CONTINUOUS NONCROSSING PARTITIONS AND THE DUAL BRAID COMPLEX

MICHAEL DOUGHERTY AND JON MCCAMMOND

Abstract. This article examines the poset of noncrossing partitions of a circle, with a particular focus on the partitions which are invariant under the standard degree-d covering map. These form a subposet with close connections to the dual braid complex, a classifying space for the d-strand braid group introduced by Tom Brady in 2001. We use this connection to prove that the dual braid complex is homeomorphic to the space of monic degree-d polynomials with distinct, centered roots and critical values on the unit circle. In particular, this identifies the dual braid complex as a spine of the classical polynomial classifying space for the braid groups. Along the way we introduce new algebraic objects which describe the points and cells in both the order complexes of generalized noncrossing partitions and the interval complexes for arbitrary dual Artin groups.

Introduction

First, let $\mathrm{NC}(\mathbf{S})$ denote the set of all partitions of the unit circle $\mathbf{S}$ for which the convex hulls of the blocks are pairwise disjoint. This set is partially ordered by refinement, and we refer to its elements as *continuous noncrossing partitions*. For example, a geodesic lamination of a hyperbolic surface lifts to a lamination of the hyperbolic plane, and this induces a noncrossing partition of the boundary at infinity. To give another example, the preimage of the coordinate axes under a degree-d complex polynomial will intersect a sufficiently large circle in $4d$ points, and this will determine a noncrossing partition of $\mathbf{S}$ in which each block has size divisible by 4 [MSS07]. In ongoing work of the authors, we provide similar constructions connecting continuous noncrossing partitions to complex polynomials [DM22, DMa]. We are particularly interested in the subposet $\mathrm{NC}_d(\mathbf{S})$ of *degree-d-invariant* partitions i.e. those which are compatible with the standard degree-d covering map $z \mapsto z^d$ for the unit circle – see Figure 1.

Our first main theorem provides an algebraic interpretation for $\mathrm{NC}_d(\mathbf{S})$ using a far more general construction. Let G be a group and suppose that X is a generating set for G which is closed under conjugation. For each $g \in G$, let $[1, g]$ be the interval poset formed by the geodesics between the identity and g in the right Cayley graph of G with respect to X. Essentially, $[1, g]$ consists of all elements of G which appear in a minimum-length factorization of g. In this article, we introduce the graded poset $\mathcal{F}(g, \mathbf{S})$ of *weighted circular factorizations* of elements in $[1, g]$. Each factorization is defined by a map $\mathbf{S} \to [1, g]$ with certain properties and the set of all such factorizations inherits a partial order from $[1, g]$. When G is the

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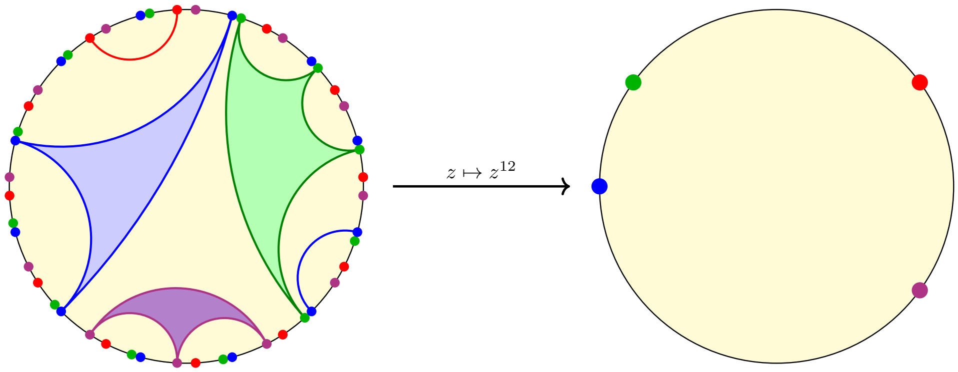

FIGURE 1. A degree-12-invariant noncrossing partition of the circle

symmetric group $\textsc{Sym}_d$, X is the set of all transpositions in $\textsc{Sym}_d$, and δ is the d-cycle $(1\ 2\ \cdots\ d)$, we obtain the poset of degree-d-invariant continuous noncrossing partitions.

Theorem A (Theorem 6.11). $\mathrm{NC}_d(\mathbf{S})$ *is isomorphic to* $\mathcal{F}(\delta, \mathbf{S})$.

Returning to the general setting, we view $\mathcal{F}(g, \mathbf{S})$ as a "topological graded poset" in the sense that it is a graded poset such that the elements in each rank form a topological space. This type of object is best described by an example. Fix a positive integer n and consider the set of all multisets (i.e. sets with repetition allowed) of at most n points in the unit interval $\mathbf{I}$. This set is a graded poset under inclusion, and the elements of rank k are the multisets which have exactly k points (including repetition). Moreover, the set of elements with rank k can naturally be identified with the cell structure of a k-dimensional simplex, so this graded poset comes with a topology in each rank. We can also describe the elements of each rank combinatorially by noting that the Boolean lattice of height k is the face poset for the k-dimensional simplex, but we must take care to not confuse the two partial orders. To get a sense of the structure, see Figure 12 for a simplified diagram.

In general, the topology for $\mathcal{F}(g, \mathbf{S})$ is closely related to a useful cell complex associated to $[1, g]$. The order complex of $[1, g]$ is a Δ-complex in which each edge inherits a label from the right Cayley graph of G with respect to X; the *interval complex* K_g is the single-vertex Δ-complex obtained by identifying faces with matching labels on their 1-skeletons. When G is the symmetric group $\textsc{Sym}_d$, X is the set of all transpositions in $\textsc{Sym}_d$, and δ is the d-cycle $(1\ 2\ \cdots\ d)$, the interval complex K_δ is known as the *dual braid complex*. This space was introduced and shown to be a classifying space for the d-strand braid group by Brady [Bra01], then given a piecewise-Euclidean metric by Brady and the second author [BM10].

Theorem B (Theorem 5.6). *The maximal elements of the topological graded poset* $\mathcal{F}(g, \mathbf{S})$ *form a subspace isometric to* K_g*, the interval complex for* $[1, g]$.

As a consequence of our first two theorems, we see that the maximal elements of $\mathrm{NC}_d(\mathbf{S})$ form a subspace which is isometric to the dual braid complex K_δ.

We also investigate some of the combinatorial structure for $\mathcal{F}(g, \mathbf{S})$. It follows quickly from the definitions that if $\mathbf{u}$ is a weighted circular factorization of $h \in [1, g]$, then the lower set $\downarrow(\mathbf{u})$ (i.e. the poset of all elements below $\mathbf{u}$ in the partial order on $\mathcal{F}(g, \mathbf{S})$) is isomorphic to a product of intervals $[1, x_1] \times \cdots \times [1, x_k]$ for some $x_1, \ldots, x_k \in [1, g]$. In particular, this means that $\mathcal{F}(g, \mathbf{S})$ is a poset with uncountably many elements in which each each lower set is finite. Our third main theorem treats the more complicated case of upper sets in $\mathcal{F}(\delta, \mathbf{S})$.

Theorem C (Corollary 5.12)**.** *Let $\mathbf{u}$ be a weighted circular factorization of $h \in [1, g]$. Then the upper set $\uparrow(\mathbf{u})$ in $\mathcal{F}(g, \mathbf{S})$ is isomorphic to $\mathcal{F}(h^{-1}g, \mathbf{S})$. Consequently, the maximal elements of $\uparrow(\mathbf{u})$ form a subspace which is isometric to the interval complex $K_{h^{-1}g}$.*

The theorems above allow us to tackle the following natural question: what is the relationship between the dual braid complex and other classifying spaces for the braid group? The quintessential example comes from arrangements of hyperplanes. Let $\mathcal{A}_d$ denote the *complex braid arrangement*, i.e. the union of all hyperplanes in $\mathbb{C}^d$ described by equations of the form $z_i = z_j$ where $i \neq j$. The complement $\mathbb{C}^d - \mathcal{A}_d$ is a connected $2d$-manifold which admits a free action of the symmetric group Sym_d by permuting coordinates, and the resulting quotient $Y_d = (\mathbb{C}^d - \mathcal{A}_d)/\mathrm{Sym}_d$ was proven to be a classifying space for the d-strand braid group by Fox and Neuwirth in 1962 [FN62].

Since classifying spaces are unique up to homotopy equivalence, we know for abstract reasons that Y_d is homotopy equivalent to the dual braid complex. Our fourth result provides a more concrete connection between the two.

Theorem D (Corollary 6.13)**.** *The dual braid complex is a spine for Y_d. That is, there is a subspace of Y_d which is homeomorphic to the dual braid complex and a deformation retraction from Y_d to this subspace.*

The main ingredient in this theorem is a result in complex dynamics which was stated by W. Thurston in an unpublished manuscript, then posthumously completed by Baik, Gao, Hubbard, Lei, Lindsey, and D. Thurston [TBY+20]. As a consequence of this result, we identify Y_d with the space of monic complex polynomials with d distinct roots centered at the origin, then find that the dual braid complex is homeomorphic to the subspace of polynomials with critical values on the unit circle. This connection was pointed out to us by Daan Krammer in 2017; in our upcoming article [DMa], we provide another proof of this fact through a more general construction. In particular, we describe a piecewise-Euclidean metric for Y_d and a polysimplicial cell structure for its completion such that the subspace of polynomials with critical values on the unit circle inherits a metric cell structure which is isometric to the dual braid complex.

We believe that the results of Theorem D can be extended to a broader class of examples. For example, let W be a finite Coxeter group with S its set of simple reflections, and let X be the set of all conjugates of S. Letting $n = |S|$, we know that W acts on $\mathbb{C}^n$ by isometries, where each element of X acts as a reflection. If we define $\mathcal{A}_W$ to be the union of all hyperplanes fixed by elements of X, then the $2n$-manifold $Y_W = (\mathbb{C}^n - \mathcal{A}_W)/W$ is a classifying space for W [Bri73, Del72]. Finally, $\delta \in W$ is a *Coxeter element* if it can be written as the product of all elements in S in some order, with each element appearing exactly once.

Conjecture 1. *Let δ be a Coxeter element for the Coxeter group W. Then the interval complex K_δ is a spine for Y_W.*

The article is structured as follows. In Sections 1 and 2, we provide background information on posets and the dual braid complex. Section 3 introduces posets of factorizations and examines their combinatorial structure, while Section 4 introduces weighted analogues of these factorizations and examines their topology and geometry. In Section 5, we introduce the topological graded posets of weighted factorizations and prove Theorems B and C. Finally, we define the poset of degree-d-invariant partitions of the circle in Section 6 and prove Theorems A and D.

1. Posets and Orthoschemes

In this section we recall some background information for partially ordered sets and a special kind of metric simplex known as an orthoscheme. See [Sta12, Ch. 3] for a standard reference on posets and [Hat02, BM10] for background on simplices.

A partially ordered set P is a *lattice* if each pair of elements has a unique meet and a unique join. If P has unique meets (but not necessarily unique joins), we say that P is a *meet-semilattice*. A *chain* in P is a collection of distinct elements $x_0, \ldots, x_k \in P$ such that $x_0 \leq \cdots \leq x_k$, and a chain is *maximal* if it is not properly contained in another chain. We say that P is *graded* if there is a *rank function* $\mathrm{rk}\colon P \to \mathbb{N}$ such that for all $x, y \in P$, we have $\mathrm{rk}(x) < \mathrm{rk}(y)$ whenever $x < y$ and $\mathrm{rk}(x)+1 = \mathrm{rk}(y)$ whenever $x < y$ and there is no $z \in P$ with $x < z < y$. The *height* of a graded poset is the length of a maximal chain. Given two elements $x, y \in P$, the set of all $z \in P$ with $x \leq z \leq y$ is the *interval* $[x, y]$. The set of all elements $y \in P$ with $x \leq y$ is called the *upper set* of x and is denoted $\uparrow(x)$. Similarly, the set of all $z \in P$ with $z \leq x$ is the *lower set* of x, denoted $\downarrow(x)$.

Example 1.1 (Boolean lattice)**.** The *Boolean lattice* $\textsc{Bool}(n)$ consists of all subsets for the n-element set $\{1, \ldots, n\}$, partially ordered under inclusion, and it is indeed a lattice: given $A, B \in \textsc{Bool}(n)$, the unique meet is $A \cap B$ and the unique join is $A \cup B$. This poset is also graded, with rank function $\mathrm{rk}\colon \textsc{Bool}(n) \to \mathbb{N}$ given by $\mathrm{rk}(A) = |A|$ for each $A \subseteq \{1, \ldots, n\}$. Fixing an element $A \in \textsc{Bool}(n)$ with rank k, the lower set $\downarrow(A)$ is isomorphic to the smaller Boolean lattice $\textsc{Bool}(k)$, whereas the upper set $\uparrow(A)$ is isomorphic to $\textsc{Bool}(n-k)$.

There is a close connection between partially ordered sets and simplicial complexes. Each cell complex has an associated *face poset* and each poset has an associated *order complex*. We will make use of both operations, and we begin by describing the first.

Definition 1.2 (Face posets)**.** Let X be a simplicial complex. The *face poset* $P(X)$ is defined to be the graded poset of all faces of X (including the empty face), ordered by inclusion.

For example, each face of an n-dimensional simplex can be specified precisely by a subset of the $n+1$ vertices, and the relation of incidence between two faces corresponds exactly to inclusion between the two subsets. In other words, the face poset for the n-simplex is isomorphic to the Boolean lattice $\textsc{Bool}(n+1)$.

