

FIXED POINTS OF PARKING FUNCTIONS

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ABSTRACT. We define an action of words in $[m]^n$ on \mathbb{R}^m to give a new characterization of rational parking functions—they are exactly those words whose action has a fixed point. We use this viewpoint to give a simple definition of Gorsky, Mazin, and Vazirani’s zeta map on rational parking functions when m and n are coprime [GMV16], and prove that this zeta map is invertible. A specialization recovers Loehr and Warrington’s sweep map on rational Dyck paths [ALW15, TW18, GMV17].

1. INTRODUCTION

1.1. Parking Words. Classical parking words have a well-known interpretation in the language of parking cars. There are n parking places and n cars, each indexed from 0 to $n-1$. As in [KW66, Section 6], car i has a preference for parking place \mathbf{p}_i , and cars attempt to park as follows: for $0 \leq i \leq n-1$, car i takes the unoccupied parking place with the lowest number larger than or equal to \mathbf{p}_i , should such a parking place exist. The *classical parking words* \mathcal{PW}_n are defined as those words for which each car is able to park.

The 16 parking words in \mathcal{PW}_3 are given on the left side of Figure 1. Garsia introduced a combinatorial interpretation of \mathcal{PW}_n as certain super-diagonal labelled paths in an $n \times n$ square, which has served as the basis of many subsequent investigations. Replacing the square by an $m \times n$ rectangle gives the (m, n) -parking words \mathcal{PW}_m^n —those words $\mathbf{p} = \mathbf{p}_0 \cdots \mathbf{p}_{n-1} \in [m]^n = \{0, 1, \dots, m-1\}^n$ such that

$$(1) \quad \left| \{j : \mathbf{p}_j < i\} \right| \geq \frac{in}{m} \text{ for } 1 \leq i \leq m.$$

The classical parking words are recovered as $\mathcal{PW}_n = \mathcal{PW}_{n+1}^n$. The 25 parking words in \mathcal{PW}_5^3 are illustrated on the right side of Figure 1.

000	001	002	011	012	000	001	002	003	011	012	013
	010	020	101	021		010	020	030	101	021	031
	100	200	110	102		100	200	300	110	201	301
				120						210	310
				201						120	130
				210						102	103

FIGURE 1. Left: the $4^{3-1} = 16$ $(4, 3)$ -parking words in \mathcal{PW}_3 (these are also the $(3, 3)$ -parking words). Right: the $5^{3-1} = 25$ $(5, 3)$ -parking words in \mathcal{PW}_5^3 . Each column is an orbit under \mathfrak{S}_3 .

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1.2. A New Characterization of Parking Words. Our main result is a new characterization of (m, n) -parking words as *piecewise-linear functions* from \mathbb{R}^m to \mathbb{R}^m . This characterization is new even for classical parking words.

Define

$$V^m := \left\{ \mathbf{x} = (x_0, \dots, x_{m-1}) \in \mathbb{R}^m : \sum_{i=0}^{m-1} x_i = 0 \text{ and } x_0 \leq x_1 \leq \dots \leq x_{m-1} \right\}.$$

A letter $i \in [m]$ acts on $\mathbf{x} \in V^m$ by adding m to x_i , subtracting the tuple $\mathbb{1}_m := (1, 1, \dots, 1)$, and then resorting. A word $\mathbf{w} \in [m]^n$ acts on $\mathbf{x} \in V^m$ by acting by its letters from left to right. The following theorem distinguishes parking words in $[m]^n$ by their action on V^m .

Theorem 1.1. *The action of $\mathbf{w} \in [m]^n$ on V^m has a fixed point if and only if \mathbf{w} is an (m, n) -parking word. More precisely, the action of $\mathbf{w} \in [m]^n$ on V^m :*

- has a **unique** fixed point iff $\mathbf{w} \in \mathcal{PW}_m^n$ and $\gcd(m, n) = 1$;
- has **infinitely many** fixed points iff $\mathbf{w} \in \mathcal{PW}_m^n$ and $\gcd(m, n) > 1$; and
- has **no** fixed points iff $\mathbf{w} \in [m]^n \setminus \mathcal{PW}_m^n$.

The motivation for [Theorem 1.1](#) comes from generalizations of the space of diagonal coinvariants and the zeta map on parking functions, as we now explain.

1.3. Coinvariants and the Symmetric Group. The Hilbert series for the space of coinvariants is the generating function for two important statistics on the $n!$ permutations in \mathfrak{S}_n :

$$(2) \quad \text{Hilb} \left(\mathbb{C}[\mathbf{x}_n] / \langle \mathbb{C}[\mathbf{x}_n]_+^{\mathfrak{S}_n} \rangle; q \right) = \sum_{w \in \mathfrak{S}_n} q^{\text{inv}(w)} = \sum_{w \in \mathfrak{S}_n} q^{\text{maj}(w)},$$

where $\mathbb{C}[\mathbf{x}_n]$ is shorthand for a polynomial ring in n variables and $\langle \mathbb{C}[\mathbf{x}_n]_+^{\mathfrak{S}_n} \rangle$ is the ideal of $\mathbb{C}[\mathbf{x}_n]$ generated by symmetric polynomials with no constant term.

Artin gave a basis for this space using the code of a permutation to reflect the first generating function of [Equation \(2\)](#) [[Art44](#)], while Garsia and Stanton found a basis using the descents of a permutation to explain the second [[GS84](#)].

A statistic with the same distribution as inv or maj is eponymously named *Mahonian* [[Mac13](#)], but Foata gave the first bijection sending one statistic to the other [[Foa68](#)]. Exploiting the fact that this bijection preserves descents of the inverse permutation, Foata and Schützenberger later found an involution that *interchanges* inv and maj [[FS78](#)].

1.4. Diagonal Coinvariants. The study of the space of diagonal coinvariants originated with Garsia and Haiman; its relationship to parking words was first suggested by Gessel [[Hai94](#), [GH96](#)]. More precisely, Carlsson and Mellit’s recent proof of the shuffle conjecture [[HHL⁺05](#), [CM18](#), [HX17](#)] implies the long-suspected fact that the bigraded Hilbert series of the space of diagonal coinvariants is encoded as a positive sum over the $(n+1)^{n-1}$ parking words \mathcal{PW}_n [[Hai02](#), [HL05](#)]:¹

$$(3) \quad \text{Hilb} \left(\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n] / \langle \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]_+^{\mathfrak{S}_n} \rangle; q, t \right) = \sum_{\mathbf{p} \in \mathcal{PW}_n} q^{\text{dinv}(\mathbf{p})} t^{\text{area}(\mathbf{p})},$$

where q records the degree of the variables \mathbf{x}_n , t the degree of \mathbf{y}_n , and area and dinv are certain statistics on parking functions. Recently, Carlsson and Oblomkov artfully merged the Artin and Garsia-Stanton bases to give an explicit basis of the

¹Carlsson and Mellit actually proved a stronger result, giving an explicit formula for the *Frobenius series* for the space of diagonal coinvariants.

space of diagonal coinvariants [CO18], explaining the generating function in Equation (3).

It is known from Equation (3) that area and dinv are symmetric, i.e.,

$$(4) \quad \sum_{p \in \mathcal{PW}_n} q^{\text{area}(p)} t^{\text{dinv}(p)} = \sum_{p \in \mathcal{PW}_n} q^{\text{dinv}(p)} t^{\text{area}(p)}.$$

However, it is a long-standing open problem to find an involution that interchanges area and dinv —in the style of Foata and Schützenberger’s involution for inv and maj —thus combinatorially proving Equation (4). This problem is still wide open, even for the alternating subspace [DMM11, Gil16]. As a first step towards this elusive involution, the equidistribution of dinv and area —obtained by setting $t = 1$ in Equation (4)—was proven combinatorially by Loehr and Remmel [LR04, HL05] [Hag08, Corollary 5.6.1]:

Theorem 1.2 ([LR04]).

$$\sum_{p \in \mathcal{PW}_n} q^{\text{area}(p)} = \sum_{p \in \mathcal{PW}_n} q^{\text{dinv}(p)}.$$

This bijection on \mathcal{PW}_n takes area to dinv , combinatorially proving the symmetry of Theorem 1.2. It was first understood, generalized, and inverted for the alternating subspace, where it was called the *zeta map* [KOP02, Hag03, HL05, GH02, Hag08, ALW15, TW18]. It has been rediscovered many times. We review the history of the zeta map in Section 5.1.

1.5. Rational Parking Words and the Affine Symmetric Group. *We now assume m and n are coprime.* The classical parking words \mathcal{PW}_n , their statistics area and dinv , and the shuffle conjecture have all been (at least combinatorially) generalized to the (m, n) -parking words \mathcal{PW}_m^n [BGLX15, ALW16, GMV16, GN15, Thi16, GMV17].

The Fuss $(nk+1, n)$ generalization of the story of diagonal coinvariants is due to Garsia and Haiman [Hai98, GH96]. Writing \mathcal{A} for the ideal generated by the alternating polynomials in $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]$, Mellit proved the rational shuffle conjecture in [Mel16], which implies that

$$\text{Hilb} \left(\mathcal{A}^{k-1} / \mathcal{A}^{k-1} \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]_+^{\mathfrak{S}_n}; q, t \right) = \sum_{p \in \mathcal{PW}_{kn+1}^n} q^{\text{area}(p)} t^{\text{dinv}(p)}.$$

The more general rational (m, n) version comes from Hikita’s study of the Borel-Moore homology of affine type A Springer fibers, which has a natural basis indexed by the m^{n-1} elements of the affine symmetric group $\tilde{\mathfrak{S}}_n$ lying inside an m -fold dilation of the fundamental alcove [Shi87, Hai94, CP02, Che03, Som05, Hik14, GMV16, Thi16]. Thus, while the space of coinvariants $\mathbb{C}[\mathbf{x}_n] / \langle \mathbb{C}[\mathbf{x}_n]_+^{\mathfrak{S}_n} \rangle$ is related to the symmetric group \mathfrak{S}_n , the diagonal coinvariants are related to the affine symmetric group $\tilde{\mathfrak{S}}_n$.

There are many bijections from these affine elements to the parking words \mathcal{PW}_m^n . Armstrong found natural interpretations of area and dinv in terms of affine permutations for the Fuss case [Arm13], and his work was extended to the rational case by Gorsky, Mazin, and Vazirani [GMV16, GMV17]. Gorsky and Negut formulated the rational shuffle conjecture in [GN15]—that Hikita’s polynomial was given by an operator from an elliptic Hall algebra (see also [BGLX15]). This operator formulation

leads to a q, t -symmetric bivariate polynomial generalizing Equation (4):²

$$(5) \quad \sum_{\mathfrak{p} \in \mathcal{PW}_m^n} q^{\text{area}(\mathfrak{p})} t^{\text{div}(\mathfrak{p})} = \sum_{\mathfrak{p} \in \mathcal{PW}_m^n} q^{\text{div}(\mathfrak{p})} t^{\text{area}(\mathfrak{p})}.$$

As a combinatorial proof of q, t -symmetry seems out of reach even in the classical $m = n+1$ case, the next best thing is the analogue of the equidistribution of Theorem 1.2. To this end, Gorsky, Mazin, and Vazirani defined a zeta map on \mathcal{PW}_m^n (a map taking area to div), and conjectured that it was a bijection by providing what they believed to be an inverse map. A curious feature of their conjectural inverse is that it appears to *converge* to the correct answer.

As a corollary to our Theorem 1.1, we prove Gorsky, Mazin, and Vazirani's conjecture and obtain a rational generalization of Theorem 1.2.

Theorem 1.3. *For m and n relatively prime,*

$$\sum_{\mathfrak{p} \in \mathcal{PW}_m^n} q^{\text{area}(\mathfrak{p})} = \sum_{\mathfrak{p} \in \mathcal{PW}_m^n} q^{\text{div}(\mathfrak{p})}.$$

1.6. Outline of the Paper. In Section 3 we define (m, n) -parking words, the action of words in $[m]^n$ on \mathbb{R}^m , and prove our characterizations in Theorem 1.1 using the Brouwer fixed point theorem.

To relate this characterization to parking functions, we introduce some notation. Fixing (m, n) relatively prime, we define (m, n) -filters as certain periodic filters of $\mathbb{Z} \times \mathbb{Z}$ in Section 4.1, and show that equivalence classes of these filters are naturally parameterized by rational (m, n) -Dyck paths and balanced (m, n) -filters. We define (m, n) -filter tuples in Section 4.3 as certain sequences of (m, n) -filters, and relate these sequences to labeled (m, n) -Dyck paths.

