CHAPTER I

de Rham Theory

§1 The de Rham Complex on $\mathbb{R}^n$

To start things off we define in this section the de Rham cohomology and compute a few examples. This will turn out to be the most important diffeomorphism invariant of a manifold. So let $x_1, \ldots, x_n$ be the linear coordinates on $\mathbb{R}^n$. We define Ω^* to be the algebra over $\mathbb{R}$ generated by $dx_1, \ldots, dx_n$ with the relations

$$\begin{cases} (dx_i)^2 = 0 \\ dx_i\, dx_j = -dx_j\, dx_i, \; i \neq j. \end{cases}$$

As a vector space over $\mathbb{R}$, Ω^* has basis

$$1,\ dx_i,\ \underset{i<j}{dx_i\, dx_j},\ \underset{i<j<k}{dx_i\, dx_j\, dx_k},\ \ldots,\ dx_1 \ldots dx_n.$$

The C^∞ *differential forms* on $\mathbb{R}^n$ are elements of

$$\Omega^*(\mathbb{R}^n) = \{C^\infty \text{ functions on } \mathbb{R}^n\} \underset{\mathbb{R}}{\otimes} \Omega^*.$$

Thus, if ω is such a form, then ω can be uniquely written as $\sum f_{i_1 \ldots i_q}\, dx_{i_1} \ldots dx_{i_q}$ where the coefficients $f_{i_1 \ldots i_q}$ are C^∞ functions. We also write $\omega = \sum f_I\, dx_I$. The algebra $\Omega^*(\mathbb{R}^n) = \bigoplus_{q=0}^{n} \Omega^q(\mathbb{R}^n)$ is naturally graded, where $\Omega^q(\mathbb{R}^n)$ consists of the C^∞ q-forms on $\mathbb{R}^n$. There is a differential operator

$$d : \Omega^q(\mathbb{R}^n) \longrightarrow \Omega^{q+1}(\mathbb{R}^n),$$

defined as follows:

i) if $f \in \Omega^0(\mathbb{R}^n)$, then $df = \sum \partial f/\partial x_i\, dx_i$
ii) if $\omega = \sum f_I\, dx_I$, then $d\omega = \sum df_I\, dx_I$.

EXAMPLE 1.1. If $\omega = x\,dy$, then $d\omega = dx\,dy$.

This d, called the *exterior differentiation*, is the ultimate abstract extension of the usual gradient, curl, and divergence of vector calculus on $\mathbb{R}^3$, as the example below partially illustrates.

EXAMPLE 1.2. On $\mathbb{R}^3$, $\Omega^0(\mathbb{R}^3)$ and $\Omega^3(\mathbb{R}^3)$ are each 1-dimensional and $\Omega^1(\mathbb{R}^3)$ and $\Omega^2(\mathbb{R}^3)$ are each 3-dimensional over the C^∞ functions, so the following identifications are possible:

$$\begin{array}{ccccc} \{\text{functions}\} & \simeq & \{0\text{-forms}\} & \simeq & \{3\text{-forms}\} \\ f & \leftrightarrow & f & \leftrightarrow & f\,dx\,dy\,dz \end{array}$$

and

$$\{\text{vector fields}\} \simeq \{1\text{-forms}\} \simeq \{2\text{-forms}\}$$

$$X = (f_1, f_2, f_3) \leftrightarrow f_1\,dx + f_2\,dy + f_3\,dz \leftrightarrow f_1\,dy\,dz - f_2\,dx\,dz + f_3\,dx\,dy.$$

On functions,

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz.$$

On 1-forms,

$$d(f_1\,dx + f_2\,dy + f_3\,dz)$$
$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) dy\,dz - \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) dx\,dz + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy.$$

On 2-forms,

$$d(f_1\,dy\,dz - f_2\,dx\,dz + f_3\,dx\,dy) = \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz.$$

In summary,

$$d(0\text{-forms}) = \text{gradient},$$
$$d(1\text{-forms}) = \text{curl},$$
$$d(2\text{-forms}) = \text{divergence}.$$

The *wedge product* of two differential forms, written $\tau \wedge \omega$ or $\tau \cdot \omega$, is defined as follows: if $\tau = \sum f_I\,dx_I$ and $\omega = \sum g_J\,dx_J$, then

$$\tau \wedge \omega = \sum f_I g_J\,dx_I\,dx_J.$$

Note that $\tau \wedge \omega = (-1)^{\deg \tau \deg \omega}\omega \wedge \tau$.

