi(ba)$ for all $a, b \in \mathcal{A}$. On the other hand one sometimes drops the non-degeneracy assumption that $\varphi(a^*a) = 0$ implies $a = 0$, or renounces to the $*$-operation and to the positivity of φ altogether (but in this lecture we will stick to the definition as given above).

(c) Another useful property of φ which could have been mentioned in Definition 1.1 is that

$$\varphi(a^*) = \overline{\varphi(a)}, \quad \forall \ a \in \mathcal{A}. \tag{1.1}$$

The reason for not mentioning (1.1) explicitly in the definition is that it follows from the positivity of φ. Indeed, it is easily seen that (1.1) is equivalent to the fact that $\varphi(a) \in \mathbb{R}$ whenever $a \in \mathcal{A}$ is such that $a = a^*$, and the latter fact follows from the positivity by writing

$$a = \frac{1}{4}\Big[\ (a + 1_{\mathcal{A}})^2 - (a - 1_{\mathcal{A}})^2 \ \Big].$$

1.3 Examples.

(a) One can take $\mathcal{A}$ to be the algebra of polynomials $\mathbb{C}[t]$, and then let $\varphi : \mathcal{A} \to \mathbb{C}$ be defined by $\varphi(f) = \int_0^1 f(t) \ dt$, for $f \in \mathcal{A}$. This is an example where the elements of $\mathcal{A}$ really are functions, and applying φ really amounts to taking an integral. (Of course, in this example one could replace $\mathbb{C}[t]$ by some larger algebra of integrable functions on $[0, 1]$.)

(b) One can take $\mathcal{A}$ to be the algebra $\mathcal{M}_n(\mathbb{C})$ of $n \times n$ matrices, and then let $\varphi : \mathcal{A} \to \mathbb{C}$ be the normalized trace,

$$\varphi(A) = \frac{1}{n}\mathrm{Tr}(A) = \frac{1}{n}\sum_{i=1}^{n} \alpha_{i,i}, \ \text{ for } A = [\alpha_{i,j}]_{i,j=1}^{n} \in \mathcal{A}.$$

(c) Let G be a group (not necessarily finite). One can take $\mathcal{A}$ to be the associated group algebra,

$$\mathcal{A} = \mathbb{C}[G] = \mathrm{span}\{\chi_g \mid g \in G\},$$

where the elements of the linear basis $\{\chi_g \mid g \in G\}$ for $\mathcal{A}$ are multiplied according to the rule $\chi_g \cdot \chi_h = \chi_{gh}$, and the $*$-operation is determined by the condition $\chi_g^* = \chi_{g^{-1}}$, $g \in G$. Then one can take $\varphi : \mathcal{A} \to \mathbb{C}$ to be the so-called *canonical trace* on $\mathbb{C}[G]$, i.e. to be the linear functional determined by the fact that

$$\varphi(\chi_g) \ = \ \begin{cases} 1, & \text{if } a = e, \text{ the unit of } G, \\ 0, & \text{otherwise.} \end{cases}$$

1.4 Remark *(how to think of free independence).* Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $\mathcal{A}_1, \ldots, \mathcal{A}_s$ be unital $*$-subalgebras of $\mathcal{A}$ (where by "$*$-subalgebra" we understand a subalgebra of $\mathcal{A}$ which is also closed under the $*$-operation). Consider the algebra (equivalently $*$-algebra) generated by $\mathcal{A}_1, \ldots, \mathcal{A}_s$ together,

$$\mathcal{U} := \mathrm{Alg}(\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_s) \subseteq \mathcal{A}.$$

Then $\mathcal{U}$ is spanned linearly by products of elements from $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_s$; since φ is not assumed to be (and typically is not) multiplicative, the knowledge of the restrictions $\varphi|\mathcal{A}_1, \ldots, \varphi|\mathcal{A}_s$ isn't usually sufficient for determining what is $\varphi|\mathcal{U}$. Well, *free independence is a recipe* which *is* allowing us to compute $\varphi|\mathcal{U}$ – if we know $\varphi|\mathcal{A}_1, \ldots, \varphi|\mathcal{A}_s$, and we know that $\mathcal{A}_1, \ldots, \mathcal{A}_s$ are freely independent.

So once again, for the purpose of this lecture, the free independence of $\mathcal{A}_1, \ldots, \mathcal{A}_s$ is a recipe which allows us to compute any value

$$\varphi(a_1 a_2 \cdots a_n), \text{ with } n \geq 1 \text{ and } a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n} \text{ (and where } 1 \leq i_1, \ldots i_n \leq s),$$

in terms of the restrictions $\varphi|\mathcal{A}_1, \ldots, \varphi|\mathcal{A}_s$. The recipe is usually stated in a special case, to which the computations can always be reduced. More precisely, we have:

1.5 Definition. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $\mathcal{A}_1, \ldots, \mathcal{A}_s$ be unital $*$-subalgebras of $\mathcal{A}$. We say that $\mathcal{A}_1, \ldots, \mathcal{A}_s$ are **freely independent** if the following implication holds:

$$\left.\begin{array}{l} 1 \leq i_1, \ldots, i_n \leq s \\ \text{such that } i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n \\ \\ a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n} \\ \text{such that } \varphi(a_1) = \cdots = \varphi(a_n) = 0 \end{array}\right\} \Rightarrow \varphi(a_1 \cdots a_n) = 0. \tag{1.2}$$

In order to see how the Equation (1.2) can be bootstrapped to a recipe which computes *any* value $\varphi(a_1 \cdots a_n)$, it is probably best to work out explicitly a few monomials of small length.

Exercise 1. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $\mathcal{B}, \mathcal{C}$ be two unital $*$-subalgebras of $\mathcal{A}$ which are freely independent. Verify that:

(a) $\varphi(bc) = \varphi(b) \cdot \varphi(c), \ \forall \ b \in \mathcal{B}, \ c \in \mathcal{C}$.