The notion of (m, n) -filters allows us to give a new, remarkably simple definition of the zeta map on (m, n) -parking words in Section 5. We summarize past work on zeta maps in Section 5.1, define the zeta map in Section 5.2, and relate our construction to Loehr and Warrington's sweep map on (m, n) -Dyck paths in Section 5.3.

In Section 6, we finally turn to the affine symmetric group. After basic definitions in Section 6.1, we use balanced (m, n) -filters to give a bijection between (m, n) -filter tuples and affine permutations whose inverses lie in the Sommers region in Section 6.2. We use this bijection in Section 6.3 to relate our constructions to the work of Gorsky, Mazin, and Vazirani, showing that our Theorem 1.1 resolves [GMV16, Conjecture 1.4].

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²Something is lost in the rational case: one statistic remains the degree, but the second statistic now appears only using a filtration.

2. WORDS AND ACTIONS

2.1. Parking Words. As in the introduction, the (m, n) -parking words \mathcal{PW}_m^n are those words $\mathbf{p} = p_0 \cdots p_{n-1} \in [m]^n$ such that

$$(6) \quad \left| \{j : p_j < i\} \right| \geq \frac{in}{m} \text{ for } 1 \leq i \leq m.$$

By definition, any (m, n) -parking word is a permutation of the column lengths of a lattice path staying above the main diagonal in an $m \times n$ rectangle, as illustrated in Figure 7. (Here, by “column lengths,” we mean the distances between the top of the rectangle and the horizontal steps of the lattice path.) We write \mathcal{DW}_m^n for the increasing (m, n) -parking words—the (m, n) -Dyck words—which are in bijection with the set of such lattice paths.

2.2. Hyperplanes. Although we defer most of the connections between parking words and the affine symmetric group to Section 6, we will require the hyperplane arrangement of the affine symmetric group $\tilde{\mathfrak{S}}_m$ immediately. For $0 \leq i, j < m$ and $k \in \mathbb{N}$, we define the hyperplane

$$\mathcal{H}_{i,j}^k = \{\mathbf{x} \in \mathbb{R}^m : x_i - x_j = mk\}$$

to be of height $j - i + mk$, so that the affine simple hyperplanes $\{\mathcal{H}_{i,i+1}^0\}_{0 \leq i < m-1} \cup \{\mathcal{H}_{m-1,0}^1\}$ each have height one. We call the set $\{\mathcal{H}_{i,i+1}^0\}_{0 \leq i < m-1}$ the simple hyperplanes. Write

$$\mathcal{H} = \bigcup_{\substack{0 \leq i < j < m \\ k \in \mathbb{Z}}} \mathcal{H}_{i,j}^k$$

for the affine hyperplane arrangement of type $\tilde{\mathfrak{S}}_m$ and let

$$\mathbb{R}_t^m = \left\{ \mathbf{x} \in \mathbb{R}^m : \sum_{i=0}^{m-1} x_i = t \right\} \cong \mathbb{R}^{m-1}.$$

The closure of each connected region of $\mathbb{R}_t^m \setminus \mathcal{H}$ is called an *alcove*. For $0 \leq i < m$, write \mathbf{e}_i for the i th standard basis vector of \mathbb{R}^m . The set of alcoves is permuted under translations by $m\mathbf{e}_i - \mathbb{1}_m$ and under reflections in any hyperplane $\mathcal{H}_{i,j}^k$. We define the norm of an m -tuple $\mathbf{x} \in \mathbb{R}_t^m$ to be

$$|\mathbf{x}| := \sum_{0 \leq i < j < m} (x_j - x_i)^2.$$

Definition 2.1. A fundamental domain for the natural action of \mathfrak{S}_m on \mathbb{R}^m is given by those points whose coordinates weakly increase. Define the cone

$$V_t^m := \left\{ \mathbf{x} \in \mathbb{R}_t^m : x_0 \leq x_1 \leq \cdots \leq x_{m-1} \right\}$$

and $V^m := V_0^m$. We may *rebalance* an element of $V_{t_1}^m$ to an element of $V_{t_2}^m$ by adding the appropriate multiple of $\mathbb{1}_m$.

2.3. Actions of Words. For each $i \in [m]$, we define piecewise linear transformations on $\mathbb{R}_t^m \setminus \mathcal{H}$ and on V_t^m .

Definition 2.2. A letter $i \in [m]$ acts on $\mathbf{x} \in \mathbb{R}_t^m \setminus \mathcal{H}$ by adding m to the i th smallest coordinate of \mathbf{x} and subtracting the tuple $\mathbb{1}_m$. The letter i acts on $\mathbf{x} \in V_t^m$ in the same way, but with a final resorting step at the end. A word $\mathbf{w} \in [m]^n$ acts on $\mathbf{x} \in \mathbb{R}_t^m \setminus \mathcal{H}$ or $\mathbf{x} \in V_t^m$ by acting by its letters from left to right.

More precisely, writing $\mathbf{y} := \text{sort}(\mathbf{x})$ for the increasing rearrangement of a point $\mathbf{x} \in \mathbb{R}_t^m$,

$$(7) \quad i(\mathbf{x}) := \mathbf{x} + \mathbf{e}_j - \mathbb{1}_m \text{ for } \mathbf{x} \in \mathbb{R}_t^m \setminus \mathcal{H} \text{ if } x_j = \mathbf{y}_i, \text{ and}$$

$$(8) \quad i(\mathbf{x}) := \text{sort}(\mathbf{x} + \mathbf{e}_i - \mathbb{1}_m) \text{ for } \mathbf{x} \in V_t^m.$$

An example of the action on $\mathbb{R}_6^3 \setminus \mathcal{H}$ is given in Figure 2.³

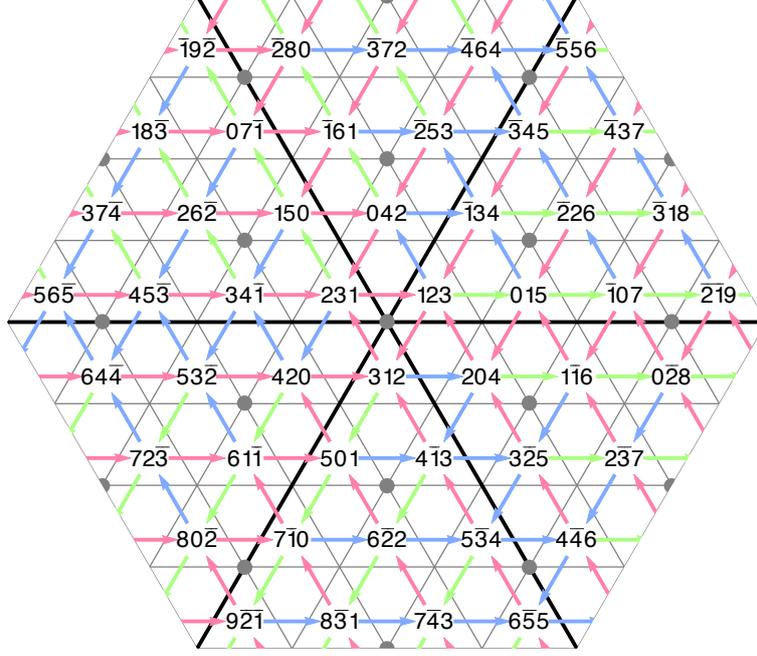


FIGURE 2. An orbit of the action of the letters 0, 1, and 2 on $\mathbb{R}_6^3 \setminus \mathcal{H}$. Acting by 0 adds 3 to the smallest coordinate and subtracts $\mathbb{1}_3$; acting by 2 adds 3 to the largest coordinate and subtracts $\mathbb{1}_3$; and acting by 1 adds 3 to the coordinate that is neither largest nor smallest and subtracts $\mathbb{1}_3$. For formatting, we have written \bar{i} for $-i$ and suppressed commas and parentheses; thus, the string $\bar{2}53$ stands for the point with coordinates $(-2, 5, 3)$.

The action of a letter $i \in [m]$ on V_t^m is the restriction to V_t^m of a piecewise-linear function from $\mathbb{R}_t^m \cong \mathbb{R}^{m-1}$ to V_t^m that sends alcoves to alcoves. By Equation (7), the letter i acts on $\mathbf{x} \in \mathbb{R}_t^m$ by the translation $\mathbf{x} + m\mathbf{e}_i - \mathbb{1}_m$, and the final resorting of the coordinates into increasing order may be interpreted geometrically by folding once along the simple hyperplane $\mathcal{H}_{i,i+1}^0$, and then folding again as needed along simple hyperplanes until all points lie in the cone V_t^m .

Lemma 2.3. *The action of a word $\mathbf{w} \in [m]^n$ on V^m sends alcoves to alcoves and only decreases distances between points: $|\mathbf{x} - \mathbf{y}| \geq |\mathbf{w}(\mathbf{x}) - \mathbf{w}(\mathbf{y})|$.*

Proof. The conclusion of the lemma is evident from the geometric description of the action of a letter. \square

³To cleanly bridge from this section to the affine symmetric group in Section 6, we will want to normalize points so that $\sum_{i=0}^{m-1} x_i = \binom{m+1}{2}$. We therefore used that normalization in Figure 2.

2.4. Affine Dimension. We say that a subset $X \subseteq V^m$ is *of affine dimension k* if it is a convex set contained in an affine subspace of dimension k and contains an open ball in that affine subspace. In particular, X is of affine dimension 0 if it consists of a single fixed point. Note that affine dimension is not defined for an arbitrary subset of V^m ; to say that a subset is of affine dimension k is to make a strong statement about the kind of subset it is.

For $w \in [m]^n$, define

$$\text{Fix}(w) := \{\mathbf{x} \in V^m : w(\mathbf{x}) = \mathbf{x}\}$$

to be the set of points that are fixed under w .

Fix $\mathfrak{p} \in \mathcal{PW}_m^n$ and write $d := \gcd(m, n)$. We will prove [Theorem 1.1](#) in [Section 3](#) by showing that $\text{Fix}(\mathfrak{p})$ is of affine dimension $d - 1$. For now, we prove that $\text{Fix}(\mathfrak{p})$ is convex and is contained in an affine subspace of dimension $d - 1$.

Lemma 2.4. $\text{Fix}(\mathfrak{p})$ is convex.

Proof. Let $\mathfrak{p} \in \mathcal{PW}_m^n$ and suppose $\mathbf{x}, \mathbf{y} \in \text{Fix}(\mathfrak{p})$. Since the application of \mathfrak{p} only decreases distances, the line between them is fixed. \square

Lemma 2.5. Let $\gcd(m, n) = d$, $\mathfrak{p} \in \mathcal{PW}_m^n$, and suppose $\mathbf{x} \in \text{Fix}(\mathfrak{p})$. Then the multiset of coordinates $\{x_i\}_{i=0}^{m-1}$ can be partitioned into disjoint multisets, each of which is of size m/d and of the form $\{a + kd + b_k m\}_{k=0}^{m/d-1}$.

Proof. Up to rebalancing, the action of each letter of \mathfrak{p} increases one coordinate of \mathbf{x} by m , but the effect of the entire parking word is to send x_i to $x_i + n$. Since each individual entry changes by a multiple of m , it does not change modulo m . This means that the multiset of remainders of $x_i \bmod m$ must be fixed under addition of n , so it must also be fixed under addition of $\gcd(n, m) = d$. \square

Lemma 2.6. Let $\gcd(m, n) = d \geq 1$ and $\mathfrak{p} \in \mathcal{PW}_m^n$. Then $\text{Fix}(\mathfrak{p})$ is contained in an affine subspace of dimension $d - 1$.

Proof. Let $\mathbf{x} \in \text{Fix}(\mathfrak{p})$, and let $\mathbf{y} \in \text{Fix}(\mathfrak{p})$ be another fixed point in a small ball around \mathbf{x} . By [Lemma 2.5](#), the coordinates of \mathbf{y} can be partitioned into d disjoint multisets of size m/d , each of which consists of a set of residues mod m which are fixed under addition of d . Because \mathbf{y} is close to \mathbf{x} , the partition we have found for the coordinates of \mathbf{y} also works for the coordinates of \mathbf{x} . For each of the parts in the partition, there is therefore some offset such that adding this offset to the coordinates of \mathbf{x} in that part, yields the coordinates of \mathbf{y} in that part. These offsets must add up to zero, since the sum of the entries of \mathbf{x} and \mathbf{y} are assumed to be the same. This shows that \mathbf{y} lies in an affine subspace of dimension $d - 1$ which also contains \mathbf{x} .