Proposition 1.3. *d is an antiderivation, i.e.,*

$$d(\tau \cdot \omega) = (d\tau) \cdot \omega + (-1)^{\deg \tau}\,\tau \cdot d\omega.$$

PROOF. By linearity it suffices to check on monomials

$$\tau = f_I\,dx_I, \quad \omega = g_J\,dx_J.$$

$$d(\tau \cdot \omega) = d(f_I g_J)\,dx_I\,dx_J = (df_I)g_J\,dx_I\,dx_J + f_I\,dg_J\,dx_I\,dx_J$$
$$= (d\tau) \cdot \omega + (-1)^{\deg \tau}\,\tau \cdot d\omega.$$

On the level of functions $d(fg) = (df)g + f(dg)$ is simply the ordinary product rule. □

Proposition 1.4. $d^2 = 0$.

PROOF. This is basically a consequence of the fact that the mixed partials are equal. On functions,

$$d^2 f = d\left(\sum_i \frac{\partial f}{\partial x_i}\,dx_i\right) = \sum_{i,j} \frac{\partial^2 f}{\partial x_j\,\partial x_i}\,dx_j\,dx_i.$$

Here the factors $\partial^2 f/\partial x_j \partial x_i$ are symmetric in i, j while $dx_j\,dx_i$ are skew-symmetric in i, j; hence $d^2 f = 0$. On forms $\omega = f_I\,dx_I$,

$$d^2\omega = d^2(f_I\,dx_I) = d(df_I\,dx_I) = 0$$

by the previous computation and the antiderivation property of d. □

The complex $\Omega^*(\mathbb{R}^n)$ together with the differential operator d is called the *de Rham complex* on $\mathbb{R}^n$. The kernel of d are the *closed* forms and the image of d, the *exact* forms. The de Rham complex may be viewed as a God-given set of differential equations, whose solutions are the closed forms. For instance, finding a closed 1-form $f\,dx + g\,dy$ on $\mathbb{R}^2$ is tantamount to solving the differential equation $\partial g/\partial x - \partial f/\partial y = 0$. By Proposition 1.4 the exact forms are automatically closed; these are the trivial or "uninteresting" solutions. A measure of the size of the space of "interesting" solutions is the definition of the de Rham cohomology.

Definition. The q-th *de Rham cohomology* of $\mathbb{R}^n$ is the vector space

$$H^q_{DR}(\mathbb{R}^n) = \{\text{closed } q\text{-forms}\}/\{\text{exact } q\text{-forms}\}.$$

We sometimes suppress the subscript DR and write $H^q(\mathbb{R}^n)$. If there is a need to distinguish between a form ω and its cohomology class, we denote the latter by $[\omega]$.

Note that all the definitions so far work equally well for any open subset U of $\mathbb{R}^n$; for instance,

$$\Omega^*(U) = \{C^\infty \text{ functions on } U\} \underset{\mathbb{R}}{\otimes} \Omega^*.$$

So we may also speak of the de Rham cohomology $H^*_{DR}(U)$ of U.

EXAMPLES 1.5.
(a) $n = 0$

$$H^q = \begin{cases} \mathbb{R} & q = 0 \\ 0 & q > 0. \end{cases}$$

(b) $n = 1$
Since $(\ker d) \cap \Omega^0(\mathbb{R}^1)$ are the constant functions,

$$H^0(\mathbb{R}^1) = \mathbb{R}.$$

On $\Omega^1(\mathbb{R}^1)$, $\ker d$ are all the 1-forms.
If $\omega = g(x)dx$ is a 1-form, then by taking

$$f = \int_0^x g(u)\, du,$$

we find that

$$df = g(x)\, dx.$$

Therefore every 1-form on $\mathbb{R}^1$ is exact and

$$H^1(\mathbb{R}^1) = 0.$$

(c) Let U be a disjoint union of m open intervals on $\mathbb{R}^1$. Then

$$H^0(U) = \mathbb{R}^m$$

and

$$H^1(U) = 0.$$

(d) In general

$$H^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } 0, \\ 0 & \text{otherwise.} \end{cases}$$

This result is called the *Poincaré lemma* and will be proved in Section 4.