Definition 1.3 (Order complexes)**.** Let P be a graded poset. The *order complex* $\Delta(P)$ is the Δ-complex with vertex set P and an ordered k-simplex on the vertices $x_0, \ldots, x_k$ whenever $x_0 < \cdots < x_k$ in P. See Hatcher's book for background on Δ-complexes [Hat02].

The order complex of $\textsc{Bool}(n)$, for example, is homeomorphic to the cell complex obtained by subdividing the cube $[0,1]^n \subset \mathbb{R}^n$ into $n!$ top-dimensional simplices via the $\binom{n}{2}$ hyperplanes with equations $x_i = x_j$ where $i \neq j$. In fact, we can promote this homeomorphism to an isometry with an appropriate choice of metric for the order complex.

Definition 1.4 (Orthoschemes)**.** The simplex spanned by points $\mathbf{p}_0, \ldots, \mathbf{p}_n$ in $\mathbb{R}^n$ is called an *n-dimensional orthoscheme* if the set of n vectors $\{\mathbf{p}_i - \mathbf{p}_{i-1} \mid i \in [n]\}$ is orthogonal. If those vectors are orthonormal, then the simplex is a *standard n-dimensional orthoscheme*, and it is isometric to the subset of points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ subject to the inequalities $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1$.

For each poset P, we give the order complex $\Delta(P)$ the *orthoscheme metric*, in which each maximal simplex is a standard orthoscheme with the order of its vertices determined by the order of the corresponding maximal chain in P. For more detail on this use of the orthoscheme metric, see [BM10, Sec. 5-6].

To conclude this section, we give a brief remark on the faces of orthoschemes.

Remark 1.5. Each face of a standard n-dimensional orthoscheme is itself an orthoscheme, but one which is not necessarily standard. Using the inequalities $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1$ to describe the standard n-dimensional orthoscheme, each nonempty $A = \{t_0, \ldots, t_k\}$ in $\textsc{Bool}(n+1)$ corresponds to the closed k-dimensional face determined by the equations

$$\begin{aligned} x_i &= 0 \text{ if } 1 \leq i \leq t_0; \\ x_i &= x_{i+1} \text{ if } t_j < i < i+1 \leq t_{j+1} \text{ for some } j \in \{0, 1, \ldots, k-1\}; \\ x_i &= 1 \text{ if } t_k < i \leq n. \end{aligned}$$

In plain language, this face is the set of points such that the first t_0 coordinates are equal to 0, the next $t_1 - t_0$ coordinates are equal to each other, the following $t_2 - t_1$ coordinates are equal, and so on until we reach a_k coordinates set equal to each other, concluding with the final $n - t_k$ coordinates set equal to 1. Furthermore, this face is isometric to the set of points $(y_1, \ldots, y_k) \in \mathbb{R}^k$ determined by the inequalities $0 \leq \sqrt{a_1} y_1 \leq \sqrt{a_2} y_2 \leq \cdots \leq \sqrt{a_k} y_k \leq 1$. Note in particular that two nonempty subsets $\{t_0, \ldots, t_k\}$ and $\{s_0, \ldots, s_k\}$ label isometric k-dimensional faces if $t_i - t_{i-1} = s_i - s_{i-1}$ for all $i \in \{1, 2, \ldots, k\}$.

2. Interval Complexes and The Dual Braid Complex

In this section, we review the definition and basic properties of interval complexes and the dual braid complex. Throughout the rest of the article (unless otherwise specified), let G be a group and let X be a generating set for G which is closed under conjugation.

Definition 2.1 (Intervals)**.** For each $g \in G$, let $\ell(g)$ denote the length of a minimal factorization of g into elements of X. This induces a partial order on G by declaring that $g \leq h$ if $\ell(g) + \ell(g^{-1}h) = \ell(h)$; in other words, $g \leq h$ if there is a minimal-length factorization of h into elements of X which has a minimal-length factorization of g as a left prefix. We denote the interval from the identity element 1 to g in this poset by $[1, g]$ and observe that the order diagram of $[1, g]$ is the graph obtained by taking the union of all geodesics from 1 to g in the right Cayley graph of G with respect to X. Since every geodesic has the same length, this makes the interval $[1, g]$ into a graded poset with height $\ell(g)$.

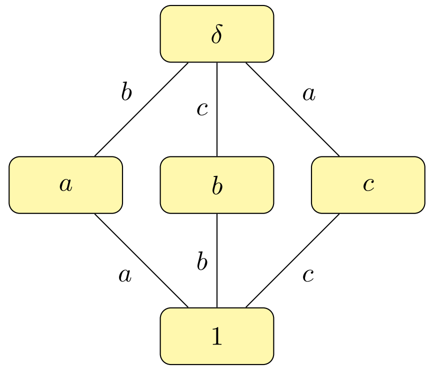

FIGURE 2. The interval $[1, \delta]$ described in Example 2.3, depicted as a subgraph of the Cayley graph $\mathrm{Cay}(\mathrm{SYM}_3, T)$

In this article, we are primarily interested in cases where G is either $\mathbb{Z}$ or SYM_d, but much of what we discuss will generalize to other settings. In a future article, we will discuss the case where G is a Coxeter group with standard generating set S and X is the set of conjugates in S (i.e. reflections) [DMb].

Example 2.2. Let $G = \mathbb{Z}$ with generating set $X = \{1\}$. Then $\ell(n) = |n|$ for all $n \in \mathbb{Z}$ and the induced partial order is the usual one for $\mathbb{Z}$. The interval $[1, n]$ consists of a single chain with n elements.

Example 2.3. Let G be the symmetric group SYM_d and let $X = T$ be the set of all transpositions. Then for all $g \in \mathrm{SYM}_d$, the length $\ell(g)$ is called the *absolute reflection length* and the induced partial order is the *absolute order* on the symmetric group. If we define δ to be the d-cycle $(1\ 2\ \cdots\ d) \in \mathrm{SYM}_d$, then the interval $[1, \delta]$ is sometimes known as the *lattice of noncrossing permutations* for reasons which will be explained in Section 6. The *dual presentation* for the d-strand braid group BRAID_d has as its generating set the nonempty elements of $[1, \delta]$, with relations consisting of all words which arise from closed loops in $[1, \delta]$ which are based at the identity [Bra01, Bes03]. For example, if $\delta = (1\ 2\ 3) \in \mathrm{SYM}_3$ and $T = \{a, b, c\}$ where $a = (1\ 2)$, $b = (2\ 3)$ and $c = (1\ 3)$, then the interval $[1, \delta]$ consists of five elements (see Figure 2) and the dual presentation for BRAID_3 is $\langle a, b, c, \delta \mid ab = bc = ca = \delta \rangle$.

The following lemma demonstrates that $[1, g]$ consists of all elements in G which appear in partial factorizations of g.

Lemma 2.4. *If $x_1, \ldots, x_n \in G$ with $x_1 \cdots x_n = g$ and $\ell(x_1) + \cdots + \ell(x_n) = \ell(g)$, then for any choice of integers $1 \leq i_1 < \cdots < i_k \leq n$, we have $x_{i_1} \cdots x_{i_k} \in [1, g]$.*

Proof. First, we can write $g = w_0 x_{i_1} w_1 \cdots w_{k-1} x_{i_k} w_k$, where $w_0 = x_1 \cdots x_{i_1 - 1}$, $w_n = x_{i_k + 1} \cdots x_n$ and $w_j = x_{i_j + 1} x_{i_j + 2} \cdots x_{i_{j+1}}$ for all $j \in \{1, \ldots, n-1\}$. By assumption, we can see that

$$\ell(g) = \ell(w_0) + \ell(x_{i_1}) + \ell(w_1) + \cdots \ell(w_{k-1}) + \ell(x_{i_k}) + \ell(w_k).$$

If we define $z_j = x_{i_j} \cdots x_{i_k}$ for each j, then we can rearrange the first product to obtain

$$g = x_{i_1} \cdots x_{i_k} (z_1^{-1} w_0 z_1)(z_2^{-1} w_1 z_2) \cdots (z_k^{-1} w_{k-1} z_k) w_k.$$

By definition, we know that $\ell(x_{i_1} \cdots x_{i_k}) \leq \ell(x_{i_1}) + \cdots + \ell(x_{i_k})$, and since the generating set X is closed under conjugation, we have $\ell(z_j^{-1} w_{j-1} z_j) = \ell(w_{j-1})$ for each j. Combining these with the equation above, we obtain

$$\ell(g) = \ell(x_{i_1} \cdots x_{i_k}) + \ell(w_0) + \cdots + \ell(w_n),$$

from which it follows that $x_{i_1} \cdots x_{i_k} \leq g$. □

Definition 2.5 (Dual braid complex)**.** Let $g \in G$. The *interval complex* K_g associated to the interval $[1, g]$ is the single-vertex Δ-complex obtained by identifying faces in the order complex $\Delta([1, g])$ as follows: the k-simplices labeled by chains $x_0 < \cdots < x_k$ and $y_0 < \cdots < y_k$ are identified if and only if $x_{i-1}^{-1} x_i = y_{i-1}^{-1} y_i$ for all $i \in \{1, \ldots, k\}$. Note that this identification is well-defined by the ordering given to each simplex. In the special case when $G = \textsc{Sym}_d$ and $X = T$ as outlined in Example 2.3, we write $\delta = (1\ 2\ \cdots\ d)$ and refer to the interval complex K_δ as the *dual braid complex*[1].

Brady introduced the dual braid complex in [Bra01] and showed that it is a classifying space for the d-strand braid group. Using the orthoscheme metric, it is conjectured that the dual braid complex is locally CAT(0) [BM10].

Remark 2.6. If $h \leq g$, then the order complex $\Delta([1, h])$ is isometric to a subcomplex of $\Delta([1, g])$ and, by following the gluing described in Definition 2.5, we can see that the interval complex K_h is isometric to a subcomplex of K_g. When $G = \textsc{Sym}_d$ and $X = T$, each permutation $\gamma \in \textsc{Sym}_d$ can be written as a product of disjoint cycles $\gamma = x_1 \cdots x_k$ and the interval $[1, \gamma]$ is isomorphic to the product of intervals $[1, x_1] \times \cdots \times [1, x_k]$, so it follows that the interval complex K_γ is a subcomplex of K_δ which is isometric to a product of smaller dual braid complexes.

3. Compositions and Factorizations

Our first goal is to develop a convenient way of describing the cells in the order complex and interval complex. To this end, we define two key posets for each $g \in G$: $\textsc{Fact}(g, \mathbf{I})$, the poset of weak-ended factorizations of g, and $\textsc{Fact}(g, \mathbf{S})$, the poset of circular factorizations of g. Throughout this section (unless otherwise specified), let G be a group with generating set X which is closed under conjugation, and let the length function $\ell\colon G \to \mathbb{Z}$ and interval $[1, g]$ be given as in Definition 2.1.

Definition 3.1 (Weak-ended factorizations)**.** Let $g \in G$. A *weak-ended factorization of* g is a row vector $\mathbf{x} = [x_0\ \cdots\ x_{k+1}]$ where $x_0, \ldots, x_{k+1}$ are elements of G such that

(1) x_i is nontrivial when $i \in \{1, \ldots, k\}$;
(2) $\ell(x_0) + \ell(x_1) + \cdots + \ell(x_k) + \ell(x_{k+1}) = \ell(g)$;
(3) $x_0 x_1 \cdots x_k x_{k+1} = g$.

For each $i \in \{0, 1, \ldots, k\}$, a weak-ended factorization

$$[x_0\ \cdots\ x_{i-1}\ x_i\ x_{i+1}\ x_{i+2} \cdots\ x_{k+1}]$$

of length $k+2$ can be *merged* at position i to obtain the weak-ended factorization

$$[x_0\ \cdots\ x_{i-1}\ (x_i x_{i+1})\ x_{i+2} \cdots\ x_{k+1}]$$

[1] In previous work of the authors, "dual braid complex" referred to the universal cover of the space described above rather than the quotient. Here, we use the term to refer to the single-vertex quotient since this space is of greater use in the present article.

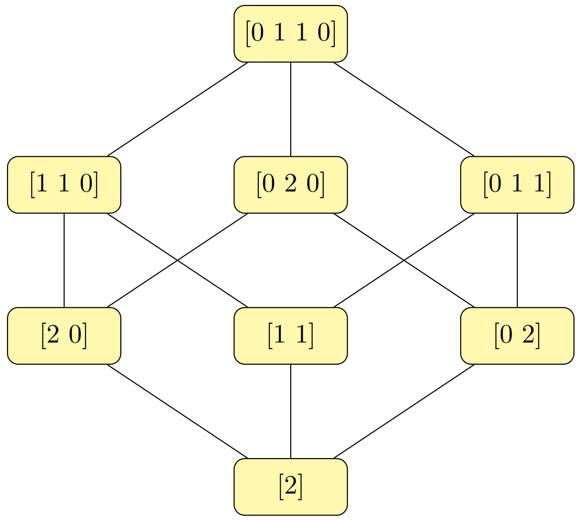

FIGURE 3. The poset $\textsc{Comp}(2, \mathbf{I})$ of weak-ended compositions of 2

of length $k+1$. Let $\textsc{Fact}(g, \mathbf{I})$ denote the set of all weak-ended factorizations of g, equipped with the partial order $\mathbf{x} \leq \mathbf{y}$ if $\mathbf{x}$ can be obtained from $\mathbf{y}$ by a sequence of merges. This partially ordered set is a meet-semilattice with minimal element $[g]$. Let $\textsc{Fact}^*(g, \mathbf{I})$ denote the subposet of all weak-ended factorizations with length at least 2.