In principle, if we chose a different $\mathbf{y}' \in \text{Fix}(\mathfrak{p})$ near \mathbf{x} , we could obtain a different affine subspace (corresponding to a different way of partitioning the coordinates of \mathbf{x}). However, convexity would then imply that the line between \mathbf{y} and \mathbf{y}' is also in $\text{Fix}(\mathfrak{p})$, and this includes points which are not on any affine subspace of the above form, which is impossible. Thus all the points in $\text{Fix}(\mathfrak{p})$ near \mathbf{x} lie in a single affine subspace.

By convexity, any point in $\text{Fix}(\mathfrak{p})$ lies in the same affine subspace. \square

3. A NEW CHARACTERIZATION OF PARKING WORDS

In this section we prove [Theorem 1.1](#), distinguishing parking words in the set of all words in $[m]^n$ using the action of a word on V^m .

Definition 3.1. For $\mathfrak{p} \in \mathcal{PW}_m^n$ and $1 \leq i < m$, define i to be a *touch point* of \mathfrak{p} if

$$\left| \{j : \mathfrak{p}_j < i\} \right| = i \frac{n}{m}.$$

Note that we do not count 0 or m as touch points, so that when n and m are coprime, no $\mathfrak{p} \in \mathcal{PW}_m^n$ has a touch point.

We break the proof of [Theorem 1.1](#) into five parts. Let $\mathfrak{p} \in \mathcal{PW}_m^n$ and $d := \gcd(m, n)$.

- [Lemma 3.2](#): if \mathfrak{p} has no touch points, then it has a fixed point.
- [Corollary 3.3](#): if $d = 1$, then \mathfrak{p} has a unique fixed point.
- [Lemma 3.4](#): if \mathfrak{p} has no touch points, then its fixed point space is bounded of affine dimension $d - 1$.
- [Lemma 3.6](#): if \mathfrak{p} has a touch point, then it has infinitely many fixed points, of arbitrarily large norm.
- [Lemma 3.8](#): if $\mathfrak{w} \in [m]^n \setminus \mathcal{PW}_m^n$, then it has no fixed points.

3.1. Parking words without touch points.

Lemma 3.2. *The action of $\mathfrak{p} \in \mathcal{PW}_m^n$ on V^m has a fixed point when \mathfrak{p} has no touch point.*

Proof. We want to show that $|\mathfrak{p}(\mathbf{x})| < |\mathbf{x}|$ for $|\mathbf{x}| > N$, provided N is sufficiently large (that is, parking contracts). By [Lemma 2.3](#), the Brouwer fixed point theorem can then be invoked on the $(m - 1)$ -ball

$$\left\{ \mathbf{x} \in \mathbb{R}^m : \sum_{i=0}^{m-1} x_i = 0, |\mathbf{x}| \leq N \right\}$$

to guarantee a fixed point.

We first consider the case that the x_i are sufficiently separated that each “resort” step does nothing—that is, the actions of \mathfrak{p} on \mathbf{x} as an element of $\mathbb{R}_0^m \setminus \mathcal{H}$ and as an element of V^m coincide. For $0 \leq i < m$, let

$$q_i = \left| \{j : \mathfrak{p}_j = i\} \right|$$

be the number of occurrences of i in the (m, n) -parking word \mathfrak{p} . The change in the norm produced by applying \mathfrak{p} to $\mathbf{x} \in V^m$ is the difference

$$\begin{aligned} (9) \quad & \sum_{0 \leq i < j < m} [(x_i + mq_i - x_j - mq_j)^2 - (x_i - x_j)^2] \\ & = \sum_{0 \leq i < j < m} m^2(q_i - q_j)^2 + \sum_{i=0}^{m-1} 2m^2q_i x_i - 2 \left(\sum_{i=0}^{m-1} x_i \right) \left(\sum_{i=0}^{m-1} q_i \right). \end{aligned}$$

Of the three terms on the right-hand side of [Equation \(9\)](#), the first sum depends only on \mathfrak{p} . The third vanishes because we have assumed that $\sum_i x_i = 0$. We want to show that the second sum on the right-hand side

$$\sum_{i=0}^{m-1} 2m^2q_i x_i$$

is sufficiently negative to dominate the first, provided $|\mathbf{x}|$ is big enough. We will do this by showing that we can add a sequence of positive numbers to the second sum to make it zero.

Suppose that $x_0 = x_1 = \dots = x_{b-1}$ (with b maximal) and $x_{m-1} = x_{m-2} = \dots = x_{m-c}$ (with c maximal). If we increase each of the b minimal coordinates of \mathbf{x} by $c\alpha$ and decrease each of the c maximal coordinates of \mathbf{x} by $b\alpha$, we have not changed the average value of \mathbf{x} . On the other hand, the value of $\sum_i q_i x_i$ changes by

$$(10) \quad \sum_{i=0}^{b-1} \alpha c q_i - \sum_{i=m-c}^{m-1} \alpha b q_i.$$

By Equation (6)—because \mathbf{p} is an (m, n) -parking word—the first term of Equation (10) is greater than $\alpha c b \frac{n}{m}$, while the second term is less than $\alpha c b \frac{n}{m}$. Thus, changing the values of \mathbf{x} in this way increases the sum $\sum_i q_i x_i$.

Choose α maximal so that, in increasing the b minimal coordinates and decreasing the c maximal coordinates, none of the values changed pass any other values of \mathbf{x} . We call this a *step*. Since at least one of b or c increases, after a finite number of steps, we will have all entries of \mathbf{x} at zero and the value of the sum $\sum_i q_i x_i$ will also be zero. But since we increased the value of the sum at each step, its initial value was negative.

In fact, we can bound the initial value of $\sum_i q_i x_i$ away from 0 by approximating its change across all steps. For $b, c < m$, we have

$$\left(\sum_{i=0}^{i=b-1} q_i \right) - \frac{bn}{m} \geq \frac{1}{m} \quad \text{and}$$

$$\frac{cn}{m} - \left(\sum_{i=m-c}^{i=m-1} q_i \right) \geq \frac{1}{m},$$

since both left-hand sides are strictly positive (because of our assumption that \mathbf{p} has no touch points) and can be expressed as a rational number with denominator m . Therefore,

$$(11) \quad \sum_{i=0}^{b-1} \alpha c q_i - \sum_{i=m-c}^{m-1} \alpha b q_i \geq \frac{\alpha c}{m} + \frac{\alpha b}{m}.$$

To approximate $\sum_i q_i x_i$, we bound the two terms on the right-hand side of Equation (11) over the entire process which moves all the x_i to zero.

Since αc is the amount that each of the minimal x_i 's were moved during each step, the sum of $\alpha c/m$ over all steps is $1/m$ times the total amount the minimal coordinates are increased over the whole process. But this begins at x_0 and terminates at 0, so the total amount they change by is $-x_0$ and the sum of the first term on the right-hand side of Equation (11) over the whole process is $-x_0/m$. Similarly, the sum of the second term on the right-hand side of Equation (11) over the whole process is x_{m-1}/m . We obtain the bound

$$\sum_{i=0}^{m-1} 2m^2 q_i x_i \leq 2m(x_0 - x_{m-1}),$$

which we can make as negative as we like by requiring $|\mathbf{x}|$ to be sufficiently large.

We now consider the case that the resorting is not necessarily trivial—that is, the action on \mathbf{x} as an element of $\mathbb{R}_0^m \setminus \mathcal{H}$ and as an element of V^m do *not* necessarily coincide. Fix a sorted tuple \mathbf{x} and distinguish this tuple as living in $\mathbb{R}_0^m \setminus \mathcal{H}$ or V^m by writing $\mathbf{x}_U \in \mathbb{R}_0^m \setminus \mathcal{H}$ and $\mathbf{x}_V \in V^m$. Write

$$\mathbf{x}_U^{(j)} := \mathbf{p}_0 \mathbf{p}_1 \cdots \mathbf{p}_{j-1}(\mathbf{x}_U) \quad \text{and}$$

$$\mathbf{x}_V^{(j)} := \mathbf{p}_0 \mathbf{p}_1 \cdots \mathbf{p}_{j-1}(\mathbf{x}_V).$$

We note that after applying a single letter i to \mathbf{x}_U and \mathbf{x}_V , the difference between any coordinate of \mathbf{x}_U and the same coordinate of \mathbf{x}_V is less than m . By induction, corresponding coordinates of $\mathbf{x}_U^{(j)}$ and $\mathbf{x}_V^{(j)}$ differ by at most mj .

On the other hand, for any tuple \mathbf{y} , applying a single letter i to \mathbf{y}_U or \mathbf{y}_V , we compute

$$\begin{aligned} |i(\mathbf{y}_U)| - |\mathbf{y}_U| &= |i(\mathbf{y}_V)| - |\mathbf{y}_V| = \sum_{\substack{0 \leq j < m \\ j \neq i}} \left[(y_i + m - y_j)^2 - (y_i - y_j)^2 \right] \\ &= (m-1)m^2 + 2m \sum_{\substack{0 \leq j < m \\ j \neq i}} (y_i - y_j) \\ &= (m-1)m^2 + 2m^2 y_i. \end{aligned}$$

By telescoping, we can now bound the difference $|\mathfrak{p}(\mathbf{x}_U)| - |\mathfrak{p}(\mathbf{x}_V)|$:

$$\begin{aligned} |\mathfrak{p}(\mathbf{x}_U)| - |\mathfrak{p}(\mathbf{x}_V)| &= \sum_{j=0}^{n-1} \left(|\mathbf{x}_U^{(j+1)}| - |\mathbf{x}_U^{(j)}| \right) - \left(|\mathbf{x}_V^{(j+1)}| - |\mathbf{x}_V^{(j)}| \right) \\ &\leq \sum_{j=0}^{n-1} 2m^3 j = n(n-1)m^3, \end{aligned}$$

using our analysis that corresponding coordinates in $\mathbf{x}_U^{(j)}$ and $\mathbf{x}_V^{(j)}$ differ by at most mj . This quantity is still a constant in the fixed parameters n and m , so we can overcome it by requiring that $|\mathbf{x}|$ be sufficiently large.

We conclude that the second term of the right-hand side of Equation (9) dominates the first if $|\mathbf{x}| > N$ for N sufficiently large, so that $|\mathfrak{p}(\mathbf{x})| < |\mathbf{x}|$ for $|\mathbf{x}| > N$. \square

In the case $\gcd(m, n) = 1$, Lemma 2.6, together with our previous results, suffices to show that the set of fixed points is of affine dimension 0 (i.e., consists of a single point).

Corollary 3.3. *Let $\gcd(m, n) = 1$ and $\mathfrak{p} \in \mathcal{PW}_m^n$. Then $\text{Fix}(\mathfrak{p})$ is of affine dimension 0. In particular, $|\text{Fix}(\mathfrak{p})| = 1$.*

We now show that $\text{Fix}(\mathfrak{p})$ is of affine dimension $d - 1$ for $d = \gcd(m, n)$ in the case that \mathfrak{p} has no touch points.

Lemma 3.4. *Let $\gcd(m, n) = d \geq 1$ and $\mathfrak{p} \in \mathcal{PW}_m^n$ with no touch points. Then $\text{Fix}(\mathfrak{p})$ is bounded of affine dimension $d - 1$.*

Proof. $\text{Fix}(\mathfrak{p})$ is bounded, since we showed in Lemma 3.2 that $|\mathfrak{p}(\mathbf{x})| < |\mathbf{x}|$ for $|\mathbf{x}| > N$ for some large N .

Let F be a face of the affine arrangement \mathcal{H} such that for any G having F as a face, we have $G \cap \text{Fix}(\mathfrak{p}) = F \cap \text{Fix}(\mathfrak{p})$. Suppose, seeking a contradiction, that F is of codimension $c \geq 1$. Consider the action of \mathfrak{p} on a small sphere S around a point x of $\text{Fix}(\mathfrak{p})$ in the plane normal to F . Since the sphere is not fixed by \mathfrak{p} , the action of \mathfrak{p} on it is by some non-trivial foldings. The image therefore misses some open ball B in the sphere. Restricting, \mathfrak{p} now defines a map from $S \setminus B$ to $S \setminus B$, and by Brouwer's fixed point theorem, it has a fixed point. This contradicts our assumption on F . Thus there must be a fixed point \mathbf{x} not lying on any hyperplane.

Lemma 2.5 divides the set of all coordinates of \mathbf{x} into d subsets of size m/d , where the elements of each set are congruent modulo d . Since \mathbf{x} lies on no hyperplane, no coordinate value modulo m is repeated, so it is unambiguous how to apportion the coordinates into these sets.