The de Rham complex is an example of a *differential complex*. For the convenience of the reader we recall here some basic definitions and results on differential complexes. A direct sum of vector spaces $C = \bigoplus_{q \in \mathbb{Z}} C^q$ indexed by the integers is called a *differential complex* if there are homomorphisms

$$\cdots \longrightarrow C^{q-1} \xrightarrow{d} C^q \xrightarrow{d} C^{q+1} \longrightarrow \cdots$$

such that $d^2 = 0$. d is the *differential operator* of the complex C. The *cohomology* of C is the direct sum of vector spaces $H(C) = \bigoplus_{q \in \mathbb{Z}} H^q(C)$, where

$$H^q(C) = (\ker d \cap C^q)/(\operatorname{im} d \cap C^q).$$

A map $f: A \to B$ between two differential complexes is a *chain map* if it commutes with the differential operators of A and B: $f\, d_A = d_B\, f$.

A sequence of vector spaces

$$\cdots \longrightarrow V_{i-1} \xrightarrow{f_{i-1}} V_i \xrightarrow{f_i} V_{i+1} \longrightarrow \cdots$$

is said to be *exact* if for all i the kernel of f_i is equal to the image of its predecessor f_{i-1}. An exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is called a *short exact sequence*. Given a short exact sequence of differential complexes

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in which the maps f and g are chain maps, there is a long exact sequence of cohomology groups

$$\begin{tikzcd} \cdots \arrow[r] & H^q(A) \arrow[r, "f^*"] & H^q(B) \arrow[r, "g^*"] & H^q(C) \arrow[dll, "d^*"] \\ & H^{q+1}(A) \arrow[r] & \cdots \end{tikzcd}$$

In this sequence f^* and g^* are the naturally induced maps and $d^*[c]$, $c \in C^q$, is obtained as follows:

$$\begin{tikzcd} & \arrow[u] & \arrow[u] & \arrow[u] & \\ 0 \arrow[r] & A^{q+1} \arrow[r, "f"] \arrow[u] & B^{q+1} \arrow[r, "g"] \arrow[u] & C^{q+1} \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & A^q \arrow[r, "f"] \arrow[u] & B^q \arrow[r, "g"] \arrow[u, "d"] & C^q \arrow[r] \arrow[u, "d"] & 0 \\ & \arrow[u] & \arrow[u] & \arrow[u] & \end{tikzcd}$$

By the surjectivity of g there is an element b in B^q such that $g(b) = c$. Because $g(db) = d(gb) = dc = 0$, $db = f(a)$ for some a in A^{q+1}. This a is easily checked to be closed. $d^*[c]$ is defined to be the cohomology class $[a]$ in $H^{q+1}(A)$. A simple diagram-chasing shows that this definition of d^* is independent of the choices made.

Exercise. Show that the long exact sequence of cohomology groups exists and is exact. (See, for instance, Munkres [2, §24].)

Compact Supports

A slight modification of the construction of the preceding section will give us another diffeomorphism invariant of a manifold. For now we again restrict our attention to $\mathbb{R}^n$. Recall that the *support* of a continuous function f on a topological space X is the closure of the set on which f is not zero, i.e., $\operatorname{Supp} f = \overline{\{p \in X \mid f(p) \neq 0\}}$. If in the definition of the de Rham complex we use only the C^∞ functions with compact support, the resulting complex is called the *de Rham complex* $\Omega_c^*(\mathbb{R}^n)$ *with compact supports*:

$$\Omega_c^*(\mathbb{R}^n) = \{C^\infty \text{ functions on } \mathbb{R}^n \text{ with compact support}\} \underset{\mathbb{R}}{\otimes} \Omega^*.$$

The cohomology of this complex is denoted by $H_c^*(\mathbb{R}^n)$.

EXAMPLE 1.6.