By Lemma 2.4, we see that the elements of $\textsc{Sym}_d$ which appear in weak-ended factorizations of g are precisely those which belong to the interval $[1, g]$. We also note that weak-ended factorizations are closely related to the "reduced products" in [McC] and the "block factorizations" in [Rip12], but with the slight variation that the first and last entries are permitted to be trivial.

Example 3.2 ($\textsc{Comp}(n, \mathbf{I})$)**.** In the special case when $G = \mathbb{Z}$ and $X = \{1\}$, we refer to the weak-ended factorizations of $n \in \mathbb{Z}$ as *weak-ended compositions*, the set of which is denoted $\textsc{Comp}(n, \mathbf{I})$. Unlike the more general case, $\textsc{Comp}(n, \mathbf{I})$ is a lattice with maximum element given by the row vector $[0\ 1\ \cdots\ 1\ 0]$ of length $n+2$. See Figure 3 for an example when $n = 2$.

Example 3.3 ($\textsc{Fact}(\delta, \mathbf{I})$)**.** If $G = \textsc{Sym}_3$ and $X = \{a, b, c\}$ as in Example 2.3, then $\textsc{Fact}(\delta, \mathbf{I})$ contains 16 elements, illustrated in Figure 4.

Proposition 3.4. *Let $g \in G$. Then $\textsc{Fact}(g, \mathbf{I})$ is isomorphic to the poset of chains for $[1, g]$, ordered by inclusion.*

Proof. Let f be the function which sends $[g]$ to the empty chain and, more generally, the weak-ended factorization $[x_0\ \cdots\ x_{k+1}]$ to the chain

$$x_0 \leq x_0x_1 \leq x_0x_1x_2 \leq \cdots \leq x_0x_1\cdots x_k,$$

noting that $x_0x_1\cdots x_i \in [1, g]$ for each i. Then f has an inverse which takes the empty chain to $[g]$ and, more generally, the chain $y_0 < y_1 < \cdots < y_k$ to the weak-ended factorization $[y_0\ \ y_0^{-1}y_1\ \cdots\ y_{k-1}^{-1}y_k\ \ y_k^{-1}g]$. Moreover, merging

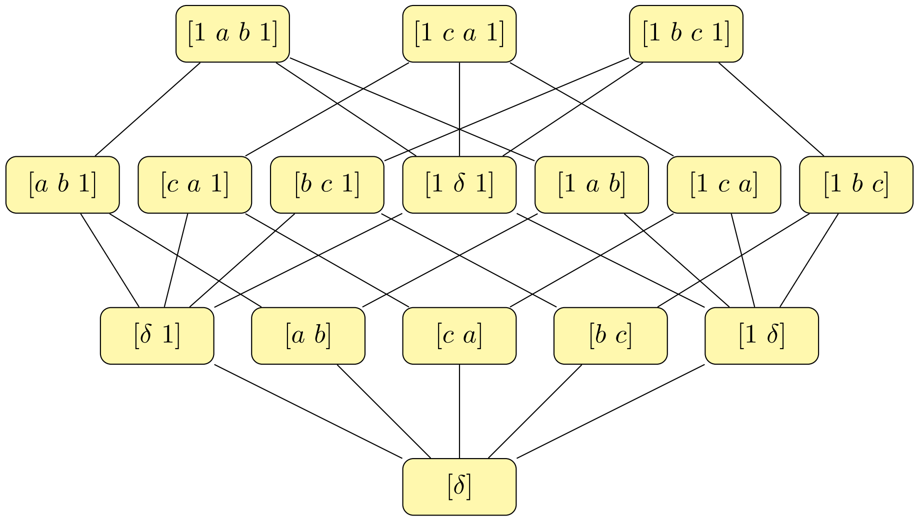

FIGURE 4. The poset $\text{FACT}(\delta, \mathbf{I})$ of weak-ended factorizations of the element $\delta \in \text{SYM}_3$ with respect to the generating set of transpositions $\{a, b, c\}$ — see Example 3.3.

factorizations in $\text{FACT}(g, \mathbf{I})$ corresponds exactly to taking subchains in $[1, g]$, so f is an order-preserving bijection with order-preserving inverse and thus the two posets are isomorphic. $\square$

Corollary 3.5. $\text{COMP}(n, \mathbf{I})$ *is isomorphic to the Boolean lattice* $\text{BOOL}(n + 1)$.

By identifying the ends of a weak-ended factorization, we obtain a new object which we call a circular factorization.

Definition 3.6 (Circular factorizations)**.** Let $g \in G$. Define an equivalence relation on $\text{FACT}(g, \mathbf{I})$ by declaring $\mathbf{x} = [x_0\ x_1\ \cdots\ x_k\ x_{k+1}]$ and $\mathbf{y} = [y_0\ y_1\ \cdots\ y_k\ y_{k+1}]$ to be equivalent if and only if $x_i = y_i$ for all $i \in \{1, \ldots, k\}$. We refer to the equivalence classes as *circular factorizations of* g, each of which is represented by the unique element with 1 as its final entry. More concretely, the equivalence class of $\mathbf{x}$ is denoted $\overline{\mathbf{x}} = [gx_{k+1}g^{-1}x_0 \mid x_1\ \cdots\ x_k \mid 1]$. Let $\text{FACT}(g, \mathbf{S})$ denote the set of all circular factorizations under the partial order $\overline{\mathbf{x}} \leq \overline{\mathbf{y}}$ if $\mathbf{x}' \leq \mathbf{y}'$ for some $\mathbf{x}' \in \overline{\mathbf{x}}$ and $\mathbf{y}' \in \overline{\mathbf{y}}$, and let $\text{FACT}^*(g, \mathbf{S})$ be the subposet with the minimum element $[g \mid\mid 1]$ removed. Define the order-preserving surjection $q\colon \text{FACT}(g, \mathbf{I}) \to \text{FACT}(g, \mathbf{S})$ by sending each weak-ended factorization to the equivalence class which contains it, i.e. $q(\mathbf{x}) = \overline{\mathbf{x}}$.

Example 3.7 ($\text{COMP}(n, \mathbf{S})$)**.** When $G = \mathbb{Z}$ and $X = \{1\}$, we denote the set $\text{FACT}(g, \mathbf{S})$ by $\text{COMP}(n, \mathbf{S})$ and refer to its elements as *circular compositions of* n. When $n = 2$, for example, the poset $\text{COMP}(2, \mathbf{S})$ of circular compositions of 2 has four elements and is isomorphic to $\text{BOOL}(2)$ — see Figure 5. More generally, it is important to note that while there is a rank-preserving bijection between $\text{COMP}(n, \mathbf{S})$ and $\text{BOOL}(n)$, the two are not isomorphic when $n > 2$ since there are relations in the former which do not appear in the latter.

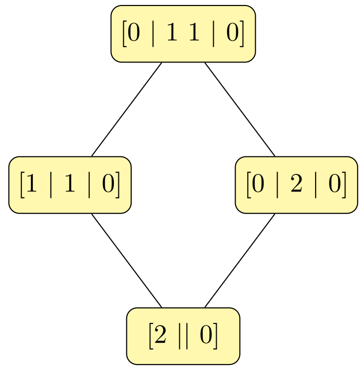

FIGURE 5. The poset $\textsc{Comp}(2, \mathbf{S})$ of circular compositions of 2

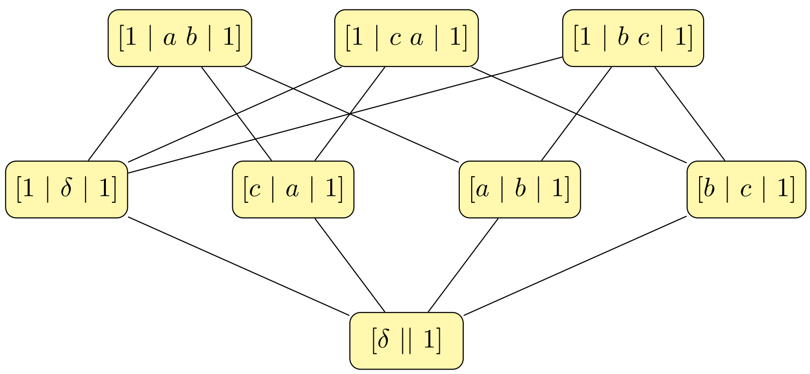

FIGURE 6. The poset $\textsc{Fact}(\delta, \mathbf{S})$ of circular factorizations of the element $\delta \in \textsc{Sym}_3$ with respect to the generating set of transpositions $\{a, b, c\}$ — see Example 3.8.

Example 3.8 ($\textsc{Fact}(\delta, \mathbf{S})$)**.** When $G = \textsc{Sym}_3$, $X = T = \{a, b, c\}$, and $\delta = (1\ 2\ 3)$ as in Example 2.3, the poset $\textsc{Fact}(\delta, \mathbf{S})$ has 8 elements — see Figure 6.

We close this section by providing combinatorial connections between the posets introduced in this section. See Figure 7 for the relevant commutative diagram.

Definition 3.9 (Factorizations to compositions)**.** Let $g \in G$ with $n = \ell(g)$ and define the function $L \colon \textsc{Fact}(g, \mathbf{I}) \to \textsc{Comp}(n, \mathbf{I})$ by sending the factorization $\mathbf{x} = [x_0\ x_1\ \cdots\ x_k\ x_{k+1}]$ to the composition $L(\mathbf{x}) = [\ell(x_0)\ \ell(x_1)\ \cdots\ \ell(x_k)\ \ell(x_{k+1})]$. Similarly, define $\overline{L} \colon \textsc{Fact}(g, \mathbf{S}) \to \textsc{Comp}(n, \mathbf{S})$ by sending $\overline{\mathbf{x}} = [x_0 \mid x_1\ \cdots\ x_k \mid 1]$ to $L(\overline{\mathbf{x}}) = [\ell(x_0) \mid \ell(x_1)\ \cdots\ \ell(x_k) \mid 0]$. Note that if $\mathbf{x}'$ is obtained from $\mathbf{x}$ by performing a merge in position i, then we know $\ell(x_i x_{i+1}) = \ell(x_i) + \ell(x_{i+1})$ by Lemma 2.4, and this means that $L(\mathbf{x}')$ is obtained from $L(\mathbf{x})$ by merging at position i. By similar reasoning, we know that if $\overline{\mathbf{x}'} \leq \overline{\mathbf{x}}$, then $\overline{L}(\overline{\mathbf{x}'}) \leq \overline{L}(\overline{x})$. Thus L and $\overline{L}$ are both order-preserving functions.

Lemma 3.10. *Let $\mathbf{x} \in \textsc{Fact}(g, \mathbf{I})$. Then L restricts to an isomorphism from $\downarrow(\mathbf{x})$ to $\downarrow(L(\mathbf{x}))$.*

$$\begin{tikzcd} \textsc{Fact}(g,\mathbf{I}) \arrow[r, "L"] \arrow[d, "q"'] & \textsc{Comp}(n,\mathbf{I}) \arrow[d, "q"] \\ \textsc{Fact}(g,\mathbf{S}) \arrow[r, "\overline{L}"] & \textsc{Comp}(n,\mathbf{S}) \end{tikzcd}$$

Figure 7. Our four key posets fit into a commutative diagram.

Proof. As described in Definition 3.9, each element of $\downarrow(\mathbf{x})$ is obtained from $\mathbf{x}$ via a sequence of merges, and the same merges can be performed on $L(\mathbf{x})$ to obtain an element of $\downarrow(L(\mathbf{x}))$. Two different sequences of merges produce the same element of $\downarrow(\mathbf{x})$ if and only if the analogous sequence produces the same element of $\downarrow(L(\mathbf{x}))$, so L restricts to a bijection and thus an isomorphism. □

Note that Lemma 3.10 does not hold for $\overline{L}$, as it is possible to have elements $\overline{\mathbf{x}} \in \textsc{Fact}(g,\mathbf{S})$ such that the restriction of $\overline{L}$ to $\downarrow(\overline{\mathbf{x}})$ yields an order-preserving surjection, but not an injection. See Figure 6 for an example.

Proposition 3.11. *The functions L and $\overline{L}$ are surjective order-preserving maps and $\overline{L}q = qL$.*

Proof. By Definition 3.9, we know the two maps are order-preserving. Also, if $\mathbf{x}$ is a maximal element of $\textsc{Fact}(g,\mathbf{I})$, then $L(\mathbf{x}) = [0\ 1\ \cdots\ 1\ 0]$, the maximum element of $\textsc{Comp}(n,\mathbf{I})$. By Lemma 3.10, we know that L restricts to an isomorphism from $\downarrow(\mathbf{x})$ to $\downarrow(L(\mathbf{x})) = \textsc{Comp}(n,\mathbf{I})$, so the (unrestricted) map L is surjective. The fact that $\overline{L}q = qL$ follows directly from Definition 3.6 and Example 3.7. Since q and L are surjective, we know qL and thus $\overline{L}q$ are surjective maps, which finally allows us to conclude that $\overline{L}$ is surjective. □

Proposition 3.12. *The poset $\textsc{Fact}(g,\mathbf{I})$ is* simplicial*, i.e. each of its intervals is isomorphic to a Boolean lattice.*

Proof. For each $\mathbf{x},\mathbf{y} \in \textsc{Fact}(g,\mathbf{I})$ with $\mathbf{x} \le \mathbf{y}$, choose a maximal element $\mathbf{z}$ with $[g] \le \mathbf{x} \le \mathbf{y} \le \mathbf{z}$. Then the interval $[\mathbf{x},\mathbf{y}]$ is contained within the larger interval $[[g],\mathbf{z}]$, which we know by Lemma 3.10 is isomorphic to $\textsc{Comp}(n,\mathbf{I})$, which is itself a Boolean lattice. Since every interval of a Boolean lattice is isomorphic to a smaller Boolean lattice, the proof is complete. □

4. Weighted Compositions and Weighted Factorizations

In this section, we introduce a way of labeling points in the order complex and interval complex. For each $g \in G$, we define $\textsc{WFact}(g,\mathbf{I})$, the space of weighted weak-ended factorizations of g, and $\textsc{WFact}(g,\mathbf{S})$, the space of weighted circular factorizations of g. The points in each space are defined as weighted versions of poset elements from the previous section and can be viewed as decorated multisets in either $\mathbf{I}$ or $\mathbf{S}$.