Now consider the action of \mathfrak{p} , omitting rebalancing. Each entry in the multiset of coordinates is changed by a multiple of d . Thus the entries in each of the d subsets are permuted among themselves by the action of \mathfrak{p} . We may translate each family

with respect to the others by some small amount without changing the relative order of any coordinates, so all such points are still fixed. This gives us an open ball around \mathbf{x} in the $(d-1)$ -dimensional affine subspace constructed in [Lemma 2.6](#) consisting entirely of fixed points. $\text{Fix}(\mathbf{p})$ is therefore of affine dimension $d-1$. \square

Example 3.5. Fix $(m, n) = (6, 9)$ with $d = \gcd(m, n) = 3$. Consider the (m, n) -parking word $\mathbf{p} = 020101151$, and note that \mathbf{p} has no touch points.

It is easily verified that \mathbf{p} has a fixed point $\mathbf{x} = (-3, 1, 2, 4, 6, 11)$, which modulo d is of the form $(0, 1, 2, 1, 0, 2)$. Let

$$\begin{aligned} v_0 &= (-3, 0, 3, 3, 6, 12), \\ v_1 &= (-2, 1, 1, 4, 7, 10), \text{ and} \\ v_2 &= (-4, 2, 2, 5, 5, 11). \end{aligned}$$

Then one can check that $\text{Fix}(\mathbf{p}) \supseteq \text{conv}(v_0, v_1, v_2)$.

3.2. Parking words with touch points. When the parking word has a touch point, we now use [Lemma 3.2](#) to also produce infinitely many fixed points. These fixed points may now be of arbitrarily large norm.

Lemma 3.6. *The action of $w \in \mathcal{PW}_m^n$ on V^m has infinitely many fixed points when w has at least one touch point. The set $\text{Fix}(w)$ has affine dimension $d-1$, and contains fixed points of arbitrarily large norm.*

Proof. Suppose that $\gcd(m, n) = d \neq 1$ and that w has $k \geq 1$ touch points. We will break \mathbf{p} into a number of smaller parking words based on its touch points, find the unique fixed points for each of those parking words, and then reassemble them in uncountably many ways to find fixed points for \mathbf{p} . To this end, list the k touch points of \mathbf{p} as m_1, \dots, m_k with

$$m_0 = 0 < m_1 < m_2 < \dots < m_k < m = m_{k+1}.$$

For $0 \leq j \leq k$, let $\mathbf{p}^{(j)}$ be the (not-necessarily consecutive) subword of \mathbf{p} containing all letters p of \mathbf{p} such that $m_j \leq p < m_{j+1}$. Let n_j be the length of $\mathbf{p}^{(j)}$ —necessarily a multiple of n/d —and note that \mathbf{p} is a shuffle of $\mathbf{p}^{(0)}, \mathbf{p}^{(1)}, \dots, \mathbf{p}^{(k-1)}$.

To define smaller parking words, we shift the individual letters of $\mathbf{p}^{(j)}$ by the previous touch point to produce the (m_j, n_j) -parking word $\mathbf{q}^{(j)} := \mathbf{p}^{(j)} - m_j$.

We can now use [Lemma 3.2](#) and the previous case to find $\mathbf{x}^{(j)} \in V^{m_j}$ that are fixed points for the $\mathbf{q}^{(j)}$. In preparation to reassemble these individual fixed points $\mathbf{x}^{(j)}$ into a fixed point for \mathbf{p} , we scale them to define

$$\mathbf{x}_N^{(j)} := \frac{n}{n_j} \mathbf{x}^{(j)} + N_j$$

for some $N_j \in \mathbb{R}$. Finally, define $\mathbf{x}_N \in V^m$ by the concatenation:

$$\mathbf{x}_N := \left(\mathbf{x}_N^{(0)}, \mathbf{x}_N^{(1)}, \dots, \mathbf{x}_N^{(k)} \right),$$

and then rebalancing so that the sum is 0.

We now check that \mathbf{x}_N is really a fixed point of \mathbf{p} , as long as the N_j give sufficient space between the $\mathbf{x}_N^{(j)}$. Since \mathbf{p} is a shuffle of the $\mathbf{p}^{(j)}$, as long as the individual coordinates of \mathbf{x}_N do not overlap during the application of the letters of \mathbf{p} (for example, we may take $N_j > mn + N_{j-1}$), we may discuss the action of \mathbf{p} on each component $\mathbf{x}_N^{(j)}$ separately. On $\mathbf{x}_N^{(j)}$, then, only the subword $\mathbf{p}^{(j)}$ of \mathbf{p} will act; the only difference from its usual action on $\mathbf{x}^{(j)}$ is that (as a piece of the larger parking word \mathbf{p}) it adds m rather than m_j —but we have compensated for this by the scaling factor $\frac{n}{n_j}$. \square

Example 3.7. We illustrate the proof of [Lemma 3.6](#). Let $(m, n) = (9, 12)$ so that $d = 3$, and let $\mathbf{p} = 531030678631$. Then there are 2 touch points of \mathbf{p} : $m_1 = 3$ and $m_2 = 6$, so that

$$\mathbf{p}^{(0)} = 1001, \mathbf{p}^{(1)} = 5333, \text{ and } \mathbf{p}^{(2)} = 6786$$

and

$$\mathbf{q}^{(0)} = 1001, \mathbf{q}^{(1)} = 2000, \text{ and } \mathbf{q}^{(2)} = 0120.$$

Fixed points for $\mathbf{q}^{(j)}$ are

$$\mathbf{x}^{(0)} = (-2, 0, 2), \mathbf{x}^{(1)} = (-1, 0, 1), \text{ and } \mathbf{x}^{(2)} = (-2, -1, 3),$$

so that

$$\mathbf{x}_N^{(0)} = (-6, 0, 6), \mathbf{x}_N^{(1)} = (-3, 0, 3) + N_2, \text{ and } \mathbf{x}_N^{(2)} = (-6, -3, 9) + N_3,$$

and before rebalancing

$$\mathbf{x}_N = (-6, 0, 6, -3 + N_2, N_2, 3 + N_2, -6 + N_3, -3 + N_3, 9 + N_3).$$

When $N_2 > 21$ and $N_3 > 21 + N_2$, we see that portion of \mathbf{p} corresponding to $\mathbf{p}^{(j)}$ acts only on the $3j - 2$, $3j - 1$, and $3j$ th coordinates of \mathbf{x} .

3.3. Non-parking words.

Lemma 3.8. *Repeated application of $w \in [m]^n \setminus \mathcal{PW}_m^n$ on V^m sends every point to infinity. In particular, w has no fixed points.*

Proof. We show that repeated application of w sends some point to infinity. Since w only decreases the distance between two points, this implies that every point is sent to infinity, so that w has no fixed points.

Suppose that $w \in [m]^n \setminus \mathcal{PW}_m^n$ is not a parking function because it has too many numbers that are at least k , and choose k maximal. Let

$$\mathbf{x} = (x_0, x_1, \dots, x_{m-1}) \in V^m$$

be a vector with sum 0. We claim that the result of applying w to \mathbf{x} has the effect of increasing the difference between the average value of x_k, \dots, x_{m-1} and the average value of x_0, \dots, x_{k-1} by a fixed quantity. Thus, after enough applications of w , the value of x_m will be arbitrarily large.

In the course of applying w to \mathbf{x} there are two ways that the difference between the average value of x_0, \dots, x_{k-1} and the average value of x_k, \dots, x_{m-1} changes. One is as a result of adding m to an entry corresponding to an element of w . By the assumption on w , these steps have the property that, on average, a more than proportionate number of these steps are applied to the entries x_k, \dots, x_{m-1} , which therefore increases the difference between the average values by a fixed positive amount. The other way that the difference between the averages increases is in the resort step. If an element in x_1, \dots, x_{k-1} is increased far enough that it moves into the top $m - k$ elements, then it is resorted into one of these positions. Whenever this happens, this also increases the difference between the average values. \square

3.4. Summary. We obtain [Theorem 1.1](#) as a corollary of [Lemmas 3.2](#), [3.4](#), [3.6](#) and [3.8](#) and [Corollary 3.3](#). Examples for $m = 3$ are given in [Figure 3](#).

The remainder of this paper is devoted to explaining the coprime case in more detail, explicitly identifying the isolated fixed points of parking words as the centers of alcoves of dominant affine permutations whose inverses lie in the Sommers region. It would be desirable to explicitly identify the regions of fixed points in the non-relatively prime case. We note that in the special (m, mk) case when the fixed regions are full dimensional, Gorsky, Mazin, and Vazirani have recently identified the set of fixed regions of an (m, mk) -parking word with the dominant regions in the k -Shi arrangement of $\tilde{\mathfrak{S}}_m$ [[GMV17](#), Section 3.4] (compare with [Figure 3](#)).

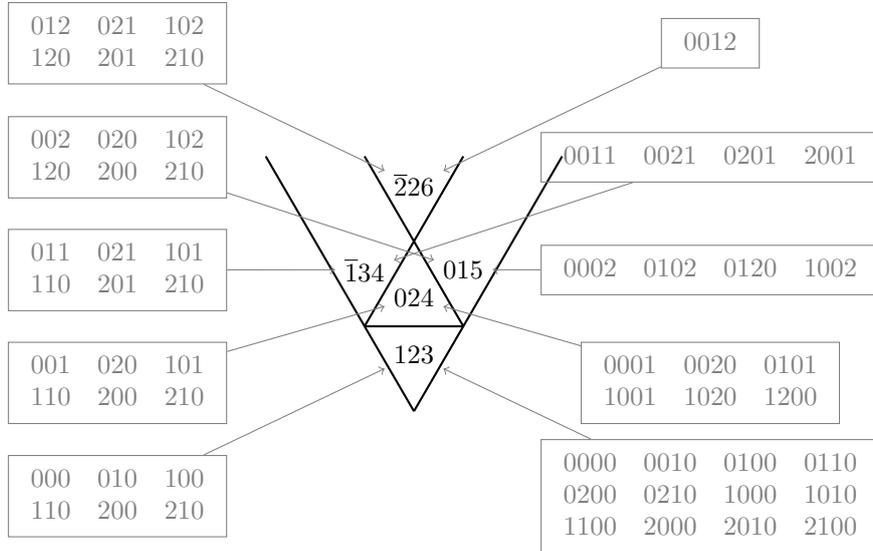


FIGURE 3. The dominant part of the $\tilde{\mathfrak{S}}_3$ Shi arrangement. Each region is labeled by a coordinate corresponding to the one-line notation of the affine permutation whose alcove is lowest in the region (see Section 6 for more details). The gray words on the left are the $(3, 3)$ -parking words that fix every point of the (closed) region to which they point; the gray words on the right are the $(4, 3)$ -parking words that fix precisely the coordinate to which they point.

4. PARKING FILTERS

For the rest of the paper, we fix m and n relatively prime.

In this section, we define the combinatorial objects—generally thought of as Dyck paths and labeled Dyck paths—that we will use to compute the zeta map defined in Section 5. These objects are all well-known; our main contribution is the simplicity of our definition of the zeta map on parking functions in Definition 5.7, and its relation with affine permutations in Section 6.

4.1. Filters. Fix m and n relatively prime, and label the point $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ by its *level*

$$\ell(i, j) = (i, j) \bullet (m, n).$$

If we draw the levels of points in the plane, rows correspond to residue classes modulo m , while columns correspond to residue classes modulo n . Any fixed row and column intersect in a unique point, and the Chinese remainder theorem ensures that the levels are distinct modulo mn in any contiguous $n \times m$ rectangle. A portion of the levels of $\mathbb{Z} \times \mathbb{Z}$ for $(m, n) = (3, 4)$ and $(3, 5)$ is illustrated in Figure 4.

Definition 4.1. An (m, n) -*filter* \mathfrak{i} is an order filter of $\mathbb{Z} \times \mathbb{Z}$ such that whenever the point (i, j) is in \mathfrak{i} , then all (i', j') for which $\ell(i', j') = \ell(i, j)$ are also in \mathfrak{i} . We write \mathcal{F}_m^n for the set of all (m, n) -filters.

Interchanging the copies of \mathbb{Z} in $\mathbb{Z} \times \mathbb{Z}$ gives a bijection between the set of (m, n) -filters and the set of (n, m) -filters; we call this the $(m \leftrightarrow n)$ -*bijection*. An (m, n) -filter \mathfrak{i} is specified in three natural ways:

- $\ell(\mathfrak{i}) := \{\ell(i, j) : (i, j) \in \mathfrak{i}\}$, the set of all its levels,

- $m(\mathbf{i}) := \{\min_{(i,j) \in \mathbf{i}} \ell(i, j) : j \in \mathbb{Z}\}$, i.e., the set formed by taking, for each row, the minimal level of a point in that row which is also in \mathbf{i} . or
- $n(\mathbf{i}) := \{\min_{(i,j) \in \mathbf{i}} \ell(i, j) : i \in \mathbb{Z}\}$, i.e., the set formed by taking, for each column, the minimal level of a point in that column which is also in \mathbf{i} .