(b) $\varphi(b_1 c b_2) = \varphi(b_1 b_2) \cdot \varphi(c), \ \forall \ b_1, b_2 \in \mathcal{B}, \ c \in \mathcal{C}$.

(c) $$\begin{aligned} \varphi(b_1 c_1 b_2 c_2) = \ & \varphi(b_1 b_2) \cdot \varphi(c_1) \cdot \varphi(c_2) + \varphi(b_1) \cdot \varphi(b_2) \cdot \varphi(c_1 c_2) \\ & - \varphi(b_1) \cdot \varphi(b_2) \cdot \varphi(c_1) \cdot \varphi(c_2), \quad \forall \ b_1, b_2 \in \mathcal{B}, \ c_1, c_2 \in \mathcal{C}. \end{aligned}$$

[Comment: A less tedious derivation for the expression in the Exercise 1(c) appears in Section 2, Exercise 6.]

For practicing the concept of free independence, here are a couple more exercises.

Exercise 2. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space.

(a) Verify that the unital $*$-subalgebra $\mathbb{C}1_{\mathcal{A}}$ of $\mathcal{A}$ is freely independent from any unital $*$-subalgebra $\mathcal{B} \subseteq \mathcal{A}$.

(b) Let $\mathcal{B}$ be a unital $*$-subalgebra of $\mathcal{A}$, such that $\mathcal{B}$ is freely independent from itself. Prove that $\mathcal{B} = \mathbb{C}1_{\mathcal{A}}$.

(c) Let $\mathcal{B}$ and $\mathcal{C}$ be unital $*$-subalgebras of $\mathcal{A}$ such that $\mathcal{B}$ is freely independent from $\mathcal{C}$, and such that $\mathcal{B}$ commutes with $\mathcal{C}$ (i.e. $bc = cb$, for every $b \in \mathcal{B}$ and $c \in \mathcal{C}$). Prove that at least one of the algebras $\mathcal{B}$, $\mathcal{C}$ is equal to $\mathbb{C}1_{\mathcal{A}}$.

[Comments: The non-degeneracy of φ is important for the parts (b) and (c) of this exercise – indeed, when we aim to show an equality $x = y$ in $\mathcal{A}$ it may be more convenient to do that by checking that $\varphi(\,(x-y)^*(x-y)\,) = 0$. Note that the Exercise 2(b) implies in particular that if the unital $*$-subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_s$ of $\mathcal{A}$ are freely independent, then $\mathcal{A}_i \cap \mathcal{A}_j = \mathbb{C}1_{\mathcal{A}}$ for every $1 \leq i < j \leq s$.]

Exercise 3. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $\mathcal{A}_1, \ldots, \mathcal{A}_s$ be unital $*$-subalgebras of $\mathcal{A}$ which are freely independent. Let $\mathcal{U}$ be the subalgebra of $\mathcal{A}$ generated by $\mathcal{A}_1, \ldots, \mathcal{A}_s$ together. Prove that if $\varphi|\mathcal{A}_i$ is a trace for every $1 \leq i \leq s$, then $\varphi|\mathcal{U}$ is a trace as well.

2. Using non-crossing cumulants to describe free independence

2.1 Remark *(how to think of non-crossing cumulants).* Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. The non-crossing cumulants of $(\mathcal{A}, \varphi)$ are a family of multilinear functionals $\kappa_n : \mathcal{A}^n \to \mathbb{C}$, $n \geq 1$. Their role is of "straightening" the formulas which we use to retrieve the values of φ on the algebra generated by $\mathcal{A}_1 \cup \cdots \cup \mathcal{A}_s$, when $\mathcal{A}_1, \ldots, \mathcal{A}_s \subseteq \mathcal{A}$ are freely independent (in the sense discussed in Section 1).

The first few of the functionals κ_n look like this:

$$\begin{cases} \kappa_1(a) = \varphi(a), \quad a \in \mathcal{A} \\ \kappa_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2), \quad a_1, a_2 \in \mathcal{A} \\ \kappa_3(a_1, a_2, a_3) = \varphi(a_1 a_2 a_3) - \varphi(a_1)\varphi(a_2 a_3) - \varphi(a_1 a_3)\varphi(a_2) \\ \qquad -\varphi(a_1 a_2)\varphi(a_3) + 2\varphi(a_1)\varphi(a_2)\varphi(a_3), \quad a_1, a_2, a_3 \in \mathcal{A}. \end{cases} \tag{2.1}$$

It is in fact nicer to re-write the above equations by expressing $\varphi(a), \varphi(a_1 a_2), \varphi(a_1 a_2 a_3)$ in terms of the functionals κ_n, because then we just have some plain sumations:

$$\begin{cases} \varphi(a) &= \kappa_1(a), \quad a \in \mathcal{A} \\ \varphi(a_1a_2) &= \kappa_2(a_1,a_2) + \kappa_1(a_1)\kappa_1(a_2), \quad a_1,a_2 \in \mathcal{A} \\ \varphi(a_1a_2a_3) &= \kappa_3(a_1,a_2,a_3) + \kappa_1(a_1)\kappa_2(a_2,a_3) + \kappa_2(a_1,a_3)\kappa_1(a_2) \\ & \quad +\kappa_2(a_1,a_2)\kappa_1(a_3) + \kappa_1(a_1)\kappa_1(a_2)\kappa_1(a_3), \quad a_1,a_2,a_3 \in \mathcal{A}. \end{cases} \tag{2.2}$$

Moreover, the way to read the right-hand sides of the Equations (2.2) is as summations over the lattices $NC(1)$, $NC(2)$, and respectively $NC(3)$. For instance in the last one of these equations, the first term on the right-hand side is indexed by the non-crossing partition 1_3 with only one block, the term coming after it is indexed by $\{ \{1\},\{2,3\} \}$, and so on until the last term, which is indexed by the partition $0_3 = \{ \{1\},\{2\},\{3\} \}$. The general rule for forming the term "$term_\pi$" indexed by $\pi = \{A_1,\ldots,A_k\} \in NC(n)$ in a summation like in (2.2) (which has $\varphi(a_1 \cdots a_n)$ on the left-hand side) is thus:

$$term_\pi = \prod_{j=1}^{k} \kappa_{|A_j|}((a_1,\ldots,a_n)|A_j),$$

where "$(a_1,\ldots,a_n)|A$" designates the $|A|$-tuple obtained by only looking at a_i's with $i \in A$ (e.g. if $n=6$ and $A=\{1,3,4\}$ then $(a_1,\ldots,a_6)|A = (a_1,a_3,a_4)$).