Definition 4.1 (G-multisets)**.** Let S be a set and let G be a group. We define a *G-multiset* on S to be a function $\mathbf{x}\colon S \to G$ such that $\mathbf{x}(s)$ is the identity in

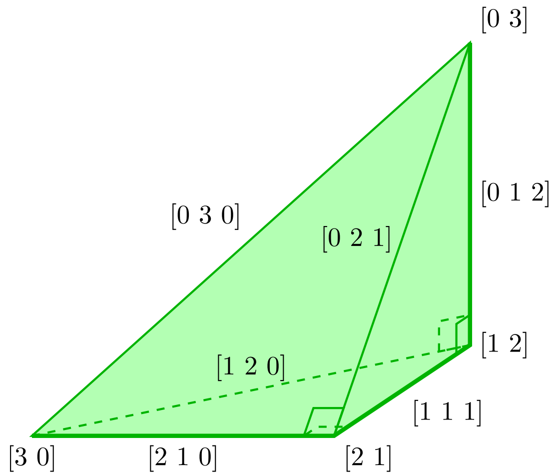

FIGURE 8. The space $\text{WComp}(3, \mathbf{I})$, with vertices and edges labeled

G for all but finitely many $s \in S$. We denote the set of all G-multisets on S by $\text{Mult}(G, S)$.

We are interested in four cases which arise from two choices: S is either the unit interval or the circle, and G is either $\mathbb{Z}$ or Sym_d. First, we consider the interval.

Definition 4.2 ($\text{WFact}(g, \mathbf{I})$)**.** Let $g \in G$. For each G-multiset $\mathbf{u} \colon \mathbf{I} \to G$, let $0 = s_0 < s_1 < \cdots < s_k < s_{k+1} = 1$ be such that $\mathbf{u}$ is nontrivial on the set $\{s_1, \ldots, s_k\}$ and trivial on its complement in $(0, 1)$. For each i, let $x_i = \mathbf{u}(s_i)$ and define $P(\mathbf{u}) = [x_0\ x_1\ \cdots\ x_k\ x_{k+1}]$; we say that $\mathbf{u}$ is a *weighted weak-ended factorization* of g if $P(\mathbf{u})$ is a weak-ended factorization of g. Note that $P(\mathbf{u})$ necessarily has length at least 2. We will often use the convenient shorthand $\mathbf{u} = 0^{x_0} s_1^{x_1} \cdots s_k^{x_k} 1^{x_{k+1}}$ to denote elements of $\text{WFact}(g, \mathbf{I})$. Denote the set of all weighted weak-ended factorizations of g by $\text{WFact}(g, \mathbf{I})$ and observe that P is a surjective function from $\text{WFact}(g, \mathbf{I})$ to $\text{Fact}^*(g, \mathbf{I})$.

Example 4.3 ($\text{WComp}(n, \mathbf{I})$)**.** When $G = \mathbb{Z}$ and $X = \{1\}$, we denote the set $\text{WFact}(n, \mathbf{I})$ by $\text{WComp}(n, \mathbf{I})$ and refer to its elements as *weighted weak-ended compositions of* n. If we consider the action of Sym_n on the n-cube $\mathbf{I}^n$ by permuting coordinates, then each element $\mathbf{s} = 0^{a_0} s_1^{a_1} \cdots s_k^{a_k} 1^{a_{k+1}}$ in $\text{WComp}(n, \mathbf{I})$ can be viewed as a point on the quotient space $\mathbf{I}^n / \text{Sym}_n$, which is isometric to a standard n-dimensional orthoscheme. For each $\mathbf{a} \in \text{Comp}(n, \mathbf{I})$, the set of weighted weak-ended compositions $\mathbf{s}$ of n with $P(\mathbf{s}) = \mathbf{a}$ forms an open face of the standard orthoscheme which we refer to as a (non-standard) *orthoscheme of shape* $\boldsymbol{a}$. This recovers what we found in Corollary 3.5: $\text{Comp}^*(n, \mathbf{I})$ is the face poset for the standard n-dimensional orthoscheme. Moreover, Remark 1.5 tells us that elements $\mathbf{a} = [a_0 \cdots a_{k+1}]$ and $\mathbf{b} = [b_0 \cdots b_{k+1}]$ in $\text{Comp}^*(n, \mathbf{I})$ label isometric faces of $\text{WComp}(n, \mathbf{I})$ if $a_i = b_i$ for all $i \in \{1, \ldots, k\}$; see Figure 8.

Using this example, we can pull back the metric on $\text{WComp}(n, \mathbf{I})$ to provide a piecewise-Euclidean metric for $\text{WFact}(g, \mathbf{I})$.

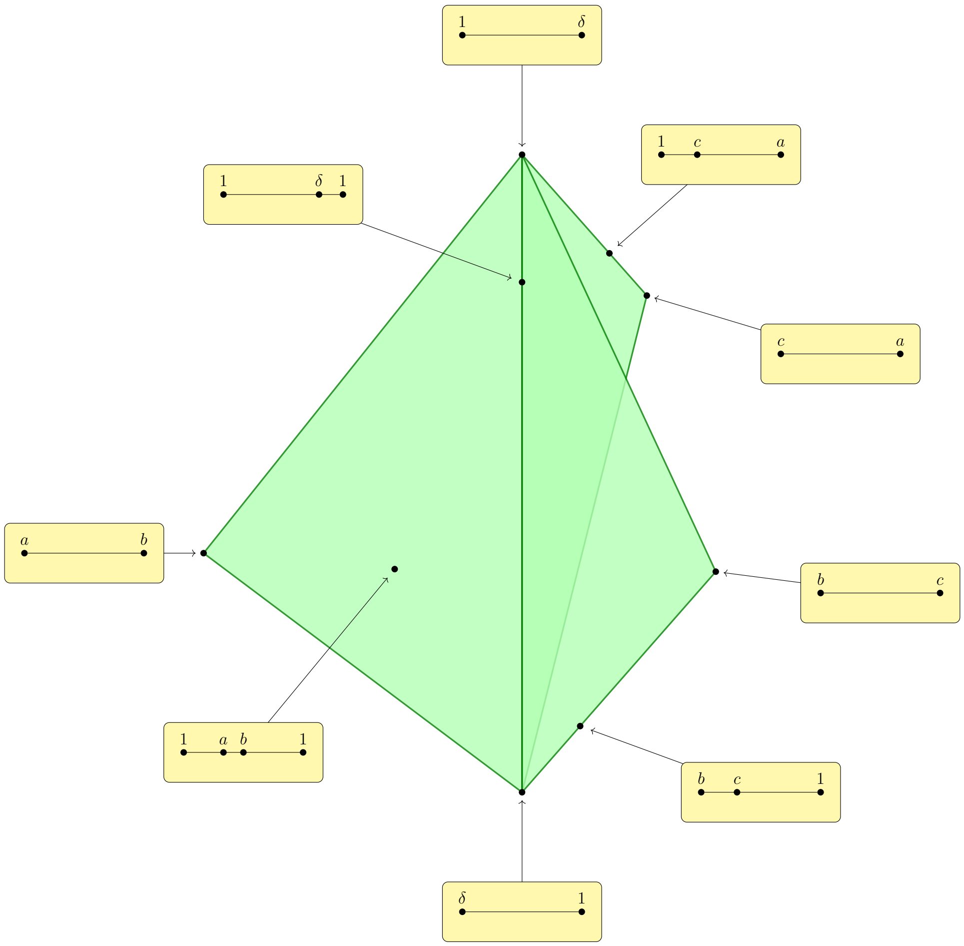

FIGURE 9. The order complex of $[1, \delta]$, where $\delta = ab = bc = ca$ in SYM_3 as described in Example 2.3. The cells are labeled by elements of $\text{FACT}(g, \mathbf{I})$ and the points are labeled by elements of $\text{WFACT}(g, \mathbf{I})$.

Definition 4.4 (Pullback metric)**.** Let $g \in G$ with $\ell(g) = n$. Abusing notation, we define the function $L\colon \text{WFACT}(g, \mathbf{I}) \to \text{WCOMP}(n, \mathbf{I})$ by $L(\mathbf{u}) = \ell \circ \mathbf{u}$ and observe that $PL = LP$, where the second occurrence of "L" denotes the function from $\text{FACT}(g, \mathbf{I})$ to $\text{COMP}(n, \mathbf{I})$ given in Definition 3.9. For each $\mathbf{x} \in \text{FACT}(g, \mathbf{I})$, the set $P^{-1}(\mathbf{x}) = \{\mathbf{u} \in \text{WFACT}(g, \mathbf{I}) \mid P(\mathbf{u}) = \mathbf{x}\}$ is sent bijectively via L to the set $P^{-1}(L(\mathbf{x})) = \{\mathbf{s} \in \text{WCOMP}(n, \mathbf{I}) \mid P(\mathbf{s}) = L(\mathbf{x})\}$, so we can pull back the metric and identify $P^{-1}(\mathbf{x})$ with an open orthoscheme of shape $L(\mathbf{x})$. By Lemma 3.10, we know that the closure of this open orthoscheme is indeed the closed orthoscheme we would expect, so this endows $\text{WFACT}(g, \mathbf{I})$ with the structure of a piecewise-Euclidean Δ-complex with $\text{FACT}(g, \mathbf{I})$ as its face poset.

Example 4.5 ($\text{WFACT}(\delta, \mathbf{I})$)**.** If $G = \text{SYM}_3$, $T = \{a, b, c\}$ and $\delta = (1\ 2\ 3)$ as in Example 2.3, then $\text{WFACT}(\delta, \mathbf{I})$ is a 2-dimensional simplicial complex which consists of three right isosceles triangles, all sharing a common hypotenuse. The three triangles, seven edges, and five vertices are labeled by the elements of $\textsc{Fact}(\delta, \mathbf{I})$; see Figures 4 and 9. Finally, note that the function L from $\mathrm{W}\textsc{Fact}(\delta, \mathbf{I})$ to $\mathrm{W}\textsc{Comp}(2, \mathbf{I})$ is a branched covering map with branch points on the shared hypotenuse.

Proposition 4.6. *The order complex of* $[1, g]$ *is isometric to* $\mathrm{W}\textsc{Fact}(g, \mathbf{I})$.

Proof. This follows immediately from Proposition 3.4 and Example 4.5. □

Identifying the endpoints of $\mathbf{I}$ to produce the circle $\mathbf{S}$ yields a "circular" quotient of $\mathrm{W}\textsc{Fact}(g, \mathbf{I})$.

Definition 4.7 ($\mathrm{W}\textsc{Fact}(g, \mathbf{S})$)**.** The equivalence relation given in Definition 3.6 transforms $\textsc{Fact}(g, \mathbf{I})$ into $\textsc{Fact}(g, \mathbf{S})$, and this determines a quotient of the cell complex $\mathrm{W}\textsc{Fact}(g, \mathbf{I})$ by isometrically identifying faces. As described in Definition 1.4, each simplex in the order complex comes with an ordering of its vertices which determines the metric, and this information determines the gluing orientation. We refer to this quotient as the space of *weighted circular factorizations of* g, denoted $\mathrm{W}\textsc{Fact}(g, \mathbf{S})$. Each point $\overline{\mathbf{u}}$ in this space can be viewed as a G-multiset $\mathbf{S} \to G$ and uniquely represented as $\overline{\mathbf{u}} = 0^{x_0} s_1^{x_1} \cdots s_k^{x_k} 1^1$, where $q(\overline{\mathbf{u}})$ is the circular factorization $[x_0 \mid x_1 \ \cdots \ x_k \mid 1]$. It follows that $\textsc{Fact}^*(g, \mathbf{S})$ is the face poset for $\mathrm{W}\textsc{Fact}(g, \mathbf{S})$.