Note that $m(\mathbf{i})$ consists of m integers, one from each congruence class mod m , while $n(\mathbf{i})$ consists of n integers, one from each congruence class mod n .

An example of [Definition 4.1](#) is given in [Figure 4](#). It is useful to identify the sets $m(\mathbf{i})$ and $n(\mathbf{i})$ with the corresponding sorted lists.

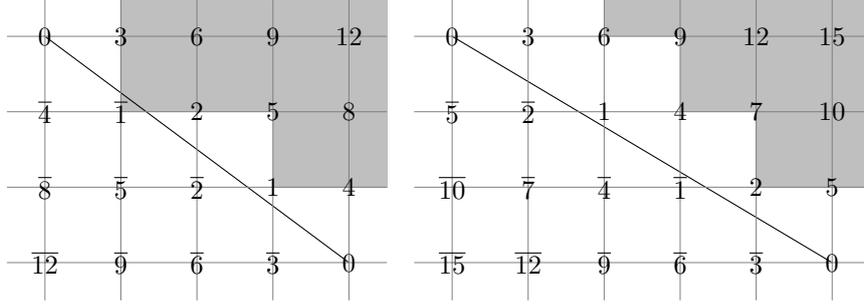


FIGURE 4. Ignoring the shading, a portion of the levels of $\mathbb{Z} \times \mathbb{Z}$ for $(m, n) = (3, 4)$ and $(3, 5)$ (the picture is extended to the rest of the plane by periodicity). The solid gray line through the points of level 0 separates the positive and negative levels. On the left, the shading specifies $\mathbf{i} \in \mathcal{F}_3^4$ as those lattice points contained in the shaded region; similarly, the shading on the right specifies $\mathbf{i}' \in \mathcal{F}_3^5$. One checks that (as sorted lists) $m(\mathbf{i}) = [-1, 1, 3]$, $n(\mathbf{i}) = [-1, 1, 2, 4]$, $m(\mathbf{i}') = [2, 4, 6]$, and $n(\mathbf{i}') = [2, 4, 5, 6, 8]$.

Definition 4.2. We say that $\mathbf{i}, \mathbf{i}' \in \mathcal{F}_m^n$ are *equivalent* if $\mathbf{i} = \mathbf{i}' + (x, y)$ for some $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, and write $\tilde{\mathcal{F}}_m^n$ for the set of equivalence classes of \mathcal{F}_m^n .

Definition 4.3. Define a directed graph \mathfrak{F}_m^n on $\tilde{\mathcal{F}}_m^n$ with a directed edge from $\tilde{\mathbf{i}} \in \tilde{\mathcal{F}}_m^n$ to $\tilde{\mathbf{i}}' \in \tilde{\mathcal{F}}_m^n$ iff there is some $\mathbf{i}' \in \tilde{\mathbf{i}}'$ and some $\mathbf{i} \in \tilde{\mathbf{i}}$ such that $\ell(\mathbf{i}')$ can be obtained from $\ell(\mathbf{i})$ by removing a single level from \mathbf{i} . We write $\tilde{\mathfrak{b}}_m^n$ for the equivalence class containing the (m, n) -filters generated by a single level.

The $(m \leftrightarrow n)$ -bijection gives an isomorphism between \mathfrak{F}_m^n and \mathfrak{F}_n^m . The graphs \mathfrak{F}_3^4 and \mathfrak{F}_3^5 are illustrated in [Figures 5](#) and [6](#).

4.2. Representatives. In this section, we introduce two natural representatives of the equivalence classes of (m, n) -filters:

- Dyck (m, n) -filters, in bijection with Dyck paths and most useful to relate our constructions to the standard combinatorial objects ([Remarks 4.5](#) and [4.13](#)); and
- balanced (m, n) -filters, which will be essential for specifying affine permutations ([Theorem 6.6](#), [Proposition 6.7](#), and [Theorem 6.10](#))

4.2.1. Dyck filters. We define a first representative of the equivalence classes in $\tilde{\mathcal{F}}_m^n$. These representatives are usually defined in the literature as lattice paths staying above or below a diagonal, and we show how our definition recovers this interpretation in [Remark 4.5](#).

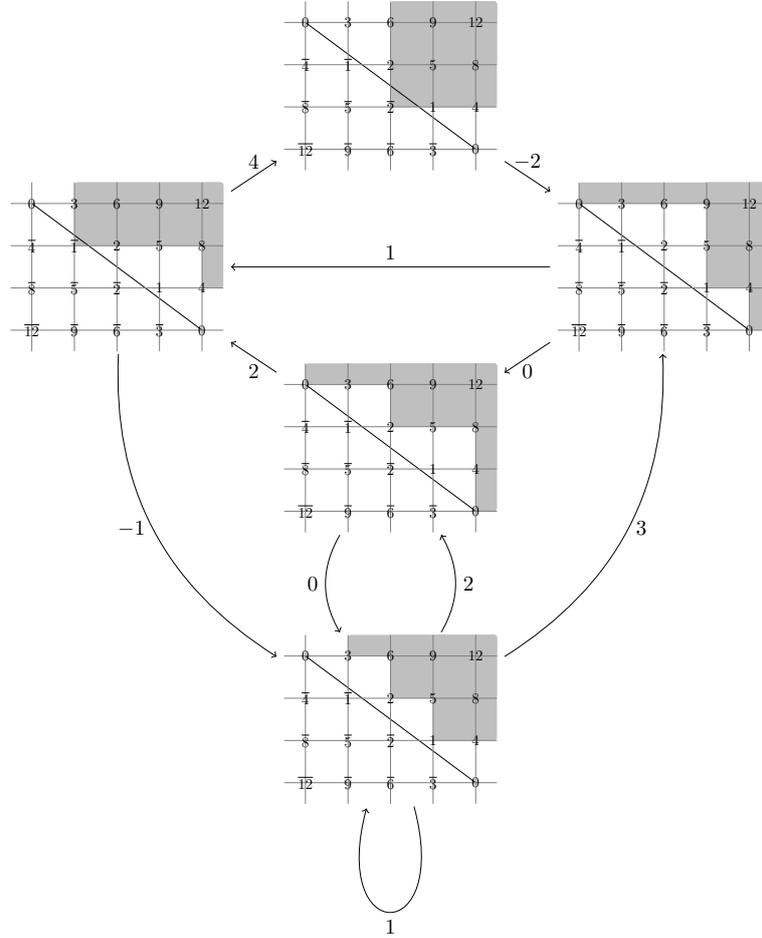


FIGURE 5. The directed graph $\mathfrak{F}_3^4 \cong \mathfrak{F}_4^3$, with equivalence classes represented by the balanced $(3, 4)$ -filters of Section 4.2.2. The edge labels record the level of minimal element removed. Compare with Figure 18.

Definition 4.4. A *Dyck (m, n) -filter* is an (m, n) -filter \mathfrak{d} such that

$$\min_{(i,j) \in \mathfrak{d}} \ell(i, j) = 0.$$

We write \mathcal{DF}_m^n for the set of all Dyck (m, n) -filters.

In particular, for $\mathfrak{d} \in \mathcal{DF}_m^n$, $\min(n(\mathfrak{d})) = \min(m(\mathfrak{d})) = 0$. Note that the $(m \leftrightarrow n)$ -bijection restricts to a bijection between \mathcal{DF}_m^n and \mathcal{DF}_n^m .

Remark 4.5. We relate Definition 4.4 to the set of *(m, n) -Dyck paths*—those lattice paths from $(0, 0)$ to $(-n, m)$ using north steps $(0, 1)$ and west steps $(-1, 0)$ and staying above the line $(x, y) \bullet (m, n) = 0$. The boundary of an (m, n) -filter of $\mathbb{Z} \times \mathbb{Z}$ traces out a periodic path in the plane. This periodicity allows us to restrict to the contiguous $n \times m$ rectangle with corners at level 0 without losing information, giving \mathcal{DF}_m^n the standard geometric interpretation as (m, n) -Dyck paths. This is illustrated in Figure 7.

All (m, n) -filters whose boundaries trace out the same path—up to translation—are equivalent to the same Dyck (m, n) -filter.

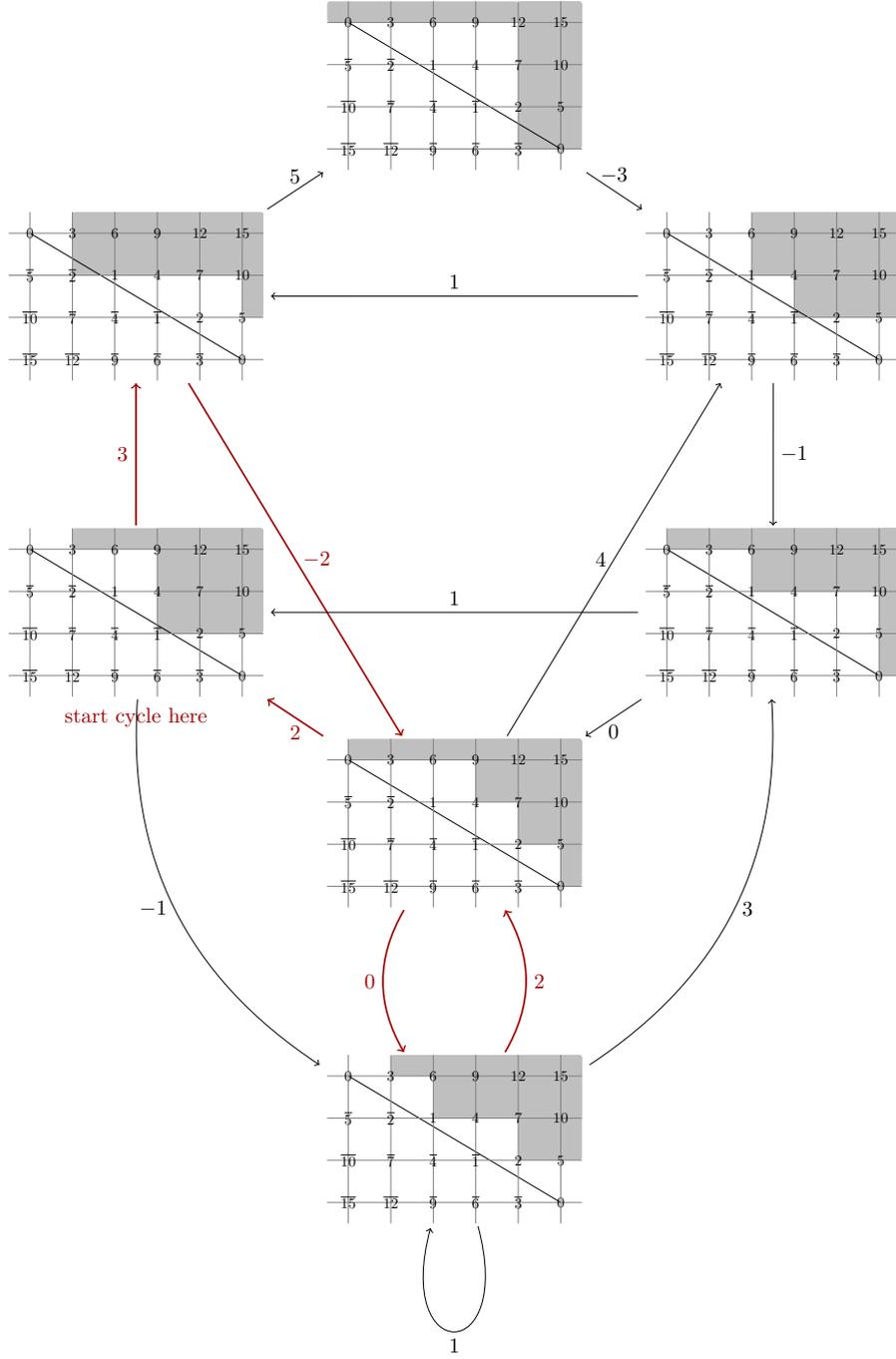


FIGURE 6. The directed graph $\mathfrak{F}_3^5 \cong \mathfrak{F}_5^3$, with equivalence classes represented by the balanced $(3, 5)$ -filters of Section 4.2.2. The edge labels record the level of the minimal element removed. The cycle consisting of the red edges corresponds to the parking (m, n) -filter tuple considered in Example 4.12. Compare with Figure 19.