So, for the record, the precise definition of the cumulant functionals κ_n goes as follows:

2.2 Definition (and Proposition). Let $(\mathcal{A},\varphi)$ be a $*$-probability space. There exists a family of multilinear functionals $\kappa_n : \mathcal{A}^n \to \mathbb{C}$, $n \geq 1$, uniquely determined by the following formula:

$$\varphi(a_1 \cdots a_n) = \sum_{\substack{\pi \in NC(n) \\ \pi =: \{A_1,\ldots,A_k\}}} \prod_{j=1}^{k} \kappa_{|A_j|}((a_1,\ldots,a_n)|A_j), \tag{2.3}$$

for every $n \geq 1$ and every $a_1,\ldots,a_n \in \mathcal{A}$. The functionals κ_n are called the **non-crossing cumulant functionals** of the space $(\mathcal{A},\varphi)$.

[The "proposition" part in 2.2 is immediate, its proof consists in observing that the Equation (2.3) is actually of the form $\varphi(a_1 \cdots a_n) = \kappa_n(a_1,\ldots,a_n)$ + a sum of products of k_m's with $m < n$.]

Of course, one can also write a general fomula which generalizes the Equations (2.1). This is obtained from Equation (2.3) by using the Moebius function on the lattice $NC(n)$, and is stated in the next exercise.

The part (b) of Exercise 4 mentions the Kreweras complementation map $\mathrm{Kr} : NC(n) \to NC(n)$. This is defined geometrically exactly as in the Exercise 2(h) of the "warm-up" set of Vic Reiner, with the difference that the new points $1',\ldots,n'$ are placed to the other side of $1, \ldots, n$, respectively (i.e. $1'$ is the midpoint of the edge from 1 to $2, \ldots, n'$ is the midpoint of the edge from n to 1). Or in other words, we have

$$\mathrm{Kr} \ = \ \psi^{-1},$$

with $\psi : NC(n) \to NC(n)$ as in the Exercise 2(h) of the warm-up set.

Exercise 4. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, let n be a positive integer, and let $a_1, \ldots, a_n$ be in $\mathcal{A}$. For every non-empty subset $A = \{a_1 < a_2 < \cdots < a_m\}$ of $[n]$ we denote $\varphi(a_{i_1} \cdots a_{i_m}) =: \varphi_A \in \mathbb{C}$.

(a) Prove that:

$$\kappa_n(a_1, \ldots, a_n) \ = \ \sum_{\substack{\pi \in NC(n) \\ \pi =: \{A_1, \ldots, A_k\}}} \alpha(\pi) \cdot \varphi_{A_1} \cdots \varphi_{A_k}, \tag{2.4}$$

where $\{\alpha(\pi) \mid \pi \in NC(n)\}$ are some universal constants (not depending on $a_1, \ldots, a_n$, or on the space $(\mathcal{A}, \varphi)$ that we started with).

(b) Prove that the universal constants $\alpha(\pi)$ from the part (a) of the exercise are described as follows: Let π be in $NC(n)$, and consider the Kreweras complement of π, $\mathrm{Kr}(\pi) =: \{B_1, \ldots, B_l\}$. Then we have

$$\alpha(\pi) = s_{|B_1|} \cdots s_{|B_l|},$$

where $(s_n)_{n=1}^{\infty}$ is the sequence of signed Catalan numbers, $s_n = (-1)^{n+1}(2n-2)!/(\,(n-1)!n!\,)$ for $n \geq 1$.

Exercise 5. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. Let $a_1, \ldots, a_n$ be elements of $\mathcal{A}$, where $n \geq 2$, and suppose that at least one of $a_1, \ldots, a_n$ is a scalar multiple of $1_{\mathcal{A}}$. Prove that $\kappa_n(a_1, \ldots, a_n) = 0$.

[Comment: This exercise becomes trivial if instead of working directly from the definitions one invokes the Theorem 2.4 below. On the other hand, the proof (or at least one of the possible proofs) of Theorem 2.4 makes use of the statement of this exercise.]

The main point about non-crossing cumulant functionals is the Theorem 2.4 below, due to Speicher [4], which describes free independence in terms of cumulants. The relevant fact to look for turns out to be *the vanishing of mixed cumulants*, as reviewed in the next definition.

2.3 Definition. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space and let $\mathcal{A}_1, \ldots, \mathcal{A}_s$ be a family of unital $*$-subalgebras of $\mathcal{A}$. We will say that $\mathcal{A}_1, \ldots, \mathcal{A}_s$ have **vanishing mixed (non-crossing) cumulants** if the following happens:

$$\left. \begin{array}{c} 1 \leq i_1, \ldots i_n, \leq s, \text{ such that} \\ \text{not all of } i_1, \ldots, i_n \text{ are equal to each other} \\ \\ a_1 \in \mathcal{A}_{i_1}, \ldots, a_n \in \mathcal{A}_{i_n} \end{array} \right\} \Rightarrow \ \kappa_n(a_1, \ldots, a_n) = 0. \tag{2.5}$$

[Comment: Note that Definition 2.3 does not require the condition $i_1 \neq i_2, \ldots, i_{n-1} \neq i_n$ among the hypotheses of the implication (2.5), nor is there any requirement on the values of $\varphi(a_m)$, $1 \leq m \leq n$.]