Example 4.8 ($\mathrm{W}\textsc{Comp}(n, \mathbf{S})$)**.** When $G = \mathbb{Z}$ and $X = \{1\}$, we denote the set $\mathrm{W}\textsc{Fact}(n, \mathbf{S})$ by $\mathrm{W}\textsc{Comp}(n, \mathbf{S})$ and refer to its elements as *weighted circular compositions of* n. As discussed in Example 4.3, $\mathrm{W}\textsc{Comp}(n, \mathbf{I})$ is isometric to a standard n-dimensional orthoscheme, so $\mathrm{W}\textsc{Comp}(n, \mathbf{S})$ is obtained by identifying faces of an orthoscheme according to the map $q \colon \textsc{Comp}(n, \mathbf{I}) \to \textsc{Comp}(n, \mathbf{S})$ given in Definition 3.6. To give another way of viewing this identification, the inequalities $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_1 + 1$ define a topological subspace of $\mathbb{R}^n$ called a *column* which is isometric to the product of $\mathbb{R}$ and an $(n-1)$-simplex (more specifically, a Coxeter simplex of type $\widetilde{A}_{n-1}$ — see [BM10, Section 8] and [DMW20, Section 8]). The infinite cyclic group generated by the isometry $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(x_1, x_2, \ldots, x_n) = (x_2, \ldots, x_n, x_1 + 1)$ acts freely on the column with a fundamental domain which is isometric to the standard n-dimensional orthoscheme. The quotient by this action is $\mathrm{W}\textsc{Comp}(n, \mathbf{S})$.

Example 4.9 ($\mathrm{W}\textsc{Fact}(\delta, \mathbf{S})$)**.** If $G = \textsc{Sym}_3$, $T = \{a, b, c\}$ and $\delta = (1\ 2\ 3)$ as in Example 4.5, then $\mathrm{W}\textsc{Fact}(\delta, \mathbf{S})$ is obtained from $\mathrm{W}\textsc{Fact}(\delta, \mathbf{I})$ by identifying edges; see Figure 10.

Proposition 4.10. *The interval complex for* $[1, g]$ *is isometric to* $\mathrm{W}\textsc{Fact}(g, \mathbf{S})$.

Proof. The interval complex for $[1, g]$ is obtained from the order complex by identifying the faces labeled by chains $x_0 < \cdots < x_k$ and $y_0 < \cdots < y_k$ if and only if $x_{i-1}^{-1} x_i = y_{i-1}^{-1} y_i$ for all $i \in \{1, \ldots, k\}$. By Proposition 3.4, these faces are labeled by the weak-ended factorizations

$$[x_0 \ \ x_0^{-1} x_1 \ \cdots \ x_{k-1}^{-1} x_k \ \ x_k^{-1} g]$$

and

$$[y_0 \ \ y_0^{-1} y_1 \ \cdots \ y_{k-1}^{-1} y_k \ \ y_k^{-1} g],$$

so the identification of faces in constructing the interval complex is identical to the equivalence relation established on $\textsc{Fact}(g, \mathbf{I})$ when defining $\textsc{Fact}(g, \mathbf{S})$. Since

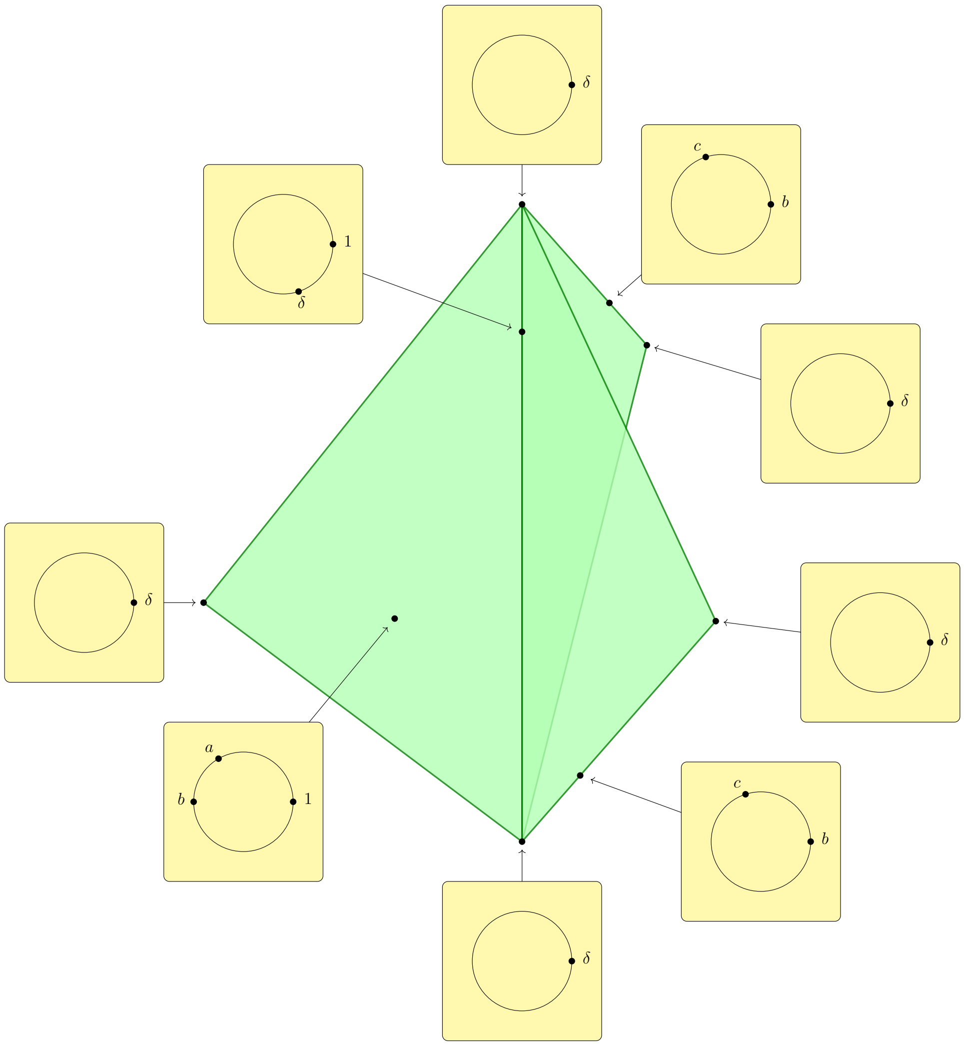

FIGURE 10. Gluing the cells of the order complex $\mathrm{WFACT}(\delta, \mathbf{I})$ as described in Definition 4.7 yields the interval complex $\mathrm{WFACT}(\delta, \mathbf{S})$. Points with the same label are glued together, which means that all five vertices are identified and the short edges are identified in pairs. As before, $\delta = ab = bc = ca$ in SYM_3 as described in Example 2.3.

this equivalence relation dictates the identification of faces in $\mathrm{WFACT}(g, \mathbf{I})$ when constructing $\mathrm{WFACT}(g, \mathbf{S})$ and $\mathrm{WFACT}(g, \mathbf{I})$ is isometric to $\Delta([1, g])$ by Proposition 4.6, the proof is complete. □

5. Topological Graded Posets

We now define two graded posets, each with a natural topology: $\mathcal{F}(g, \mathbf{I})$, the poset of weighted weak-ended factorizations of elements in $[1, g]$, and $\mathcal{F}(g, \mathbf{S})$, the set of weighted circular factorizations of elements in $[1, g]$. With these in hand, we then prove Theorems B and C.

Definition 5.1 ($\mathcal{F}(g, \mathbf{I})$)**.** For each $g \in G$, define $\mathcal{F}(g, \mathbf{I})$ to be the disjoint union of topological spaces

$$\bigsqcup_{h \in [1,g]} \mathrm{WF{\scriptsize ACT}}(h, \mathbf{I}).$$

This space comes with a partial order: for $\mathbf{u}, \mathbf{v} \in \mathcal{F}(g, \mathbf{I})$, we say that $\mathbf{v}$ is a *subfactorization* of $\mathbf{u}$ and write $\mathbf{v} \subseteq \mathbf{u}$ if $\mathbf{v}(r) \leq \mathbf{u}(r)$ in G for all $r \in I$. Define $\rho \colon \mathcal{F}(g, \mathbf{I}) \to G$ by $\rho(\mathbf{u}) = \prod_{r \in [0,1]} \mathbf{u}(r)$, where the factors are arranged from left to right in increasing order of r and note that the product is well-defined since all but finitely many factors are trivial. Then $\mathcal{F}(g, \mathbf{I})$ is a graded poset of height $\ell(g)$ with rank function $\ell \circ \rho$.

Definition 5.2 ($\mathcal{F}(g, \mathbf{S})$)**.** Define an equivalence relation on $\mathcal{F}(g, \mathbf{I})$ by declaring $\mathbf{u} = 0^{x_0} s_1^{x_1} \cdots s_k^{x_k} 1^{x_{k+1}}$ and $\mathbf{v} = 0^{y_0} s_1^{y_1} \cdots s_k^{y_k} 1^{y_{k+1}}$ if $x_i = y_i$ for all $i \in \{1, \ldots, k\}$ and $g x_{k+1} g^{-1} x_0 = g y_{k+1} g^{-1} y_0$. We denote the set of all such equivalence classes by $\mathcal{F}(g, \mathbf{S})$ and observe that it inherits the partial order by subfactorizations from $\mathcal{F}(g, \mathbf{I})$. In particular, each equivalence class can be uniquely represented by an element of the form $\mathbf{w} = 0^{z_0} s_1^{z_1} \cdots s_k^{z_k} 1^1$, and applying $\ell \circ \rho$ to these representatives provides a rank function for $\mathcal{F}(g, \mathbf{S})$. Finally, note that the quotient map $q \colon \mathcal{F}(g, \mathbf{I}) \to \mathcal{F}(g, \mathbf{S})$ given by $q(0^{x_0} s_1^{x_1} \cdots s_k^{x_k} 1^{x_{k+1}}) = 0^{g x_{k+1} g^{-1} x_0} s_1^{x_1} \cdots s_k^{x_k} 1^1$ is an order-preserving surjection.

Some words of caution are required when dealing with $\mathcal{F}(g, \mathbf{S})$. It is possible that $\mathbf{u}$ and $\mathbf{v}$ could belong to the same equivalence class even though $\rho(\mathbf{u}) \neq \rho(\mathbf{v})$, which is why we need to take care when defining the rank function above. To give a small example, the weighted weak-ended factorizations $[1\ a]$ and $[gag^{-1}\ 1]$ belong to the same equivalence class, but will have different values of ρ if a and g do not commute. Similarly, the maximal elements of $\mathcal{F}(g, \mathbf{I})$ form the set $\mathrm{WF{\scriptsize ACT}}(g, \mathbf{I})$ and the maximal elements of $\mathcal{F}(g, \mathbf{S})$ form the quotient $\mathrm{WF{\scriptsize ACT}}(g, \mathbf{S})$, but this pattern does not continue into the lower ranks. The elements of rank k in $\mathcal{F}(g, \mathbf{I})$ form the disjoint union of spaces $\mathrm{WF{\scriptsize ACT}}(h, \mathbf{I})$ with $h \in [1, g]$ and $\ell(h) = k$, whereas the elements of rank k may not be a disjoint union of $\mathrm{WF{\scriptsize ACT}}(h, \mathbf{S})$ spaces. See Example 5.5.

Moving on, both $\mathcal{F}(g, \mathbf{I})$ and $\mathcal{F}(g, \mathbf{S})$ combine the structure of a graded poset with a topology in each rank. Without making a general definition, we refer to these as "topological graded posets" and remark that a similar class of posets was considered by Živaljević in [Ž98]. To give a well-known example, the set $\mathcal{L}_n(\mathbb{R})$ of linear subspaces in $\mathbb{R}^n$ is partially ordered by inclusion, has a rank function which sends each subspace to its dimension, and can be topologically viewed as the disjoint union of the Grassmannians $\mathrm{Gr}(i, \mathbb{R}^n)$, where $i \in \{0, \ldots, n\}$.

Remark 5.3. It is straightforward to compute the lower sets of elements in $\mathcal{F}(g, \mathbf{I})$ and $\mathcal{F}(g, \mathbf{S})$. For example, if $\mathbf{u} \in \mathcal{F}(g, \mathbf{I})$ with $\mathbf{u} = 0^{x_0} s_1^{x_1} \cdots s_k^{x_k} 1^{x_{k+1}}$, then the lower set $\downarrow(\mathbf{u})$ is isomorphic to the product $[1, x_0] \times [1, x_1] \times \cdots \times [1, x_k] \times [1, x_{k+1}]$ in G. Similarly, if $\overline{\mathbf{u}} \in \mathcal{F}(g, \mathbf{S})$ with $\overline{\mathbf{u}} = 0^{x_0} s_1^{x_1} \cdots s_k^{x_k} 1^1$, then $\downarrow(\overline{\mathbf{u}})$ is isomorphic to the product $[1, x_0] \times [1, x_1] \times \cdots \times [1, x_k]$.

Example 5.4. When $G = \mathbb{Z}$ and $X = \{1\}$, we denote $\mathcal{F}(n, \mathbf{I})$ and $\mathcal{F}(n, \mathbf{S})$ by $\mathcal{C}(n, \mathbf{I})$ and $\mathcal{C}(n, \mathbf{S})$ respectively and refer to subfactorizations as *subcompositions*.