(a) $H_c^*(\text{point}) = \begin{cases} \mathbb{R} & \text{in dimension } 0, \\ 0 & \text{elsewhere.} \end{cases}$

(b) *The compact cohomology of* $\mathbb{R}^1$. Again the closed 0-forms are the constant functions. Since there are no constant functions on $\mathbb{R}^1$ with compact support,

$$H_c^0(\mathbb{R}^1) = 0.$$

To compute $H_c^1(\mathbb{R}^1)$, consider the integration map

$$\int_{\mathbb{R}^1} : \Omega_c^1(\mathbb{R}^1) \longrightarrow \mathbb{R}^1.$$

This map is clearly surjective. It vanishes on the exact 1-forms df where f has compact support, for if the support of f lies in the interior of $[a,b]$, then

$$\int_{\mathbb{R}^1} \frac{df}{dx}\, dx = \int_a^b \frac{df}{dx}\, dx = f(b) - f(a) = 0.$$

If $g(x)\, dx \in \Omega_c^1(\mathbb{R}^1)$ is in the kernel of the integration map, then the function

$$f(x) = \int_{-\infty}^{x} g(u)\, du$$

will have compact support and $df = g(x)\, dx$. Hence the kernel of $\int_{\mathbb{R}^1}$ are precisely the exact forms and

$$H_c^1(\mathbb{R}^1) = \frac{\Omega_c^1(\mathbb{R}^1)}{\ker \int_{\mathbb{R}^1}} = \mathbb{R}^1.$$

REMARK. If $g(x)\, dx \in \Omega_c^1(\mathbb{R}^1)$ does not have total integral 0, then

$$f(x) = \int_{-\infty}^{x} g(u)\, du$$

will not have compact support and $g(x)\, dx$ will not be exact.

(c) More generally,

$$H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{in dimension } n \\ 0 & \text{otherwise.} \end{cases}$$

This result is the *Poincaré lemma for cohomology with compact support* and will be proved in Section 4.

Exercise 1.7. Compute $H_{DR}^*(\mathbb{R}^2 - P - Q)$ where P and Q are two points in $\mathbb{R}^2$. Find the closed forms that represent the cohomology classes.

§2 The Mayer-Vietoris Sequence

In this section we extend the definition of the de Rham cohomology from $\mathbb{R}^n$ to any differentiable manifold and introduce a basic technique for computing the de Rham cohomology, the Mayer-Vietoris sequence. But first we have to discuss the functorial nature of the de Rham complex.

The Functor Ω^*

Let $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ be the standard coordinates on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A smooth map $f: \mathbb{R}^m \to \mathbb{R}^n$ induces a pullback map on C^∞ functions $f^*: \Omega^0(\mathbb{R}^n) \to \Omega^0(\mathbb{R}^m)$ via

$$f^*(g) = g \circ f.$$

We would like to extend this pullback map to all forms $f^*: \Omega^*(\mathbb{R}^n) \to \Omega^*(\mathbb{R}^m)$ in such a way that it commutes with d. The commutativity with d defines f^* uniquely:

$$f^*\left(\sum g_I\, dy_{i_1} \ldots dy_{i_q}\right) = \sum (g_I \circ f)\, df_{i_1} \ldots df_{i_q},$$

where $f_i = y_i \circ f$ is the i-th component of the function f.

Proposition 2.1. *With the above definition of the pullback map f^* on forms, f^* commutes with d.*

PROOF. The proof is essentially an application of the chain rule.

$$df^*(g_I\, dy_{i_1} \ldots dy_{i_q}) = d((g_I \circ f)\, df_{i_1} \ldots df_{i_q}) = d(g_I \circ f)\, df_{i_1} \ldots df_{i_q}.$$

$$\begin{aligned} f^* d(g_I\, dy_{i_1} \ldots dy_{i_q}) &= f^*\left(\sum_{i=1}^{n} \frac{\partial g_I}{\partial y_i}\, dy_i\, dy_{i_1} \ldots dy_{i_q}\right) \\ &= \sum_{i=1}^{n} \left(\left(\frac{\partial g_I}{\partial y_i} \circ f\right) df_i\right) df_{i_1} \ldots df_{i_q} \\ &= d(g_I \circ f)\, df_{i_1} \ldots df_{i_q}. \end{aligned}$$

□