2.4 Theorem. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space and let $\mathcal{A}_1, \ldots, \mathcal{A}_s$ be a family of unital $*$-subalgebras of $\mathcal{A}$. Then $\mathcal{A}_1, \ldots, \mathcal{A}_s$ are freely independent if and only if they have vanishing mixed cumulants.

The upshot of Theorem 2.4 is thus: let $(\mathcal{A}, \varphi)$ be a $*$-probability space and let $\mathcal{A}_1, \ldots, \mathcal{A}_s$ be a family of unital $*$-subalgebras of $\mathcal{A}$, which are freely independent. Let $\mathcal{U} \subseteq \mathcal{A}$ be the subalgebra generated by $\mathcal{A}_1, \ldots, \mathcal{A}_s$ together. Then (as discussed in Section 1) $\varphi|\mathcal{U}$ can be recaptured if we know the restrictions $\varphi|\mathcal{A}_1, \ldots, \varphi|\mathcal{A}_s$. The best way to do this isn't, however, via the route suggested in Section 1 (and started in Exercise 1), but goes via non-crossing cumulants – we look at the non-crossing cumulant functionals restricted to $\mathcal{A}_1, \ldots, \mathcal{A}_s$, then we "fill in" all the possible mixed cumulants of these algebras as being equal to 0, and finally we go back to computing moments via the Equation (2.3). For someone who wants to practice such a computation, here is a concrete one in length $n = 4$.

Exercise 6. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $\mathcal{B}, \mathcal{C}$ be two unital $*$-subalgebras of $\mathcal{A}$, which are freely independent. Verify that for $b_1, b_2 \in \mathcal{B}$ and $c_1, c_2 \in \mathcal{C}$ we have

$$\begin{aligned} \varphi(b_1 c_1 b_2 c_2) = \ & \kappa_2(b_1, b_2)\kappa_1(c_1)\kappa_1(c_2) + \kappa_1(b_1)\kappa_1(b_2)\kappa_2(c_1, c_2) \\ & + \kappa_1(b_1)\kappa_1(b_2)\kappa_1(c_1)\kappa_1(c_2), \end{aligned}$$

and use this to give a shorter solution to the part (c) of Exercise 1.

3. The R-transform, and the addition of free elements

3.1 Definition *(moments and moment series).* Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. It is customary (in line with the probabilistic origin of the terminology we use) to refer to the elements of $\mathcal{A}$ as **random variables**. In the same vein it is customary that, for a given random variable $a \in \mathcal{A}$, we refer to the number $\varphi(a^n)$ by calling it the **moment** of order n of a. The generating series for the moments of a,

$$M_a(z) := \sum_{n=1}^{\infty} \varphi(a^n) z^n, \tag{3.1}$$

is called the **moment series** of a.

This terminology extends to the situation when we deal with an s-tuple $a_1, \ldots, a_s$ of elements of $\mathcal{A}$. The family

$$\{\varphi(a_{i_1} \cdots a_{i_n}) \mid n \geq 1,\ 1 \leq i_1, \ldots, i_n \leq s\}$$

is called the family of **joint moments** of $a_1, \ldots, a_s$. These joint moments are the coefficients of a formal power series in s non-commuting indeterminates $z_1, \ldots, z_s$, which will be denoted by $M_{a_1,\ldots,a_s}$ and will be called the **joint moment series** of $a_1, \ldots, a_s$:

$$M_{a_1,\ldots,a_s}(z_1, \ldots, z_s) \ := \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=1}^{s} \varphi(a_{i_1} \cdots a_{i_n}) z_{i_1} \cdots z_{i_n}. \tag{3.2}$$

3.2 Definition *(the R-transform).* Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $\kappa_n : \mathcal{A}^n \to \mathbb{C}$, $n \geq 1$, be the non-crossing cumulant functionals for this space. On the other hand, let $a_1, \ldots, a_s$ be an s-tuple of elements of $\mathcal{A}$. The family

$$\{\kappa_n(a_{i_1}, \ldots, a_{i_n}) \mid n \geq 1, \ 1 \leq i_1, \ldots, i_n \leq s\}$$

is called the family of **joint (non-crossing) cumulants** of $a_1, \ldots, a_s$. The series in $\mathbb{C}\langle\langle z_1, \ldots, z_s \rangle\rangle$ which has these numbers as coefficients is called the **joint R-transform** of $a_1, \ldots, a_s$, and will be denoted by $R_{a_1,\ldots,a_s}$:

$$R_{a_1,\ldots,a_s}(z_1, \ldots, z_s) \ := \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=1}^{s} \kappa_n(a_{i_1}, \ldots, a_{i_n}) z_{i_1} \cdots z_{i_n}. \tag{3.3}$$

In particular, if a is an element of $\mathcal{A}$, then the generating series

$$R_a(z) := \sum_{n=1}^{\infty} \kappa_n(\underbrace{a, a, \ldots, a}_{n \ times}) z^n, \tag{3.4}$$

is called the **R-transform** of a.

3.3 Remark *(free independence for random variables, and the R-transform).* Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $a_1, \ldots, a_s$ be a family of elements of $\mathcal{A}$. For the ease of the discussion we will assume that the a_i's are selfadjoint ($a_i = a_i^*$, $1 \leq i \leq s$).

For every $1 \leq i \leq s$, let $\mathcal{P}_i$ denote the unital subalgebra (equivalently, $*$-subalgebra) of $\mathcal{A}$ generated by a_i; thus $\mathcal{P}_i$ consists of the elements of $\mathcal{A}$ which can be written as polynomial expressions in a_i. If the $*$-subalgebras $\mathcal{P}_1, \ldots, \mathcal{P}_s$ are freely independent, then we will say that **the random variables** $a_1, \ldots, a_s$ **are freely independent**.