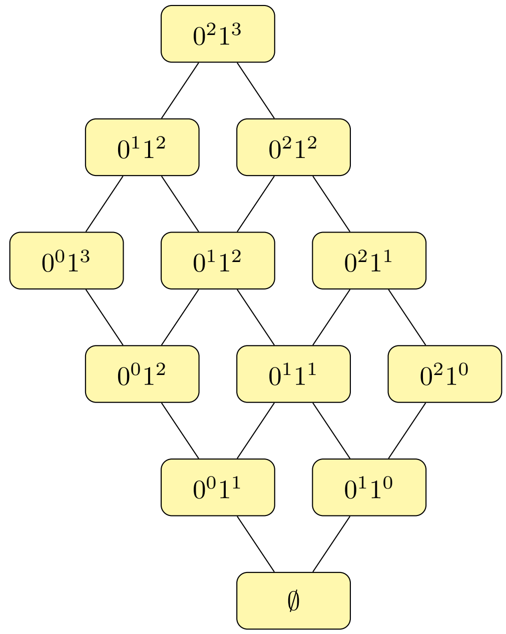

FIGURE 11. The lower set $\downarrow(\mathbf{s})$ in $\mathcal{C}(5, \mathbf{I})$, where $\mathbf{s} = 0^2 1^3$

In this case, the elements of rank k in $\mathcal{C}(n, \mathbf{I})$ are precisely those in $\mathrm{WC{\small OMP}}(k, \mathbf{I})$ — see Figure 12 for an example when $n = 3$. Similarly, the elements of rank k in $\mathcal{C}(n, \mathbf{S})$ form $\mathrm{WC{\small OMP}}(k, \mathbf{S})$. Thus, we can think of $\mathcal{C}(n, \mathbf{I})$ (or $\mathcal{C}(n, \mathbf{S})$) as consisting of one k-dimensional orthoscheme (or orthoscheme quotient) at rank k, for all $k \in \{0, \ldots, n\}$. Note that if $\mathbf{s} = 0^{a_0} s_1^{a_1} \cdots s_k^{a_k} 1^{a_{k+1}}$, then the lower set $\downarrow(\mathbf{s})$ is isomorphic to the product of path posets with lengths $a_0, \ldots, a_{k+1}$. In particular, if $\mathbf{s} = 0^1 s_1^1 \cdots s_k^1 1^1$, then $\downarrow(\mathbf{s})$ is isomorphic to $\mathrm{B{\small OOL}}(k+2)$. See Figure 11.

Example 5.5. Let $G = \mathrm{S{\small YM}}_d$, let $X = T$ and let $\delta = (1\ 2\ \cdots\ d)$ as in Example 2.3. Then $\mathcal{F}(\delta, \mathbf{I})$ is the disjoint union of order complexes for the intervals $[1, \gamma]$, where $\gamma \leq \delta$ in $\mathrm{S{\small YM}}_d$, and the maximal elements form $\mathrm{WF{\small ACT}}(\delta, \mathbf{I})$, the order complex of $[1, \delta]$. In $\mathcal{F}(\delta, \mathbf{S})$, the maximal elements form $\mathrm{WF{\small ACT}}(\delta, \mathbf{S})$, the interval complex of $[1, \delta]$, but the lower-rank structure is more complicated. For example, when $d = 3$ and $\delta = ab = bc = ca$ as in Example 2.3, the elements of rank 1 in $\mathcal{F}(\delta, \mathbf{I})$ can be viewed as the disjoint union of three closed intervals with length 1: $\mathrm{WF{\small ACT}}(a, \mathbf{I})$, $\mathrm{WF{\small ACT}}(b, \mathbf{I})$ and $\mathrm{WF{\small ACT}}(c, \mathbf{I})$. Meanwhile, the elements of rank 1 in $\mathcal{F}(\delta, \mathbf{S})$ make up a single circle of length 3, formed by identifying endpoints of the three unit intervals in pairs. Finally, we note that if $\mathbf{u} \in \mathrm{WF{\small ACT}}(\delta, \mathbf{I})$ and $\mathbf{u} = 0^{x_0} s_1^{x_1} \cdots s_k^{x_k} 1^{x_{k+1}}$, then the fact that each x_i can be written as a product of disjoint cycles tells us that $\downarrow(\mathbf{u})$ is isomorphic to a product of intervals, each of which isomorphic to a product of intervals of the form $[1, (1\ 2\ \cdots\ m)]$ where $m \leq d$.

From Example 5.5, we obtain Theorem B as a rephrasing of Proposition 4.10.

Theorem 5.6 (Theorem B)**.** *The maximal elements of the topological graded poset $\mathcal{F}(\delta, \mathbf{S})$ form a subspace isometric to K_δ, the dual braid complex for $[1, \delta]$.*

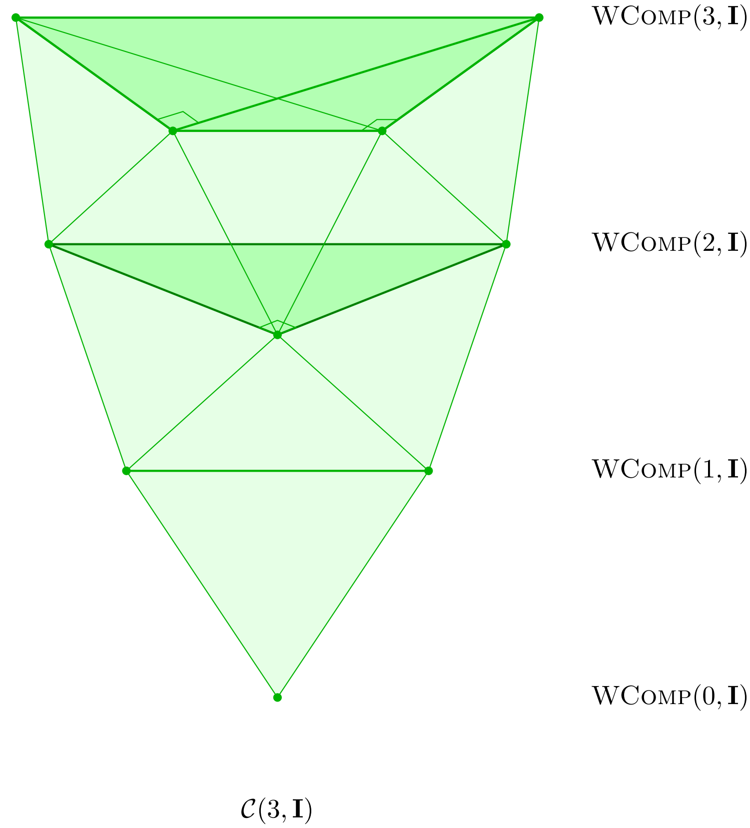

Figure 12. In the topological graded poset $\mathcal{C}(3, \mathbf{I})$, the elements of rank k have the metric topology of the standard k-dimensional orthoscheme $\mathrm{WC{\scriptstyle OMP}}(k, \mathbf{I})$. That is, each point is labeled by an element of $\mathrm{WC{\scriptstyle OMP}}(k, \mathbf{I})$ and each cell is labeled by an element of $\mathrm{C{\scriptstyle OMP}}(k, \mathbf{I})$. Moreover, elements in rank k are above only a finite number of elements in rank $k-1$, and they are below a continuum of elements in rank $k+1$.

It is straightforward to see that $\mathcal{F}(n, \mathbf{I})$ and $\mathcal{F}(n, \mathbf{S})$ are meet-semilattices, but not lattices. Next, we give a complete characterization for the upper sets in these graded posets, which will require two operations on weighted factorizations.

Definition 5.7 (Multiset operations)**.** Given G-multisets $\mathbf{u} \colon \mathbf{I} \to G$ and $\mathbf{v} \colon \mathbf{I} \to G$, define the product G-multiset $\mathbf{uv}$ by $(\mathbf{uv})(r) = \mathbf{u}(r)\mathbf{v}(r)$ and the inverse G-multiset $\mathbf{u}^{-1}$ by $\mathbf{u}^{-1}(r) = (\mathbf{u}(r))^{-1}$ for all $r \in \mathbf{I}$. Note that if $\mathbf{u}, \mathbf{v} \in \mathcal{F}(g, \mathbf{I})$ with $\mathbf{v} \subseteq \mathbf{u}$, then $\mathbf{v}^{-1}\mathbf{u}$ is a G-multiset such that $\mathbf{v}(\mathbf{v}^{-1}\mathbf{u}) = \mathbf{u}$, and by Lemma 2.4, $\mathbf{v}^{-1}\mathbf{u}$ is also an element of $\mathcal{F}(g, \mathbf{I})$.

Lemma 5.8. *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{F}(g, \mathbf{I})$ *with* $\mathbf{v} \subseteq \mathbf{u}$ *and* $\mathbf{v} \subseteq \mathbf{w}$*. Then* $\mathbf{u} \subseteq \mathbf{w}$ *if and only if* $\mathbf{v}^{-1}\mathbf{u} \subseteq \mathbf{v}^{-1}\mathbf{w}$*.*

Proof. Since $\mathbf{v} \subseteq \mathbf{u}$ and $\mathbf{v} \subseteq \mathbf{w}$, we know that $\ell(\mathbf{v}^{-1}(r)\mathbf{u}(r)) = \ell(\mathbf{u}(r)) - \ell(\mathbf{v}(r))$ and $\ell(\mathbf{v}^{-1}(r)\mathbf{w}(r)) = \ell(\mathbf{w}(r)) - \ell(\mathbf{v}(r))$. By definition, $\mathbf{v}^{-1}\mathbf{u} \subseteq \mathbf{v}^{-1}\mathbf{w}$ if and only

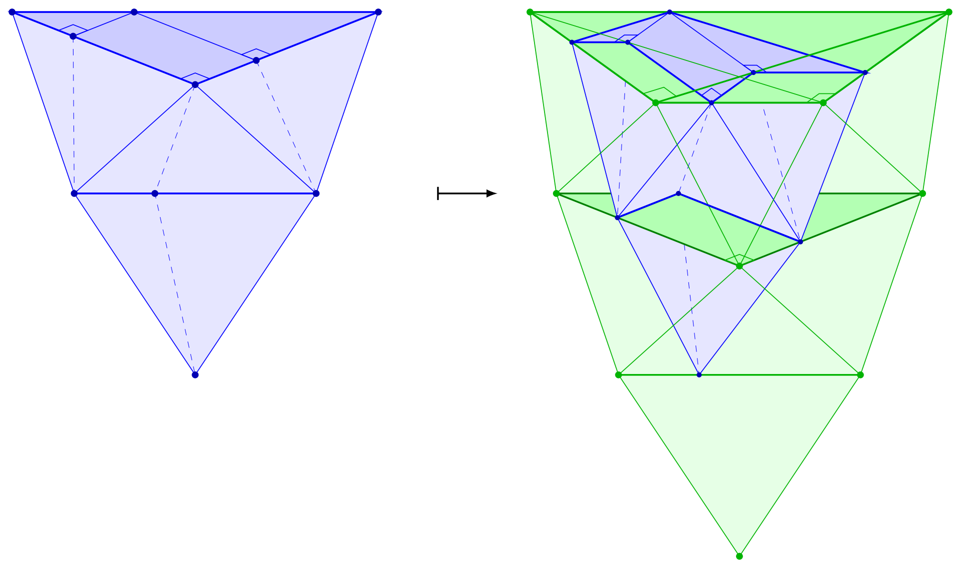

FIGURE 13. In $\mathcal{C}(3, \mathbf{I})$, the upper set of a rank-1 element (i.e. a multiset of size 1 in the unit interval) is a copy of $\mathcal{C}(2, \mathbf{I})$. The bending of the image is caused the inequalities between the new multiset elements and the original multiset element.

if $\mathbf{v}^{-1}(r)\mathbf{u}(r) \leq \mathbf{v}^{-1}(r)\mathbf{w}(r)$ for all $r \in \mathbf{I}$, which is equivalent to saying that

$$\ell(\mathbf{v}^{-1}(r)\mathbf{u}(r)) + \ell(\mathbf{u}^{-1}(r)\mathbf{w}(r)) = \ell(\mathbf{v}^{-1}(r)\mathbf{w}(r)).$$

Plugging in, we find that $\ell(\mathbf{u}(r)) + \ell(\mathbf{u}^{-1}(r)\mathbf{w}(r)) = \ell(\mathbf{w}(r))$, which means that $\mathbf{u}(r) \leq \mathbf{w}(r)$ for all $r \in \mathbf{I}$, i.e. $\mathbf{u} \subseteq \mathbf{w}$. $\square$

One can easily see that for all $h \in [1, g]$, the lower set $\downarrow(h)$ is isomorphic to the interval $[1, h]$ and the upper set $\uparrow(h)$ is isomorphic to $[1, h^{-1}g]$. We have already seen in Remark 5.3 that lower sets in $\mathcal{F}(g, \mathbf{I})$ are products of intervals in $[1, g]$ and thus finite. On the other hand, the upper sets in $\mathcal{F}(g, \mathbf{I})$ are uncountable, but with a familiar structure.

Theorem 5.9. *Let $\mathbf{v} \in \mathcal{F}(g, \mathbf{I})$ with $\rho(\mathbf{v}) = h$. Then the upper set $\uparrow(\mathbf{v})$ is isomorphic to $\mathcal{F}(h^{-1}g, \mathbf{I})$. Moreover, the maximal elements in $\uparrow(\mathbf{v})$ form a subspace which is isometric to the order complex for the interval $[1, h^{-1}g]$.*

Proof. For the first claim, note that for each $r \in I$, the product $\prod_{s>r} \mathbf{v}(s)$, in which the factors are arranged left to right in increasing order of s, is a well-defined element of $[1, g]$ since there are only finitely many $s \in I$ such that $\mathbf{v}(s)$ is nontrivial. Using this, we define $\phi\colon \uparrow(\mathbf{v}) \to \mathcal{F}(h^{-1}g, \mathbf{I})$ by

$$\phi(\mathbf{u}) = \left(\prod_{s>r} \mathbf{v}(s)\right)^{-1} (\mathbf{v}^{-1}\mathbf{u})(r) \left(\prod_{s>r} \mathbf{v}(s)\right).$$

If $\rho(\mathbf{u}) = h'$, then we can see by definition that $\rho(\phi(\mathbf{u})) = h^{-1}h'$ and therefore we do indeed have $\phi(\mathbf{u}) \in \mathcal{F}(h^{-1}g, \mathbf{I})$. This function is a bijection with inverse

$\phi^{-1}\colon \mathcal{F}(h^{-1}g,\mathbf{I}) \to {\uparrow}(\mathbf{v})$ given by

$$\phi^{-1}(\mathbf{w}) = \mathbf{v}(r)\left(\prod_{s>r}\mathbf{v}(s)\right)\mathbf{w}(r)\left(\prod_{s>r}\mathbf{v}(s)\right)^{-1},$$

and both ϕ and ϕ^{-1} are order-preserving maps by Lemma 5.8 and the fact that conjugation preserves the partial order on G. Therefore, ϕ is a poset isomorphism.