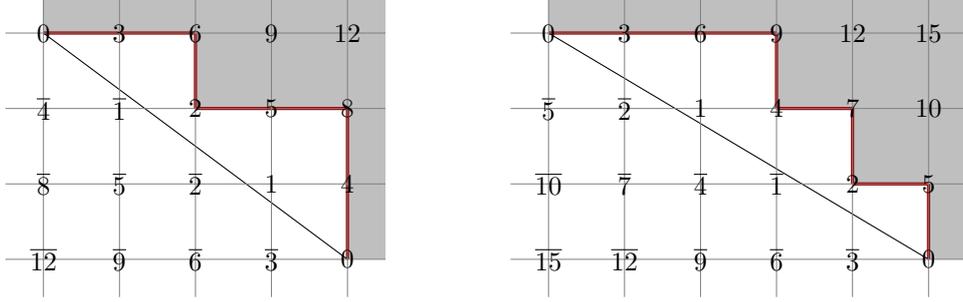


FIGURE 7. The Dyck $(3, 4)$ - and $(3, 5)$ -filters corresponding to the filters in Figure 4 (they also happen to be balanced). The boundary between two consecutive points with level 0 traces an (m, n) -Dyck path (marked in red). Recording the column lengths of the left path gives the $(3, 4)$ -Dyck word $[0, 0, 1, 1]$, while the right path corresponds to the $(3, 5)$ -Dyck word $[0, 0, 0, 1, 2]$

Proposition 4.6. *Each equivalence class in $\tilde{\mathcal{F}}_m^n$ contains a unique element of \mathcal{DF}_m^n .*

Proof. To a (m, n) -filter \mathfrak{i} we associate the unique equivalent Dyck (m, n) -filter \mathfrak{d} obtained by translating \mathfrak{i} so that it touches the line $(x, y) \bullet (m, n) = 0$, but does not go below. \square

Lemma 4.7. *There is a directed path in $\tilde{\mathfrak{F}}_m^n$ from the equivalence class $\tilde{\mathfrak{b}}_m^n$ to any other equivalence class.*

Proof. Starting from $\tilde{\mathfrak{b}}_m^n$, the directed graph $\tilde{\mathfrak{F}}_m^n$ contains a copy of the distributive lattice (whose Hasse diagram is thought of as a directed graph) consisting of the (m, n) -Dyck paths ordered by inclusion. (Note that this is via the identification with Dyck paths which lie below the diagonal, not above it.) \square

The following enumeration is a well-known application of the cycle lemma.

Proposition 4.8.

$$|\tilde{\mathcal{F}}_m^n| = |\tilde{\mathcal{F}}_n^m| = \frac{1}{n+m} \binom{n+m}{n}.$$

4.2.2. *Balanced Filters.* We define a second representative of the equivalence classes in $\tilde{\mathcal{F}}_m^n$. These objects appear to have been much less studied, and will allow us to relate $\tilde{\mathcal{F}}_m^n$ and affine permutations.

Definition 4.9. We call an (m, n) -filter $\mathfrak{b} \in \mathcal{F}_m^n$ satisfying

$$\sum_{i \in m(\mathfrak{b})} i = \binom{m+1}{2} \text{ and } \sum_{j \in n(\mathfrak{b})} j = \binom{n+1}{2}$$

a *balanced (m, n) -filter*. We write \mathcal{BF}_m^n for the set of balanced (m, n) -filters

It is a simple check that this set is nonempty—it contains the (m, n) -filter \mathfrak{b}_m^n generated by the points with level $\ell = \frac{1+m+n-mn}{2}$.

Proposition 4.10. *Each equivalence class in $\tilde{\mathcal{F}}_m^n$ contains a unique element of \mathcal{BF}_m^n . Furthermore, for $\mathfrak{b} \in \mathcal{F}_m^n$,*

$$\sum_{i \in m(\mathfrak{b})} i = \binom{m+1}{2} \text{ if and only if } \sum_{j \in n(\mathfrak{b})} j = \binom{n+1}{2}.$$

Proof. Suppose we have an (m, n) -filter \mathbf{i} satisfying both conditions. Removing a minimal element from \mathbf{i} to make a new (m, n) -filter \mathbf{i}' has the effect of adding m to the sum of elements of $m(\mathbf{i})$ and the effect of adding n to the sum of the elements of $n(\mathbf{i})$. The (m, n) -filter defined by $\ell(\mathbf{i}') = \ell(\mathbf{i}) - 1$ therefore also satisfies both conditions. The conclusions now follow from [Lemma 4.7](#). \square

The five balanced $(3, 4)$ -filters are illustrated in [Figure 5](#).

4.3. Filter Tuples. We define (m, n) -filter tuples as certain sequences of (m, n) -filters, and we explain in [Remark 4.13](#) how (m, n) -filter tuples are in bijection with the usual definition of parking functions as labeled Dyck paths.

Definition 4.11. An (m, n) -filter tuple \mathbf{p} is a tuple of $n+1$ (m, n) -filters

$$\mathbf{p} = (\mathbf{p}^{(0)}, \mathbf{p}^{(1)}, \dots, \mathbf{p}^{(n)})$$

such that:

- $m(\mathbf{p}^{(i+1)}) = (m(\mathbf{p}^{(i)}) \setminus \{p_i\}) \cup \{p_i + m\}$ for $0 \leq i < n$, and
- $m(\mathbf{p}^{(0)}) + n = m(\mathbf{p}^{(n)})$.

We write \mathcal{T}_m^n for the set of all (m, n) -filter tuples and we say that two (m, n) -filter tuples \mathbf{p}_1 and \mathbf{p}_2 are *equivalent* if $\mathbf{p}_1^{(i)} = \mathbf{p}_2^{(i)} + (x, y)$ for all $0 \leq i \leq n$ and some fixed $(x, y) \in \mathbb{Z} \times \mathbb{Z}$.

[Definition 4.11](#) is illustrated in [Figure 8](#); the caption is explained in the next few paragraphs.

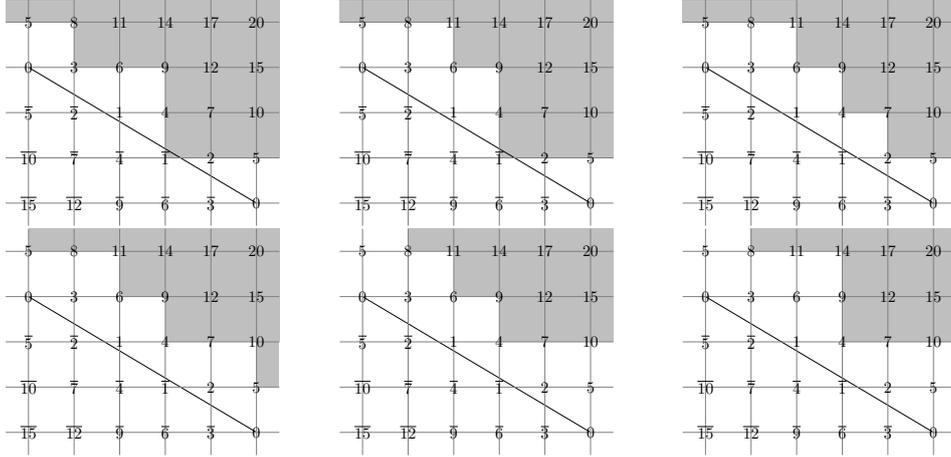


FIGURE 8. A balanced $(3, 5)$ -filter tuple \mathbf{p} with $n(\mathbf{p}) = [3, -1, 2, 5, 6]$.

An (m, n) -filter tuple may be equivalently thought of as a cycle of length n in the directed graph \mathfrak{F}_m^n of [Definition 4.3](#) with a choice of initial representative in the first equivalence class, as in [Example 4.12](#).

Example 4.12. [Figure 6](#) illustrates a cycle in \mathfrak{F}_3^5 : start at the vertex labeled by the balanced $(3, 5)$ -filter \mathbf{b} with $m(\mathbf{b}) = [-1, 3, 4]$ and then follow the red edges. This cycle corresponds to the $(3, 5)$ -filter tuple in [Figure 8](#).

An (m, n) -filter tuple \mathbf{p} is specified by the sequence of the n levels removed:

$$(12) \quad n(\mathbf{p}) = [p_0, p_1, \dots, p_{n-1}].$$

The first condition of [Definition 4.11](#) ensures that $n(\mathbf{p})$ is a permutation of $n(\mathbf{p}^{(0)})$, such that levels in the same residue class modulo m appear in increasing order.

We call $\mathbf{p} \in \mathcal{T}_m^n$ *parking* if $\mathbf{p}^{(0)} \in \mathcal{DF}_m^n$ and write \mathcal{PT}_m^n for the set of all parking (m, n) -filter tuples. We call $\mathbf{p} \in \mathcal{T}_m^n$ *balanced* if $\mathbf{p}^{(0)} \in \mathcal{BF}_m^n$ and write \mathcal{BT}_m^n for the set of all balanced (m, n) -filter tuples. [Propositions 4.8](#) and [4.10](#) show that any (m, n) -filter tuple is equivalent to a unique parking (m, n) -filter tuple and a unique balanced (m, n) -filter tuple.

Remark 4.13. We relate [Definition 4.11](#) to the definition of *(m, n) -parking paths*— (m, n) -Dyck paths whose n horizontal edges are labeled $1, 2, \dots, n$, such that levels in the same row increase from left to right. Fix $\mathbf{p} \in \mathcal{PT}_m^n$, so that $\mathbf{p}^{(0)}$ may be thought of as an (m, n) -Dyck path by [Remark 4.5](#). Number each horizontal step in this path by the order in which its left endpoint is removed in \mathbf{p} . Since $\mathbf{p}^{(i)}$ is an (m, n) -filter, points in the same row must be removed in order—this recovers the condition on levels for parking paths, as illustrated in [Figure 9](#) (which corresponds to the parking (m, n) -filter tuple of [Figure 8](#)). Thus, we may represent a parking (m, n) -filter as an (m, n) -parking path.

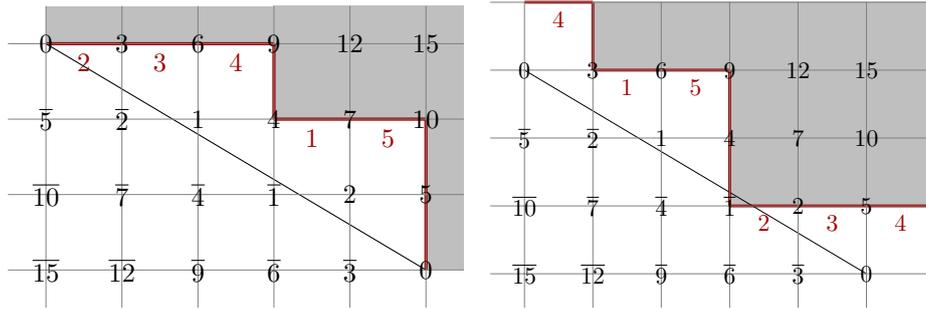


FIGURE 9. On the left is the parking $(3, 5)$ -filter tuple corresponding to the balanced $(3, 5)$ -filter tuple in [Figure 8](#), encoded in the traditional manner as a Dyck path with labeled horizontal steps (the path is marked in red). The labels record the order in which the points to their left were removed. On the right is the corresponding balanced $(3, 5)$ -filter tuple.

The following enumerative result follows from the cycle lemma, and is given geometric meaning in [Section 6.2.2](#).

Proposition 4.14 ([\[ALW16, Corollary 4\]](#), [\[AB15\]](#)).

$$|\mathcal{PT}_m^n| = m^{n-1} \text{ and } |\mathcal{PT}_n^m| = n^{m-1}.$$

5. THE ZETA MAP

After reviewing the state of the art for zeta maps in [Section 5.1](#), we use the combinatorial objects of [Section 4](#) to define two (different) bijections between parking (m, n) -filter tuples and (m, n) -parking words ([Definitions 5.1](#) and [5.3](#))—the first map is trivially a bijection, but we only conclude that the second map is a bijection as a corollary of [Theorem 1.1](#) in [Theorem 5.5](#). The composition of these two bijections defines the zeta map for rational parking words ([Definition 5.7](#)).

In [Section 5.3](#), we show that our zeta map on rational words recovers Armstrong, Loehr, and Warrington’s sweep map on rational Dyck paths using a canonical injection of Dyck paths inside parking paths.