Suppose now that $a_1, \ldots, a_s$ *are* freely independent. By paraphrasing (and paralleling) the discussion in Section 1, one can say that their free independence amounts to a recipe for how to retrieve the joint moment series $M_{a_1,\ldots,a_s}$, from the knowledge of the individual moment series $M_{a_1}, \ldots, M_{a_s}$. On the other hand, the discussion made in Section 2 tells us that the best way for retrieving $M_{a_1,\ldots,a_s}$ is by passing from moments to non-crossing cumulants – i.e. to R-transforms. Indeed, at the level of R-transforms, the condition that mixed cumulants vanish gives an extremely simple formula for $R_{a_1,\ldots,a_s}$ (cf. Equation (3.5) below). So the recommended way for getting at $M_{a_1,\ldots,a_s}$ is by getting first the individual R-transforms $R_{a_1}, \ldots, R_{a_s}$, then by using Equation (3.5) to get $R_{a_1,\ldots,a_s}$, and finally by passing back from cumulants to moments, to obtain what we need to know about $M_{a_1,\ldots,a_s}$.

The formula announced above for the joint R-transform of a free family is stated as follows (and is a version of the Theorem 2.4 in Section 2).

3.4 Theorem. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $a_1, \ldots, a_s$ be a family of selfadjoint elements of $\mathcal{A}$. Then $a_1, \ldots, a_s$ are freely independent if and only if the following equation holds:

$$R_{a_1,\ldots,a_s}(z_1, \ldots, z_s) = R_{a_1}(z_1) + \cdots + R_{a_s}(z_s) \tag{3.5}$$

(equality of formal power series in $\mathbb{C}\langle\langle z_1, \ldots, z_s\rangle\rangle$).

The R-transform was originally pinned down by Voiculescu [6] because of its property of "linearizing the addition" of freely independent random variables. In the approach to R-transforms used here (which is not the same as in [6]) this property is an easy consequence of what was discussed above.

Exercise 7. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space.

(a) Let a and b be two selfadjoint elements of $\mathcal{A}$, such that a is freely independent from b. Prove that

$$R_{a+b}(z) = R_a(z) + R_b(z). \tag{3.6}$$

(b) More generally, suppose that $\{a_1, \ldots, a_s\}$ and $\{b_1, \ldots, b_s\}$ are two sets of selfadjoint elements of $\mathcal{A}$, such that the unital algebras generated by these two sets are freely independent. Prove that we have

$$R_{a_1+b_1, a_2+b_2, \ldots, a_s+b_s} = R_{a_1,\ldots,a_s} + R_{b_1,\ldots,b_s} \tag{3.7}$$

(equality of formal power series in $\mathbb{C}\langle\langle z_1, \ldots, z_s\rangle\rangle$).

The next exercise is an illustration of how much a computation can be simplified by going to R-transforms, as opposed to sticking exclusively to moments. We will use the following definition.

3.5 Definition. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let a be a selfadjoint element of $\mathcal{A}$. We will say that a is **semicircular** of radius $r > 0$ if its moments are given by the formula:

$$\varphi(a^n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \left(\frac{r^2}{4}\right)^m \cdot \frac{(2m)!}{m!(m+1)!} & \text{if } n \text{ is even, } n = 2m. \end{cases} \tag{3.8}$$

[Comments: The justification of the name "semicircular" is better understood if one writes the right-hand side of (3.8) in the form $\int_{-r}^{r} t^n \cdot \frac{2}{\pi r^2}\sqrt{r^2 - t^2}\, dt$, which involves (the normalization of) a density with a semicircular graph of radius r.
If $r = 2$, then the element a is said to be **standard semicircular**. In this case the even moments of a are plainly given by the Catalan numbers.]

Exercise 8. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space.

(a) Suppose that $a = a^* \in \mathcal{A}$ is semicircular of radius r. Prove that $R_a(z) = \frac{r^2}{4} z^2$. Conversely, prove that if $a = a^* \in \mathcal{A}$ has R-transform $R_a(z) = \frac{r^2}{4} z^2$, then a is semicircular of radius r.

(b) Let $a_1, \ldots, a_s$ be a freely independent family of selfadjoint elements of $\mathcal{A}$, and suppose that a_i is semicircular of radius r_i, for $1 \leq i \leq s$. By using R-transforms, prove that the element $a := a_1 + \cdots + a_s$ is semicircular of radius $r := \sqrt{r_1^2 + \cdots + r_s^2}$.

(c) Consider the possibility of solving the part (b) of this exercise solely in terms of moments (by just using the Definition 3.5 and the material from Section 1), and compare the relative difficulty of the two solutions.

3.6 Remark. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, let a be an element of $\mathcal{A}$, and let us consider the moment series $M_a(z)$ and the R-transform $R_a(z)$. These are two generating functions, $\sum_{n=1}^{\infty} \alpha_n z^n$ and respectively $\sum_{n=1}^{\infty} \beta_n z^n$, which (according to the Equation (2.3) expressing moments in terms of cumulants) are related by the formula:

$$\alpha_n = \sum_{\substack{\pi \in NC(n) \\ \pi =: \{A_1, \ldots, A_k\}}} \beta_{|A_1|} \cdots \beta_{|A_k|}, \quad n \geq 1. \tag{3.9}$$

It is worth recording that the Equations (3.9) for all $n \geq 1$ can be consolidated in one functional equation relating M_a and R_a, namely

$$M_a(z) = R_a\Big(z(1 + M_a(z)) \Big). \tag{3.10}$$

Moreover, if $\varphi(a) \neq 0$, then the series M_a and R_a are invertible under composition and the Equation (3.10) can be restated in the very pleasing form

$$R_a^{<-1>}(w) = (1 + w) M_a^{<-1>}(w), \tag{3.11}$$

where "$f^{<-1>}$" stands for the inverse under composition [1] of a series $f(z) = \sum_{n=1}^{\infty} \gamma_n z^n$ with $\gamma_1 \neq 0$. For a more detailed discussion about this, see e.g. the Lecture 13 in [3].