To describe this another way, we can write $\mathbf{u} = \mathbf{v}(\mathbf{v}^{-1}\mathbf{u})$ and then deform $\mathbf{u}$ by pushing the terms appearing from $\mathbf{v}$ to the left end of the interval, conjugating the elements of $\mathbf{v}^{-1}\mathbf{u}$ along the way to preserve the product of h'. The weighted factorization obtained by applying these conjugations to $\mathbf{v}^{-1}\mathbf{u}$ is what we call $\phi(\mathbf{u})$, and the effect on the group elements is an example of a Hurwitz move as described in Lemma 2.4. Viewing this as a continuous deformation makes clear that ϕ is not just an isomorphism, but an isometry as well.

Finally, we know that $\mathrm{WF\small{ACT}}(h^{-1}g,\mathbf{I})$ forms the set of maximal elements in $\mathcal{F}(h^{-1}g,\mathbf{I})$, and this is isometric to the order complex of $[1,h^{-1}g]$ by Proposition 4.6, so the proof is complete. $\square$

Replacing $\mathbf{I}$ with $\mathbf{S}$ provides Theorem C.

Corollary 5.10 (Theorem C)**.** *Let $\overline{\mathbf{v}} \in \mathcal{F}(g,\mathbf{S})$ with $\rho(\mathbf{v}) = h$. Then the upper set ${\uparrow}(\overline{\mathbf{v}})$ is isomorphic to $\mathcal{F}(h^{-1}g,\mathbf{S})$. Moreover, the maximal elements in ${\uparrow}(\mathbf{v})$ form a subspace which is isometric to the interval complex $K_{h^{-1}g}$ for $[1,h^{-1}g]$.*

Proof. By definition of the subfactorization order on $\mathcal{F}(g,\mathbf{S})$ and the associated quotient map, we have ${\uparrow}(\overline{\mathbf{v}}) = q({\uparrow}(\mathbf{v}))$, where $\mathbf{v} \in \mathcal{F}(g,\mathbf{I})$ is any representative of the equivalence class $\overline{\mathbf{v}}$. Thus, the first claim follows from Theorem 5.9. The second, following similar reasoning to the proof above, follows from Proposition 4.10. $\square$

Finally, we state two corollaries obtained from Theorem 5.9 and Corollary 5.10 for the special cases described in Examples 5.4 and 5.5.

Corollary 5.11. *Let $\mathbf{s} \in \mathcal{C}(n,\mathbf{I})$ with $\rho(\mathbf{s}) = k$. Then ${\uparrow}(\mathbf{s})$ is isomorphic to $\mathcal{C}(n-k,\mathbf{I})$, and the maximal elements of ${\uparrow}(\mathbf{s})$ form a subspace which is isometric to an orthoscheme of dimension $n-k$. Similarly, the upper set ${\uparrow}(\overline{\mathbf{s}})$ in $\mathcal{C}(n,\mathbf{S})$ is isomorphic to $\mathcal{C}(n-k,\mathbf{S})$, and the maximal elements of ${\uparrow}(\overline{\mathbf{s}})$ form a subspace which is isometric to a quotient of the standard orthoscheme of dimension $n-k$.*

Corollary 5.12. *Let δ be the d-cycle $(1\ \cdots\ d) \in \mathrm{S\small{YM}}_d$ as in Example 5.5 and let $\mathbf{u}$ be a weighted circular factorization of $\gamma \in [1,\delta]$. Then the upper set ${\uparrow}(\mathbf{u})$ in $\mathcal{F}(\delta,\mathbf{S})$ is isomorphic to $\mathcal{F}(\gamma^{-1}\delta,\mathbf{S})$. Moreover, the maximal elements of ${\uparrow}(\mathbf{u})$ form a subspace of the dual braid complex which is isometric to a product of dual braid complexes with smaller dimension.*

6. Continuous Noncrossing Partitions

We now examine $\mathcal{F}(\delta,\mathbf{S})$ in the special case when $G = \mathrm{S\small{YM}}_d$, $X = T$, and $\delta = (1\ 2\ \cdots\ d)$ as described in Example 2.3. In particular, we introduce a new type of noncrossing partition and use this to prove Theorems A and D.

Definition 6.1 (Noncrossing partitions)**.** Let P be a subset of the complex plane and let $\Pi(P)$ denote the lattice of all partitions of P, partially ordered by refinement. The elements of each partition are subsets of P called blocks, and we say that a partition is *noncrossing* if the convex hulls of its blocks are pairwise disjoint regions in $\mathbb{C}$. We define the *poset of noncrossing partitions for P* to be the subposet of noncrossing elements in $\Pi(P)$, denoted $\mathrm{NC}(P)$.

When P is the vertex set for a convex n-gon, $\mathrm{NC}(P)$ is the *classical lattice of noncrossing partitions* $\mathrm{NC}(n)$, originally defined by Kreweras in 1972 [Kre72]. The following theorem, proven 25 years later by Biane, illustrates our interest in $\mathrm{NC}(n)$.

Theorem 6.2 ([Bia97])**.** *Let $\psi\colon \textsc{Sym}_d \to \Pi(d)$ be the function which sends each permutation to the partition formed by the orbits of its action on $\{1,\ldots,d\}$. Then ψ restricts to an isomorphism from $[1,\delta]$ to $\mathrm{NC}(d)$.*

The decade following Biane's theorem produced a flurry of connections between the absolute order on the symmetric group and the combinatorics of noncrossing partitions - see the survey articles [BBG+19] and [McC06] for more background. One of these connections, involving a variation on noncrossing partitions due to Armstrong [Arm09], will be of use to us later in this section.

Definition 6.3 (Shuffle partitions)**.** Let $\pi \in \Pi(dk)$. Then π is a *k-shuffle partition* if $a \equiv b \pmod{k}$ whenever a and b belong to the same block of π. When this is the case, note that for each $j \in \{1,\ldots,k\}$, we may identify the set $\{1,\ldots,d\}$ with the equivalence class of j via the map $m \mapsto (m-1)k+j$ to obtain a partition $\pi_j \in \Pi(d)$ which is induced by π. Then π is uniquely determined by the k-tuple $(\pi_1,\ldots,\pi_k)$.

Theorem 6.4 ([Arm09, Theorem 4.3.5])**.** *Let $x_1,\ldots,x_k \in [1,\delta]$, let $\pi_i = \psi(x_i)$ for each i, and define π to be the k-shuffle partition in $\Pi(dk)$ which is determined by the k-tuple $(\pi_1,\ldots,\pi_k)$. Then π is noncrossing if and only if $\ell(x_1)+\cdots+\ell(x_k) = \ell(x_1\cdots x_k)$.*

Some recent progress on the structure of $\mathrm{NC}(P)$ when P is finite (but not convex) can be found in [CDHM], but seemingly little attention has been devoted to cases where P is infinite. Here, we are interested in the case where P is the unit circle $\mathbf{S}$. We refer to $\mathrm{NC}(\mathbf{S})$ as the *poset of continuous noncrossing partitions*, and we are particularly interested in a subposet of these which are compatible with a covering map for the circle.

Definition 6.5 (Degree-d-invariant partitions)**.** Let $f\colon \mathbf{S} \to \mathbf{S}$ be the standard degree-d covering map $f(z) = z^d$. We say that a partition $\pi \in \Pi(\mathbf{S})$ is *degree-d-invariant* if $f(z) = f(w)$ whenever z and w belong to the same block of π. When this is the case, we may fix numbers $0 = s_0 < s_1 < \cdots < s_k < 1$ such that z belongs to a nontrivial block of π only if $f(z) = e^{2\pi i s_j}$ for some j, then identify each preimage $f^{-1}(e^{2\pi i s_j})$ with the set $\{1,\ldots,d\}$ by reading off the preimages in increasing order of argument in $[0,2\pi)$. If we let π_j be the partition in $\Pi(d)$ determined by π in this way, then π can be uniquely denoted by the expression $\pi = 0^{\pi_0} s_1^{\pi_1} \cdots s_k^{\pi_k}$. It is clear from this construction that π is noncrossing if and only if the $(k+1)$-shuffle partition determined by $(\pi_0,\pi_1,\ldots,\pi_k)$ is noncrossing. Denote the subposet of all degree-d-invariant partitions by $\Pi_d(\mathbf{S})$ and the subposet of all degree-d-invariant noncrossing partitions by $\mathrm{NC}_d(\mathbf{S})$. Note that one may replace f with any covering map $\mathbf{S} \to \mathbf{S}$ of degree d, and the resulting poset of "f-invariant" noncrossing partitions will be isomorphic to $\mathrm{NC}_d(\mathbf{S})$.

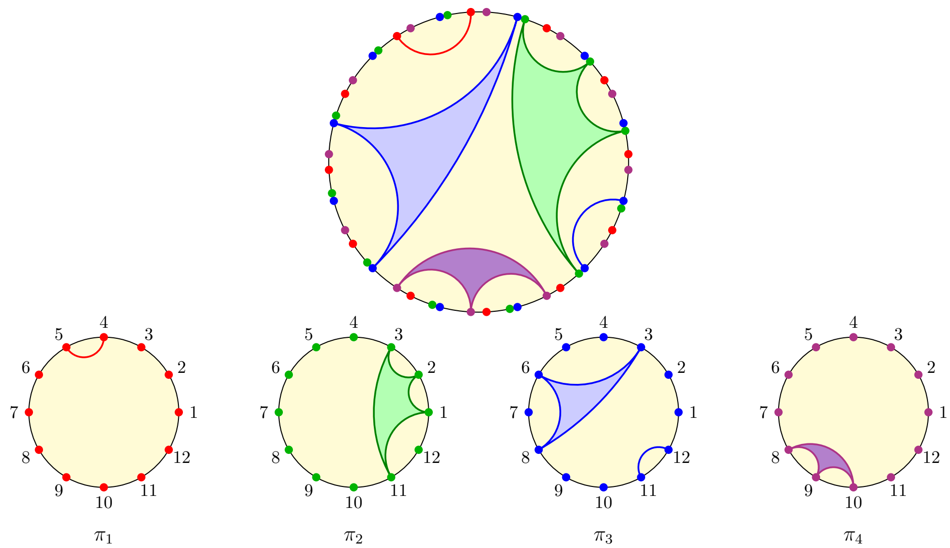

FIGURE 14. A degree-12-invariant noncrossing partition π together with its component partitions $\pi_1, \pi_2, \pi_3, \pi_4 \in \mathrm{NC}(d)$, as described in Example 6.6.

Example 6.6. The degree-12-invariant noncrossing partition in Figure 1 can described by the shorthand $\pi = 0^{\pi_0}0.1^{\pi_1}0.4^{\pi_2}0.5^{\pi_3}0.9^{\pi_4}$, where

$$\pi_0 = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\},\{11\},\{12\}\};$$
$$\pi_1 = \{\{1\},\{2\},\{3\},\{4,5\},\{6\},\{7\},\{8\},\{9\},\{10\},\{11\},\{12\}\};$$
$$\pi_2 = \{\{1,2,3,11\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\},\{10\},\{12\}\};$$
$$\pi_3 = \{\{1\},\{2\},\{3,6,8\},\{4\},\{5\},\{7\},\{9\},\{10\},\{11,12\}\};$$
$$\pi_4 = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8,9,10\},\{11\},\{12\}\}.$$

See Figure 14 for an illustration (omitting the discrete partition π_0) and note that when superimposed in the proper order, these partitions form a noncrossing hypertree on the vertex set $\{1,\ldots,12\}$ as described in [McC].

It would have been reasonable to refer to the elements of $\mathrm{NC}_d(\mathbf{S})$ as "continuous shuffle partitions" considering their resemblance to Definition 6.3. Instead, we use the descriptor *degree-d-invariant* to recognize their previous appearance in unpublished work of the late W. Thurston, which was later completed by Baik, Gao, Hubbard, Lei, Lindsey, and D. Thurston [TBY$^+$20]. In this article, Thurston and his collaborators described a spine for the space of monic complex polynomials with d distinct roots, where each point is labeled by a "primitive major" of a degree-d-invariant lamination of the disk.

The following definition and lemma rephrase a useful observation of Thurston's regarding the maximum number of non-singleton blocks in a degree-d invariant noncrossing partition.

Definition 6.7 (Total criticality)**.** Let $\pi \in \Pi_d(\mathbf{S})$ and suppose that π has k non-singleton blocks, denoted $A_1,\ldots,A_k$. The *total criticality* of the partition π is defined to be $|A_1| + \cdots + |A_k| - k$. It is straightforward to see that the total criticality gives a rank function $\mathrm{rk}\colon \Pi_d(\mathbf{S}) \to \mathbb{N}$, and this descends to a rank function for $\mathrm{NC}_d(\mathbf{S})$.