5.1. Context and History. The classical zeta map ζ is a bijection from $(n+1, n)$ -Dyck paths to themselves developed by Garsia, Haglund, and Haiman to explain the equidistribution on Dyck paths of $(\text{area}, \text{bounce})$ and $(\text{dinv}, \text{area})$. The statistic bounce is due to Haglund, while dinv is due to Haiman; we shall not review their definitions here. This equidistribution expresses the agreement of the two formulas on the righthand side of the following combinatorial expansion of the Hilbert series of the alternating subspace of the space of diagonal coinvariants [[GH02](#), [Hai02](#), [CM18](#)]:

$$\begin{aligned} \text{Hilb} \left(\left(\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n] / \langle \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]_+^{\mathfrak{S}_n} \rangle \right)^\epsilon ; q, t \right) &= \sum_{d \in \mathcal{DW}_{n+1}^n} q^{\text{area}(d)} t^{\text{bounce}(d)} \\ &= \sum_{d \in \mathcal{DW}_{n+1}^n} q^{\text{dinv}(d)} t^{\text{area}(d)}, \end{aligned}$$

With the proper conventions⁴, the map ζ explains the equidistribution in the sense that $\text{area}(d) = \text{dinv}(\zeta(d))$ and $\text{bounce}(d) = \text{area}(\zeta(d))$.

From the point of view of lattice path combinatorics, the Dyck paths encoding the Hilbert series of the alternating subspace of the space of diagonal coinvariants are much simpler than the parking paths encoding the full Hilbert series of the space of diagonal coinvariants. Presumably due to this difference in complexity, the definition and study of the zeta map was restricted to Dyck paths at first [[Hag03](#), [GH02](#)], and its extension by Haglund and Loehr [[HL05](#)]⁵ and by Loehr and Remmel [[LR04](#)] to parking paths only came later:

$$\begin{aligned} \text{Hilb} \left(\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n] / \langle \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]_+^{\mathfrak{S}_n} \rangle ; q, t \right) &= \sum_{p \in \mathcal{PW}_{n+1}^n} q^{\text{area}(p)} t^{\text{pmaj}(p)} \\ &= \sum_{p \in \mathcal{PW}_{n+1}^n} q^{\text{dinv}(p)} t^{\text{area}(p)}, \end{aligned}$$

where pmaj is a generalization of bounce , $\text{area}(p) = \text{dinv}(\zeta(p))$, and $\text{pmaj}(p) = \text{area}(\zeta(p))$. When restricted to Dyck paths, this result generalizes the zeta map on Dyck paths [[Hag08](#), Exercise 5.7]; see also our [Proposition 5.9](#).

As Dyck paths and parking paths were generalized to the Fuss $(kn+1, n)$, Dölgoln $(kn-1, n)$, and rational (m, n) cases, extensions of the zeta map were again first defined on Dyck paths, and only later for parking paths. The definition of zeta on rational parking words turns out to be surprisingly simple, as we show in [Definitions 5.1](#), [5.3](#) and [5.7](#). This definition appears in [[GMV16](#)]—but in a different language that we postpone to [Section 6.3](#).

The table in [Figure 10](#) contains a historical summary of the definitions of zeta, where for brevity we have suppressed some details as to the exact generality of the maps involved—in the column with heading “Type,” we use “Dyck” or “Parking” to refer to the unlabeled or labeled case of lattice paths, respectively. (We recommend [[ALW15](#)] for a thorough survey of the literature on zeta maps defined on lattice paths, at least when the dimensions of the bounding rectangle are coprime.)

⁴For consistency with its generalization to parking paths, we are using the inverse of the zeta map from [[Hag08](#), Theorem 3.15].

⁵Although this bijection is between two slightly different manifestations of parking paths.

Authors	Reference	Type	Generality	Proof of Bijjectivity
Garsia Haiman Haglund	[GH02]	Dyck	$(n+1, n)$	[GH02]
Loehr	[Loe05]	Dyck	$(kn+1, n)$	[Loe05]
Haglund Loehr	[HL05]	Parking	$(n+1, n)$	[HL05]
Gorsky Mazin	[GM13] [GM14]	Dyck	coprime (m, n)	[GM13] for $(kn\pm 1, n)$ [TW18] This paper
Amstrong Loehr Warrington	[ALW15]	Dyck	N -dim. box	[TW18]
Gorsky Mazin Vazirani	[GMV16]	Parking	coprime (m, n)	[GM13] for $(kn\pm 1, n)$ This paper
Gorsky Mazin Vazirani	[GMV17]	Dyck	(m, n)	[GMV17]

FIGURE 10. A brief overview of various definitions and work on zeta maps.

5.2. The Zeta Map. We define the zeta map using two bijections A, B from parking (m, n) -filter tuples to (m, n) -parking words. The zeta map is then defined to be the map $\zeta := B \circ A^{-1}$.

Following Gorsky, Mazin, and Vazirani, the q and t statistics may be read off these (m, n) -parking words [Arm13, GMV16]:

$$\begin{aligned} \text{area}(\mathbf{p}) &:= \frac{(n-1)(m-1)}{2} - \sum_{i=0}^{n-1} A(\mathbf{p})_i, \\ \text{dinv}(\mathbf{p}) &:= \frac{(n-1)(m-1)}{2} - \sum_{i=0}^{n-1} B(\mathbf{p})_i. \end{aligned}$$

5.2.1. Area (A). Our first map is a simple application of the interpretation in Remark 4.13 of an (m, n) -filter tuple as an (m, n) -parking path.

Definition 5.1. Define $A : \mathcal{PT}_m^n \rightarrow \mathcal{PW}_m^n$ to be the (m, n) -parking word recording the column lengths (in the order of the edge labels) of the (m, n) -parking path associated to \mathbf{p} by Remark 4.13.

It is easy to see that A may be equivalently defined by

$$n(\mathbf{p}) = [p_1, p_2, \dots, p_n] \xrightarrow{A} [ap_1, ap_2, \dots, ap_n] \bmod m,$$

where $an = -1 \bmod m$.

Example 5.2. The parking $(3, 5)$ -filter tuple $\mathbf{p} \in \mathcal{PT}_3^5$ encoded by the $(3, 5)$ -parking path in Figure 9 is mapped to the $(3, 5)$ -parking word

$$A(\mathbf{p}) = 10001$$

(there is one gray box in the columns containing the horizontal edges with labels 1 and 5, and no gray boxes in the other columns). We may also compute it using the

word $n(\mathbf{p})$ from [Figure 9](#):

$$n(\mathbf{p}) = [4, 0, 3, 6, 7] \mapsto [1 \cdot 4, 1 \cdot 0, 1 \cdot 3, 1 \cdot 6, 1 \cdot 7] \bmod 3 = [1, 0, 0, 0, 1],$$

since $1 \cdot 5 = -1 \bmod 3$. We compute

$$\text{area}(\mathbf{p}) = \frac{2 \cdot 4}{2} - (1 + 0 + 0 + 0 + 1) = 2.$$

On the other hand, we compute the $(5, 3)$ -parking word for the element $\mathbf{p} \in \mathcal{PT}_5^3$ with $n(\mathbf{p}) = [0, 7, 5]$:

$$n(\mathbf{p}) = [0, 7, 5] \mapsto [3 \cdot 0, 3 \cdot 7, 3 \cdot 5] \bmod 5 = [0, 1, 0],$$

since $3 \cdot 3 = -1 \bmod 5$.

It is immediate from [Remark 4.13](#) that [Definition 5.1](#) is a bijection from \mathcal{PT}_m^n to \mathcal{PW}_m^n .

5.2.2. $\text{Divv}(B)$.

Definition 5.3. Define $B : \mathcal{PT}_m^n \rightarrow [m]^n$ by recording the relative order in the word $m(\mathbf{p}^{(i-1)})$ of the number p_i removed for $1 \leq i \leq n$. That is, we define a word $\mathbf{w} = w_1 \cdots w_n \in [m]^n$ by $w_i = j$ if p_i is the j th smallest number in $m(\mathbf{p}^{(i-1)})$.

Example 5.4. As in [Example 4.12](#), we compute $m(\mathbf{p}^{(i)})$ for each $(3, 5)$ -filter in [Figure 8](#) to be

$$[-1, \mathbf{3}, 4] \rightarrow [-1, 4, 6] \rightarrow [\mathbf{2}, 4, 6] \rightarrow [4, \mathbf{5}, 6] \rightarrow [4, \mathbf{6}, 8] \rightarrow [4, 8, 9],$$

where we haven't rebalanced. Recording the relative order of the elements removed (marked in bold above) gives the $(3, 5)$ -parking word

$$B(\mathbf{p}) = 10011.$$

We compute

$$\text{divv}(\mathbf{p}) = \frac{2 \cdot 4}{2} - (1 + 0 + 0 + 1 + 1) = 1.$$

It is not immediately obvious that [Definition 5.3](#) really does produce (m, n) -parking words.

Theorem 5.5. *The map B is a bijection from \mathcal{PT}_m^n to \mathcal{PW}_m^n .*

Proof. Let $\mathbf{p} \in \mathcal{PT}_m^n$. For $0 \leq i \leq n$, we define a point $\mathbf{x}^{(i)} \in V^m$ by $\mathbf{x}^{(i)} = m(\mathbf{p}^{(i)})$, and adding a multiple of $\mathbb{1}$ so that the sum of the elements in $\mathbf{x}^{(i)}$ is zero (since every element in $m(\mathbf{p}^{(i)})$ changes by the same amount, their relative order is preserved). So the action of $B(\mathbf{p})$ on $\mathbf{x}^{(0)} \in V^m$ (as defined in [Section 3](#)) is recorded by the sequences $\mathbf{x}^{(i)}$. Finally, $\mathbf{x}^{(0)} = \mathbf{x}^{(n)}$ because \mathbf{p} is a parking (m, n) -tuple. In particular, we have shown that the word $B(\mathbf{p})$ has a fixed point. Now [Theorem 1.1](#) tells us that $B(\mathbf{p})$ is a parking word.

Further, [Theorem 1.1](#) tells us that the fixed point of $B(\mathbf{p})$ is unique. Therefore, from $B(\mathbf{p})$, we can identify its unique fixed point $\mathbf{x}^{(0)}$, from which we can reconstruct $\mathbf{x}^{(i)}$ for all i , and thus $\mathbf{p}^{(i)}$ for all i . That is to say, from $B(\mathbf{p})$, we can reconstruct \mathbf{p} . This implies that the map B is an injection from \mathcal{PT}_m^n to \mathcal{PW}_m^n . We have already established that the map A is a bijection between these two sets, so the fact that B is an injection means that it must also be surjective. \square

Given an (m, n) -filter tuple \mathbf{p} , the fixed point for $B(\mathbf{p})$ in V^m is the word $m(\mathbf{p}^{(0)})$ —up to addition of a multiple of $\mathbb{1}$.

Example 5.6. Continuing [Example 5.4](#) (and recalling [Example 4.12](#)), balancing each $(3, 5)$ -filter $\mathbf{p}^{(i)}$ gives

$$[-1, \mathbf{3}, 4] \rightarrow [-\mathbf{2}, 3, 5] \rightarrow [\mathbf{0}, 2, 4] \rightarrow [1, \mathbf{2}, 3] \rightarrow [0, \mathbf{2}, 4] \rightarrow [-1, 3, 4].$$

Balancing doesn't change the relative order of the elements, and thinking of $\mathbf{p}^{(0)}$ as an element of V_6^3 , we observe that $\mathbf{p}^{(0)}$ is a fixed point for the action of $B(\mathbf{p}) = 10011$.

5.2.3. *The Zeta Map ($A \mapsto B$).* The zeta map ζ sends the first method of associating an (m, n) -parking word to a parking (m, n) -filter tuple in [Definition 5.1](#) to the second in [Definition 5.3](#). By [Theorem 5.5](#), we conclude that ζ is a bijection.

Definition 5.7. The *zeta map* is the bijection from \mathcal{PW}_m^n to itself defined by

$$\begin{aligned} \zeta : \mathcal{PW}_m^n &\rightarrow \mathcal{PW}_m^n \\ \mathbf{p} &\mapsto B \circ A^{-1}(\mathbf{p}) \end{aligned}$$

Examples are illustrated in [Figures 11](#) and [12](#).