The Exercise 8(a) consisted essentially in verifying that if in Equation (3.9) we have $\beta_2 = r^2/4$ and $\beta_n = 0$ for $n \neq 2$, then the α_n's have the form appearing on the right-hand side of Equation (3.8). This is a very special case when no reformulation of Equation (3.9) is needed (indeed, the only thing which one has to know in this special case is that the non-crossing pairings of $\{1, \ldots, 2m\}$ are counted by the Catalan number $(2m)!/(m!(m+1)!)$). But there are other – still very reasonable – situations when the moment series or R-transform which is needed does not pop out directly from (3.9), and the use of the Equations (3.10) or (3.11) is useful. Such an example is given in the next exercise.

[1] That is, $f^{<-1>}(w)$ is determined by the relations $f(\, f^{<-1>}(w)\,) = w$, and/or $f^{<-1>}(\, f(z)\,) = z$.

Exercise 9. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, let $p \in \mathcal{A}$ be an idempotent element ($p = p^2$), and denote $\varphi(p) =: \alpha$. Verify that the R-transform of a satisfies the quadratic equation

$$R_a(z)^2 + (1-z)R_a(z) - \alpha z = 0.$$

4. Convolution in the lattices of non-crossing partitions, and multiplication of free elements

4.1 Definition. Let $\mathbb{C}_o[[z]]$ denote the set of generating series of the form $f(z) = \sum_{n=1}^{\infty} \alpha_n z^n$, where $\alpha_1, \alpha_2, \alpha_3, \ldots \in \mathbb{C}$. On $\mathbb{C}_o[[z]]$ we define an operation called **boxed convolution**, $\boxed{\star}$, by the following rule: given $f(z) = \sum_{n=1}^{\infty} \alpha_n z^n$ and $g(z) = \sum_{n=1}^{\infty} \beta_n z^n$ in $\mathbb{C}_o[[z]]$, we set $f \boxed{\star}\, g =: \sum_{n=1}^{\infty} \gamma_n z^n$ with

$$\gamma_n \;=\; \sum_{\substack{\pi \in NC(n) \\ \pi =: \{A_1, \ldots, A_k\} \\ Kr(\pi) =: \{B_1, \ldots, B_l\}}} \alpha_{|A_1|} \cdots \alpha_{|A_k|}\, \beta_{|B_1|} \cdots \beta_{|B_l|}, \quad n \geq 1. \tag{4.1}$$

4.2 Remark. It can be argued that the operation $\boxed{\star}$ of Definition 4.1 "lies at mid-distance" between the convolution operation (in lattice sense) on the $NC(n)$'s, and the multiplication of free random variables.

On one hand, concerning the connection to convolution on the $NC(n)$'s: A basic fact about the structure of the lattices $NC(n)$ is that every interval $[\pi, \rho] \subseteq NC(n)$ has a canonical factorization as a direct product,

$$[\pi, \rho] \simeq NC(1)^{k_1} \times NC(2)^{k_2} \times \cdots \times NC(n)^{k_n}, \tag{4.2}$$

for some $k_1, k_2, \ldots, k_n \geq 0$. By using (4.2), we can associate to every series $f(z) = \sum_{n=1}^{\infty} \alpha_n z^n \in \mathbb{C}_o[[z]]$ a function

$$\widetilde{f} : \cup_{n \geq 1} \{\, [\pi, \rho] \mid \pi \leq \rho \text{ in } NC(n)\,\} \to \mathbb{C},$$

where for $[\pi, \rho]$ factored as in (4.2) we set

$$\widetilde{f}(\,[\pi, \rho]\,) \;=\; \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_n^{k_n}.$$

This $\widetilde{f}$ is called the **multiplicative function** associated to f. It is easy to verify that if $f \boxed{\star}\, g = h$ in $\mathbb{C}_o[[z]]$, then the corresponding multiplicative functions satisfy

$$\widetilde{f} \star \widetilde{g} \;=\; \widetilde{h}, \tag{4.3}$$

where the operation "$*$" in Equation (4.3) is convolution in the large incidence algebra of the lattices of non-crossing partitions.

Thus, in a certain sense, the operation $\boxed{\star}$ introduced in Definition 4.1 is just convolution in lattice sense, for a special class of functions on (intervals in) the $NC(n)$'s.

But on the other hand, $\boxed{\star}$ is equally close the operation of multiplying free random variables; indeed, we have (Nica and Speicher [1], [2]):

4.3 Theorem. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ be two unital $*$-subalgebras which are freely independent. Then we have

$$R_{bc} \ = \ R_b \boxed{\star} R_c, \tag{4.4}$$

for every $b \in \mathcal{B}$, $c \in \mathcal{C}$.

4.4 Remarks and Notations. The operation $\boxed{\star}$ is *associative* and *commutative*. These properties can be verified by using either the connection between $\boxed{\star}$ and the convolution on $NC(n)$'s, or the Theorem 4.3. For instance the associativity follows naturally from Theorem 4.3 and the fact that there exist "free product constructions" which allow us to build a triplet of freely independent elements with prescribed R-transforms (in some suitably large $*$-probability space). On the other hand it is more natural to check the commutativity of $\boxed{\star}$ directly from the Equation (4.1) – indeed, the only point that has to be observed there is that (for any given $\pi \in NC(n)$) the partitions $\mathrm{Kr}(\pi)$ and $\mathrm{Kr}^{-1}(\pi)$ have the same block sizes.

It is moreover obvious that the series

$$\Delta(z) := z \in \mathbb{C}_o[[z]]$$

is the unit for $\boxed{\star}$, and that $f(z) = \sum_{n=1}^{\infty} \alpha_n z^n \in \mathbb{C}_o[[z]]$ is invertible with respect to $\boxed{\star}$ if and only if $\alpha_1 \neq 0$.