The total criticality of a generic degree-d-invariant partition can be arbitrarily large, but this is not the case for their noncrossing counterparts.

Lemma 6.8 ([TBY+20, Proposition 2.1])**.** *If $\pi \in \mathrm{NC}_d(\mathbf{S})$, then the total criticality of π is at most $d-1$. Consequently, $\mathrm{NC}_d(\mathbf{S})$ is a graded poset of height $d-1$.*

It follows from Lemma 6.8 that a degree-d invariant noncrossing partition has at most $d-1$ non-singleton blocks (in which case each has two elements). We can also give a clear characterization of the maximal elements in $\mathrm{NC}_d(\mathbf{S})$ as follows.

Definition 6.9 (Complementary regions)**.** Let $\pi \in \mathrm{NC}_d(\mathbf{S})$ and identify $\mathbf{S}$ with the boundary of a disk. The *complementary regions* of π are the connected components of the disk after removing the convex hulls of the blocks of π.

Lemma 6.10. *The partition $\pi \in \mathrm{NC}_d(\mathbf{S})$ has $rk(\pi)+1$ complementary regions. Consequently, the maximal elements of $\mathrm{NC}_d(\mathbf{S})$ are those which have exactly d complementary regions.*

Proof. If π is a degree-d-invariant noncrossing partition, then we can illustrate π in the disk and define a dual bipartite tree for π by placing a black vertex in each convex hull and a white vertex in each complementary region, then connecting a black vertex to a white vertex when the two corresponding regions are adjacent. If π has k non-singleton blocks, then the tree must have k black vertices and $\mathrm{rk}(\pi)+k = d-1+k$ edges. By the Euler characteristic, the number of vertices for a tree is always one more than the number of edges, so it follows that the number of white vertices, and thus the number of complementary regions, is $\mathrm{rk}(\pi)+1$. By Lemma 6.8, we may thus conclude that π is a maximal element of $\mathrm{NC}_d(\mathbf{S})$ if and only if it has d complementary regions. □

Given $\pi \in \mathrm{NC}_d(\mathbf{S})$, we can draw the convex hulls of the blocks of π in the disk with unit circumference, then deformation retract each convex hull to a point. Under this transformation, the boundary of the disk becomes a metric graph known as a *cactus*, and in the special case where π is maximal, Lemma 6.10 tells us that this graph can be built by gluing together d circles, each of length $1/d$. We studied these graphs in [DM22], where we associated such a graph to each complex polynomial with distinct roots and critical values on the unit circle. In [DM22, Section 8], we also described a connection between continuous noncrossing partitions (then referred to as "real" noncrossing partitions) and a type of metric tree that we called a *banyan*. These connections between $\mathrm{NC}_d(\mathbf{S})$ and complex polynomials are of central importance in our ongoing paper series [DMa].

We are now ready to prove Theorem A. Recall from Definition 4.1 that the set of all G-multisets from S to G is denoted by $\textsc{Mult}(G,S)$.

Theorem 6.11 (Theorem A)**.** *Define $\Psi\colon \textsc{Mult}(\textsc{Sym}_d,\mathbf{S}) \to \Pi_d(\mathbf{S})$ by sending the $\textsc{Sym}_d$-multiset $0^{x_0}s_1^{x_1}\cdots s_k^{x_k}1^1$ to the partition $0^{\psi(x_0)}s_1^{\psi(x_1)}\cdots s_k^{\psi(x_k)}$. Then Ψ restricts to an isomorphism from $\mathcal{F}(\delta,\mathbf{S})$ to $\mathrm{NC}_d(\mathbf{S})$.*

Proof. Let $\mathbf{x} = 0^{x_0}s_1^{x_1}\cdots s_k^{x_k}1^1 \in \textsc{Mult}(\textsc{Sym}_d,\mathbf{S})$, define $\pi_j = \psi(x_j)$ for each j and consider the partition $\Psi(\mathbf{x}) = 0^{\pi_0}s_1^{\pi_1}\cdots s_k^{\pi_k}$. As outlined in Definition 6.5, $\Psi(\mathbf{x})$ is noncrossing if and only if the shuffle partition in $\Pi(dk+d)$ determined by

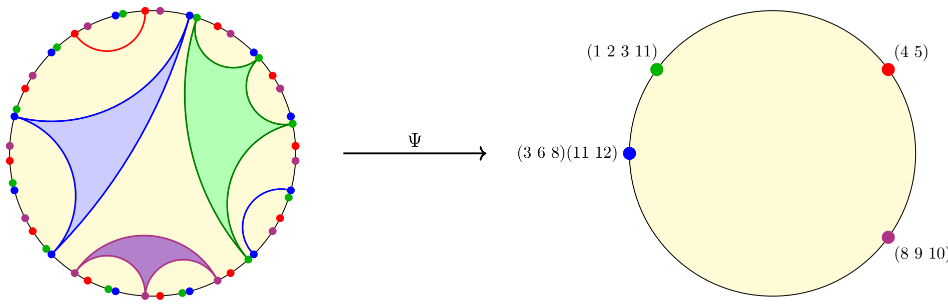

FIGURE 15. A continuous noncrossing partition in $\mathrm{NC}_{12}(\mathbf{S})$ and its corresponding weighted circular factorization in $\mathcal{F}(\delta,\mathbf{S})$

$(\pi_0,\ldots,\pi_k)$ is noncrossing, and this is equivalent to having $x_0,\ldots,x_k \in [1,\delta]$ and $\ell(x_0)+\cdots+\ell(x_k) = \ell(x_0\cdots x_k)$ by Theorem 6.4. By Lemma 2.4, this condition is satisfied by all elements of $\mathcal{F}(\delta,\mathbf{S})$, so Ψ restricts to a map $\mathcal{F}(\delta,\mathbf{S}) \to \mathrm{NC}_d(\mathbf{S})$. See Figure 15.

To see that Ψ is surjective, let $\pi = 0^{\pi_0}s_1^{\pi_1}\cdots s_k^{\pi_k}$ be an element of $\mathrm{NC}_d(\mathbf{S})$ and note that $k \le d-1$ by Lemma 6.8. By Theorem 6.2, we may define $x_i = \psi^{-1}(\pi_i)$ for each i and again apply Theorem 6.4 to see that $0^{x_0}s_1^{x_1}\cdots s_k^{x_k}1^1$ is an element of $\mathcal{F}(\delta,\mathbf{S})$ which is sent to π. Injectivity of Ψ then follows from Theorem 6.2 and by the definitions of the partial orders, $\mathbf{x} \le \mathbf{y}$ in $\mathcal{F}(\delta,\mathbf{S})$ if and only if $\Psi(\mathbf{x}) \le \Psi(\mathbf{y})$ in $\mathrm{NC}_d(\mathbf{S})$. Therefore, Ψ is an isomorphism and the proof is complete. $\square$

As a consequence of Theorem 6.11, we can import the topology and cell structure from $\mathcal{F}(\delta,\mathbf{S})$ to $\mathrm{NC}_d(\mathbf{S})$ - see Figure 16 for an example using the maximal elements of $\mathrm{NC}_3(\mathbf{S})$. In [TBY+20], Thurston and his collaborators gave a topology for the maximal elements of $\mathrm{NC}_d(\mathbf{S})$ using a slightly different metric: all edges of the dual braid complex would have equal length in their metric, whereas the lengths differ in the orthoscheme metric as described in Remark 1.5. Nevertheless, the two topologies are homeomorphic.

Let $\textsc{Poly}_d^{mc}(U)$ denote the space of monic degree-d complex polynomials for which the roots are centered at the origin and the critical values lie in the subspace $U \subseteq \mathbb{C}$. The following theorem from [TBY+20] uses the topology above to provide a spine for $\textsc{Poly}_d^{mc}(\mathbb{C}^*)$, the space of polynomials with distinct roots.

Theorem 6.12 ([TBY+20, Theorem 9.2]). *The space of maximal elements in* $\mathrm{NC}_d(\mathbf{S})$ *is homeomorphic to* $\textsc{Poly}_d^{mc}(\mathbf{S})$*, and there is a deformation retraction from* $\textsc{Poly}_d^{mc}(\mathbb{C}^*)$ *to* $\textsc{Poly}_d^{mc}(\mathbf{S})$.

Recall that Y_d is the quotient space $(\mathbb{C}^d - \mathcal{A}_d)/\textsc{Sym}_d$.

Corollary 6.13 (Theorem D). *There is a topological embedding of the dual braid complex K_δ into Y_d such that Y_d deformation retracts onto the image of the embedding.*

Proof. Noting that a polynomial has distinct roots if and only if its critical values are nonzero, there is a homeomorphism from Y_d to $\textsc{Poly}_d^m(\mathbb{C}^*)$ which sends the equivalence class of the d-tuple $(z_1,\ldots,z_d)$ to the polynomial $(z-z_1)\cdots(z-z_d)$. By translating the roots so that their centroid lies at the origin, we obtain a deformation retraction from $\textsc{Poly}_d^m(\mathbb{C}^*)$ to $\textsc{Poly}_d^{mc}(\mathbb{C}^*)$. From here, Theorem 6.12 tells us that

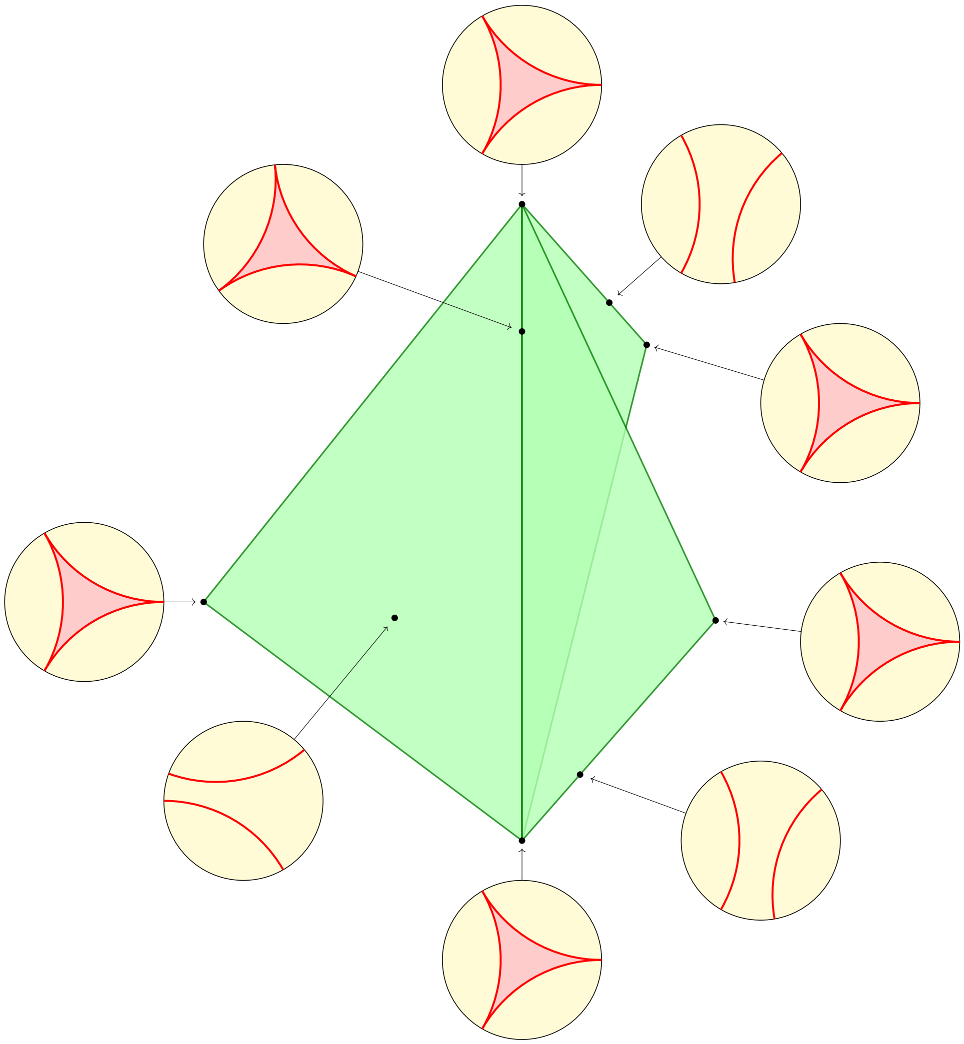

FIGURE 16. The isomorphism described in Theorem 6.11 allows us to label the points in the dual braid complex from Figure 10 by maximal elements of $\mathrm{NC}_3(\mathbf{S})$. As before, points with the same label are glued together, which means that all five vertices are identified and the short edges are identified in pairs.

$\textsc{Poly}_d^{mc}(\mathbb{C}^*)$ deformation retracts to $\textsc{Poly}_d^{mc}(\mathbf{S})$, which is homeomorphic to the maximal elements of $\mathrm{NC}_d(\mathbf{S})$, which in turn is homeomorphic to the dual braid complex K_δ by Theorems 5.6 and 6.11, so the proof is complete. $\square$

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Email address: `doughemj@lafayette.edu`

Department of Mathematics, Lafayette College, Easton, PA 18042

Email address: `jon.mccammond@math.ucsb.edu`

Department of Mathematics, UC Santa Barbara, Santa Barbara, CA 93106