We will use the notation

$$\mathrm{Zeta}(z) \ = \ \sum_{n=1}^{\infty} z^n \in \mathbb{C}_o[[z]].$$

The inverse of Zeta with respect to $\boxed{\star}$ will be denoted by Moeb, and is written explicitly as

$$\mathrm{Moeb}(z) \ = \ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-2)!}{(n-1)!n!} z^n.$$

The boxed convolution with Zeta and with Moeb had in fact already appeared in the Section 3 – indeed, it is clear (by looking e.g. at the Equation (3.9) in Remark 3.6) that we have

$$M_a \ = \ R_a \boxed{\star} \mathrm{Zeta}, \tag{4.5}$$

and hence

$$R_a \ = \ M_a \boxed{\star} \mathrm{Moeb}, \tag{4.6}$$

for every element a in a $*$-probability space.

Exercise 10. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. Suppose that $a = a^* \in \mathcal{A}$ is a standard semicircular element, and that $e \in \mathcal{A}$ is a selfadjoint projection (that is, $e^* = e = e^2$), such that a and e are freely independent. We denote $\varphi(e) =: \alpha$. Consider the new selfadjoint element $b := aea \in \mathcal{A}$. Prove that

$$R_b(z) \;=\; \frac{\alpha z}{1-z}.$$

[Comments: It is easy to see, by invoking the Exercise 3, that $M_b = M_{a^2 e}$, and hence that $R_b = R_{a^2 e}$; the reason for prefering b to $a^2 e$ in the statement of the exercise is that $b = b^*$. On the other hand, a quick solution can be given by applying Theorem 4.3 to the product $a^2 \cdot e$, and by observing that $R_{a^2} =$ Zeta.]

4.5 Remark *(the S-transform).* An efficient method for computing the moments of a product of two freely independent random variables is via the **S-transform**. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $a \in \mathcal{A}$ be such that $\varphi(a) \neq 0$. In [7], Voiculescu introduced the S-transform of a by the formula

$$S_a(w) \;=\; \frac{1+w}{w}\, M_a^{<-1>}(w), \tag{4.7}$$

where (same as in Section 3) the superscript "$< -1 >$" indicates inverse under composition ($S_a(w)$ is thus a power series in w, with constant term equal to $1/\varphi(a)$). The main result of [7] is that one has the equation

$$S_{bc} \;=\; S_b \cdot S_c, \tag{4.8}$$

holding in exactly the same framework as in Theorem 4.3. (Hence if we know the moment series M_b, M_c and we want to compute M_{bc}, then the most efficient way of doing that can be – at least in some examples – via the S-transforms S_b, S_c.) The formula (4.8) is proved in [7] with an argument of analytic inspiration.

The way how the S-transform fits in the framework of the boxed convolution operation $\boxed{\star}$ (and in particular what is the connection between the equations (4.4) and (4.8)) was explained by Nica and Speicher in [2]. For a series $f(z) = \sum_{n=1}^{\infty} \alpha_n z^n \in \mathbb{C}_o[[z]]$ which has $\alpha_1 \neq 0$, let us denote by $\mathcal{F}(f)$ the series $\frac{1}{w} f^{<-1>}(w)$. The main result of [2] is that $\mathcal{F}(\cdot)$ converts the boxed convolution into plain multiplication of series; that is, we have the equality

$$\mathcal{F}(\, f \,\boxed{\star}\, g \,) \;=\; \mathcal{F}(f) \cdot \mathcal{F}(g), \tag{4.9}$$

holding for all $f, g \in \mathbb{C}_o[[z]]$ with non-vanishing coefficients of degree 1. On the other hand, let us observe that if a is an element in the $*$-probability space $(\mathcal{A}, \varphi)$, with $\varphi(a) \neq 0$, then

$$S_a \;=\; \mathcal{F}(R_a); \tag{4.10}$$

indeed, we have

$$[\,\mathcal{F}(R_a)\,](w) \;=\; \frac{1}{w} R_a^{<-1>}(w) \;=\; \frac{1+w}{w} M_a^{<-1>}(w) \;=\; S_a(w)$$

(where we also took into account the Equation (3.11) of Remark 3.6). But then the multiplicativity of the R-transform is an immediate consequence of (4.4), (4.9) and (4.10).

Exercise 11. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and suppose that $p, q \in \mathcal{A}$ are such that $p = p^* = p^2$, $q = q^* = q^2$, and such that p is freely independent from q. Consider the new selfadjoint element $b = pqp \in \mathcal{A}$. Prove that the moment series of b satisfies the quadratic equation

$$(z-1)M_b(z)^2 + \Big((\alpha+\beta)z - 1\Big)M_b(z) + \alpha\beta z = 0,$$

where we denoted $\varphi(p) =: \alpha$, $\varphi(q) =: \beta$.

4.6 Definition and Remarks *(the multivariable version of* $\boxed{\star}$ *).* Let s be a positive integer, and let us consider the set $\mathbb{C}_o\langle\langle z_1, \ldots, z_s\rangle\rangle$ of formal power series of the kind which appeared in Equations (3.2), (3.3) of Section 3. That is, $\mathbb{C}_o\langle\langle z_1, \ldots, z_s\rangle\rangle$ consists of the series of the form:

$$f(z_1, \ldots, z_s) = \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n = 1}^{s} \alpha_{(i_1, \ldots, i_n)} z_{i_1} \cdots z_{i_n}, \tag{4.11}$$

where $\{\alpha_{(i_1,\ldots,i_n)} \mid n \geq 1,\ 1 \leq i_1, \ldots, i_n \leq s\}$ is a family of complex coefficients. The operation of boxed convolution $\boxed{\star}$ from Definition 4.1 can be naturally generalized to a binary operation $\boxed{\star}_s$ on $\mathbb{C}_o\langle\langle z_1, \ldots, z_s\rangle\rangle$ (where $\boxed{\star}_s = \boxed{\star}$ if it happens that our s is $s = 1$). In order to give the precise definition of $\boxed{\star}_s$ let us introduce the following notation: if $f \in \mathbb{C}_o\langle\langle z_1, \ldots, z_s\rangle\rangle$ has the coefficients denoted as in Equation (4.11), and if we are given a partition $\pi = \{A_1, \ldots, A_k\} \in NC(n)$ and some values $i_1, \ldots, i_n \in \{1, \ldots, s\}$, then we will denote

$$[\, f\,]_{i_1, \ldots, i_n; \pi} := \prod_{j=1}^{k} \alpha_{(\ (i_1, \ldots, i_n)|A_j\ )}. \tag{4.12}$$

(For instance if $\pi = \{\, \{1,3\}, \{2\}, \{4\}\,\} \in NC(4)$, then $[\, f\,]_{i_1,i_2,i_3,i_4;\pi} = \alpha_{(i_1,i_3)}\alpha_{(i_2)}\alpha_{(i_4)}$.) In terms of the notation in (4.12), the operation $\boxed{\star}_s$ is defined by the following formula:

$$\text{coefficient of } z_{i_1} \cdots z_{i_n} \text{ in } f \boxed{\star}_s\, g \;=\; \sum_{\pi \in NC(n)} [\, f\,]_{i_1, \ldots, i_n; \pi} \cdot [\, g\,]_{i_1, \ldots, i_n; Kr(\pi)}, \tag{4.13}$$

for f, g in $\mathbb{C}_o\langle\langle z_1, \ldots, z_s\rangle\rangle$, and for $n \geq 1$, $1 \leq i_1, \ldots, i_n \leq s$.

It is still true that $\boxed{\star}_s$ is associative, but $\boxed{\star}_s$ is not commutative for $s \geq 2$. The unit for $\boxed{\star}_s$ is the series

$$\Delta_s(z_1, \ldots, z_s) \;=\; z_1 + \cdots + z_s,$$

and a series in $\mathbb{C}_o\langle\langle z_1, \ldots, z_s\rangle\rangle$ is invertible if and only if all its s coefficients of degree 1 are different from zero. The multivariable Zeta series is

$$\text{Zeta}_s(z_1, \ldots, z_s) \;=\; \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n = 1}^{s} z_{i_1} \cdots z_{i_n},$$

and its inverse with respect to $\boxed{\star}_s$ is

$$\text{Moeb}_s(z_1, \ldots, z_s) \;=\; \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=1}^{s} \frac{(-1)^{n+1}(2n-2)!}{(n-1)!n!} z_{i_1} \cdots z_{i_n}.$$

It is easily checked that Zeta_s and Moeb_s are central with respect to $\boxed{\star}_s$ (i.e. they $\boxed{\star}_s$–commute with every $f \in \mathbb{C}_o\langle\langle z_1, \ldots, z_s\rangle\rangle$). For a more detailed discussion about this, see [1] or the Lecture 12 in [3].

The good relations between boxed convolution and free random variables extend to the multivariable case. On one hand, the connection between the joint moment series and joint R-transform (appearing in the Equations (3.2) and (3.3) of Section 3) can now be stated as follows.

4.7 Proposition. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $a_1, \ldots, a_s$ be elements of $\mathcal{A}$. Then the joint moment series $M_{a_1,\ldots,a_s}$ and the R-transform $R_{a_1,\ldots,a_s}$ are related by

$$M_{a_1,\ldots,a_s} \;=\; R_{a_1,\ldots,a_s} \boxed{\star}_s \text{Zeta}_s, \tag{4.14}$$

or equivalently by

$$R_{a_1,\ldots,a_s} \;=\; M_{a_1,\ldots,a_s} \boxed{\star}_s \text{Moeb}_s. \tag{4.15}$$

On the other hand, the Theorem 4.3 also generalizes to the multivariable case.

4.8 Theorem. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ be two unital $*$-subalgebras which are freely independent. Then for any s-tuples $b_1, \ldots, b_s \in \mathcal{B}$ and $c_1, \ldots, c_s \in \mathcal{C}$ we have

$$R_{b_1,\ldots,b_s} \boxed{\star}_s R_{c_1,\ldots,c_s} \;=\; R_{b_1c_1,\ldots,b_sc_s}. \tag{4.16}$$

As an illustration on how the multivariable boxed convolution can work, here is a generalization of Exercise 10.

Exercise 12. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. Let $a = a^* \in \mathcal{A}$ be a standard semicircular element, and let $e_1, \ldots, e_s \in \mathcal{A}$ be a family of mutually orthogonal selfadjoint projections (that is, $e_i^* = e_i = e_i^2$ for $1 \le i \le s$, and $e_ie_j = 0$ for $1 \le i < j \le s$), such that the unital algebra generated by a is freely independent from the unital algebra generated by $\{e_1, \ldots, e_s\}$. We denote $\varphi(e_i) =: \alpha_i$, for $1 \le i \le s$. Consider the new selfadjoint elements $b_i := ae_ia \in \mathcal{A}$, $1 \le i \le s$.

(a) Prove that

$$R_{b_1,\ldots,b_s}(z_1, \ldots, z_s) \;=\; \frac{\alpha_1 z_1}{1 - z_1} + \cdots + \frac{\alpha_s z_s}{1 - z_s}.$$

(b) Conclude that the random variables $b_1, \ldots, b_s$ are freely independent.

Since the considerations about $\boxed{\star}$ seem to generalize so nicely to several variables, one would hope to also have a multi-variable version of the S-transform of Voiculescu. But it is not known how to obtain such an object.

Problem. Find a multi-variable generalization for the S-transform.

A possible way to approach this problem could go by finding a multi-variable analogue for the operator $\mathcal{F}(\cdot)$ which appeared in the Equations (4.9), (4.10) of the Remark 4.5 – or in other words, by gaining a better understanding for what is the group of invertibles in the semigroup $\Big(\mathbb{C}_o\langle\langle z_1, \ldots, z_s\rangle\rangle, \ \boxed{\star} \Big)$.

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