$n(\mathbf{p})$	$A(\mathbf{p})$	$B(\mathbf{p})$	$n(\mathbf{p})$	$A(\mathbf{p})$	$B(\mathbf{p})$	$n(\mathbf{p})$	$A(\mathbf{p})$	$B(\mathbf{p})$
123	012	000	$\bar{1}34$	001	011	015	011	002
132	021	010	$\bar{1}43$	010	021	105	101	102
213	102	100	$4\bar{1}3$	100	201	150	110	120
231	120	110	024	020	001	$\bar{2}26$	000	012
312	201	200	042	002	020			
321	210	210	204	200	101			

		0	0	0	1			1	2	2	1
$\sum_{\mathfrak{d} \in \mathcal{DF}_4^3}$	$q^{\text{area}(\mathfrak{d})} t^{\text{dinv}(\mathfrak{d})} =$	0	1	1	0	$\sum_{\mathbf{p} \in \mathcal{PT}_4^3}$	$q^{\text{area}(\mathbf{p})} t^{\text{dinv}(\mathbf{p})} =$	2	3	1	0
		0	1	0	0			2	1	0	0
		1	0	0	0			1	0	0	0

FIGURE 11. The zeta map on \mathcal{PW}_4^3 . The rows shaded in gray correspond to the canonical embedding of \mathcal{DF}_4^3 in \mathcal{PT}_4^3 from [Remark 5.8](#).

5.3. **The Sweep Map.** In this section, we relate the zeta map on (m, n) -parking words to the sweep map on (m, n) -Dyck paths.

Having fixed m and n coprime, define the *level* of a step of a lattice path in $\mathbb{Z} \times \mathbb{Z}$ to be the level of its north/west endpoint. In [\[ALW15\]](#), Armstrong, Loehr, and Warrington defined the *sweep map* on (m, n) -Dyck paths by *sorting* the steps of a given path by their levels.⁶ See [Figure 13](#) for an example. One can visualize this procedure geometrically as a *sweep* of the line $\mathcal{H}_{\mathbf{a}, k} := \{\mathbf{x} : \mathbf{x} \bullet (m, n) = k\}$ up from $k = 0$ to $k = \infty$, as illustrated in [Figure 14](#) for $(m, n) = (4, 7)$.

It is not hard to argue that the sweep map sends an (m, n) -Dyck path to another (m, n) -Dyck path [\[TW18, Theorem 6.7\]](#), but invertibility is considerably more difficult. The sweep map and its various generalizations were first shown to be bijective by Thomas and Williams in [\[TW18\]](#).

⁶This is a special case of the general definition of the sweep map, which is on general lattice paths in an N -dimensional box.

$n(\mathbf{p})$	$A(\mathbf{p})$	$B(\mathbf{p})$	$n(\mathbf{p})$	$A(\mathbf{p})$	$B(\mathbf{p})$	$n(\mathbf{p})$	$A(\mathbf{p})$	$B(\mathbf{p})$	$n(\mathbf{p})$	$A(\mathbf{p})$	$B(\mathbf{p})$
123	031	000	024	012	001	$\bar{2}35$	001	012	015	030	002
132	013	010	042	021	020	$\bar{2}53$	010	031	051	003	030
312	103	200	204	102	101	$5\bar{2}3$	100	301	105	300	102
321	130	210	240	120	120	$\bar{1}34$	020	011	$\bar{1}16$	011	003
213	301	100	402	201	300	$\bar{1}43$	002	021	$\bar{1}\bar{1}6$	101	103
231	310	110	420	210	310	$3\bar{1}4$	200	201	$16\bar{1}$	110	130
									$\bar{3}27$	000	013

	0	0	0	0	1		0	1	2	2	1	
	0	0	1	1	0		1	4	3	1	0	
$\sum_{\mathbf{p} \in \mathcal{DW}_5^3} q^{\text{area}(\mathbf{p})} t^{\text{dinv}(\mathbf{p})} =$	0	1	1	0	0	,	$\sum_{\mathbf{p} \in \mathcal{PW}_5^3} q^{\text{area}(\mathbf{p})} t^{\text{dinv}(\mathbf{p})} =$	2	3	1	0	0
	0	1	0	0	0			2	1	0	0	0
	1	0	0	0	0			1	0	0	0	0

FIGURE 12. The zeta map on \mathcal{PW}_5^3 . The rows shaded in gray correspond to the canonical embedding of \mathcal{DF}_5^3 in \mathcal{PT}_5^3 from Remark 5.8.

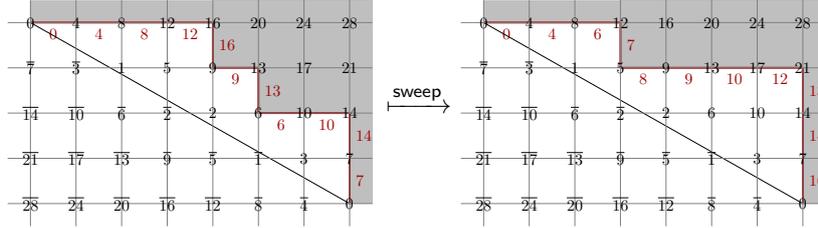


FIGURE 13. The $(4, 7)$ -filters corresponding to a path \mathfrak{d} (left) and the corresponding path $\text{sweep}(\mathfrak{d})$ (right). The horizontal steps of \mathfrak{d} are labeled by the level to their west, while the vertical steps are labeled by the level to their north; we have preserved these labels on the steps of $\text{sweep}(\mathfrak{d})$. To form the path $\text{sweep}(\mathfrak{d})$, the steps of the path \mathfrak{d} are rearranged according to the order in which they are encountered by a line of slope $-4/7$ sweeping up from below. Compare with Figure 14

Remark 5.8. There is a canonical injection

$$\begin{aligned} \mathcal{DF}_m^n &\hookrightarrow \mathcal{PT}_m^n \\ \mathfrak{d} &\mapsto \mathbf{p}_{\mathfrak{d}}, \end{aligned}$$

where $\mathbf{p}_{\mathfrak{d}}$ is the unique element of \mathcal{PT}_m^n such that $n(\mathfrak{d}) = n(\mathbf{p}_{\mathfrak{d}})$ (recall that we think of $n(\mathfrak{d})$ as a sorted list). We call $\mathbf{p}_{\mathfrak{d}}$ a *Dyck (m, n) -filter tuple*. By replacing \mathfrak{d} by $\mathbf{p}_{\mathfrak{d}}$, we may consider \mathcal{DF}_m^n as a subset of \mathcal{PT}_m^n .

We can rephrase this injection using the interpretation of \mathfrak{d} as an (m, n) -Dyck path and elements of \mathcal{PT}_m^n as (m, n) -parking paths. Thinking of \mathfrak{d} as an (m, n) -Dyck path from $(0, 0)$ to $(-n, m)$ (as in Figure 7), we label each horizontal edge by the relative order of the level of its left endpoint. This associates a canonical (m, n) -parking path to \mathfrak{d} , which corresponds to a parking (m, n) -filter tuple $\mathbf{p}_{\mathfrak{d}}$

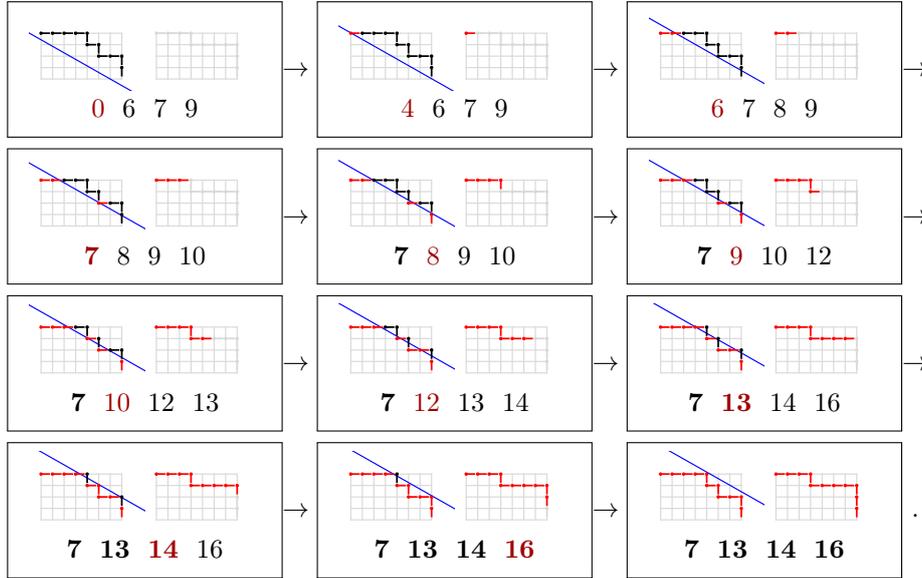


FIGURE 14. An illustration of the geometric interpretation of sweep for $(m, n) = (4, 7)$ and the paths of Figure 13. Each box in the figure corresponds to a step of the sweeping procedure. Each box contains the path \mathfrak{d} with the steps already swept marked in red (top left) and the steps of the new path $\text{sweep}(\mathfrak{d})$ already built (top right). The 4-tuples record the levels visible from the west in \mathfrak{d} if the red (swept) steps are rendered invisible—the level to be swept next is colored red. Sweeping either increases the level by 4 if it corresponds to the level to the west of a swept horizontal step, or freezes the level (indicated by bold styling) if it corresponds to the level to the north of a swept vertical step. See Remark 5.11.

by Remark 4.13. Note that the lattice path used to compute $\text{sweep}(\mathfrak{d})$ is the same as the (unlabeled) lattice path associated to $\mathfrak{p}_{\mathfrak{d}}$ in Remark 4.13, whose column heights are counted by $A(\mathfrak{p}_{\mathfrak{d}})$.

The injection of Remark 5.8 allows us to relate the zeta and sweep maps as follows.

Proposition 5.9. *For \mathfrak{d} an (m, n) -Dyck path, $B(\mathfrak{p}_{\mathfrak{d}})$ is an increasing word that records the column heights of $\text{sweep}(\mathfrak{d})$.*

Proof. We check that $B(\mathfrak{p}_{\mathfrak{d}})$ encodes $\text{sweep}(\mathfrak{d})$: by construction of $\mathfrak{p}_{\mathfrak{d}}$, the number p_i being removed when passing from $\mathfrak{p}_{\mathfrak{d}}^{(i-1)}$ to $\mathfrak{p}_{\mathfrak{d}}^{(i)}$ is the minimal level among those levels in $m(\mathfrak{p}_{\mathfrak{d}}^{(i-1)})$ which are the levels of horizontal edges. Meanwhile, the levels in $m(\mathfrak{p}_{\mathfrak{d}}^{(i)})$ with value less than that minimal horizontal edge level keep track of the vertical steps in the construction of $\text{sweep}(\mathfrak{d})$. The number of such vertical edge levels which are present as we pass from $\mathfrak{p}_{\mathfrak{d}}^{(i-1)}$ to $\mathfrak{p}_{\mathfrak{d}}^{(i)}$ tells us, on the one hand, the height of the i -th column in $\text{sweep}(\mathfrak{d})$ and on the other hand the relative order in $m(\mathfrak{p}_{\mathfrak{d}}^{(i-1)})$ of p_i (which is what B records). This is illustrated in Figure 14. \square

By Theorem 5.5, since $B : \mathcal{PT}_m^n \rightarrow \mathcal{PW}_m^n$ is a bijection, we obtain a new proof that the sweep map on (m, n) -Dyck paths is invertible.

Theorem 5.10. *For m, n coprime, the sweep map on (m, n) -Dyck paths is invertible.*

Remark 5.11. In [ALW15, Section 5.2], Armstrong, Loehr, and Warrington remark that the sweep map can be inverted if the individual levels of the steps on the path specified by $\text{sweep}(\mathfrak{d})$ can be determined. The last two authors gave an algorithm to determine these levels in [TW18].

This strategy of determining levels can be related to the fixed point of a parking word as follows. Proposition 5.9 shows that the fixed point of the (increasing) parking word $B(\mathfrak{p}_{\mathfrak{d}})$ encodes the levels of the vertical steps of $\text{sweep}(\mathfrak{d})$. For example, the left path \mathfrak{d} in Figure 14 corresponds to the (m, n) -filter tuple $\mathfrak{p}_{\mathfrak{d}}$ specified on the left of Figure 13. Then $m(\mathfrak{p}_{\mathfrak{d}}^{(n)}) = [7, 13, 14, 16]$ records the levels that should be assigned (from top left to bottom right) to the vertical steps of $\text{sweep}(\mathfrak{d})$, as illustrated on the right of Figure 13. The remaining levels—corresponding to the horizontal steps (again from top left to bottom right)—are determined from by the word $n(\mathfrak{p}_{\mathfrak{d}}^{(0)}) = [0, 4, 6, 8, 9, 10, 12]$.

6. THE AFFINE SYMMETRIC GROUP

In Section 3, we interpreted (m, n) -parking words as transformations of \mathbb{R}^m . In this section, we recall their standard interpretation as alcoves in \mathbb{R}^n .

The coincidence between the number of regions in the type $\tilde{\mathfrak{S}}_n$ Shi arrangement (Section 6.2) and the number of $(n+1, n)$ -parking words has led to many purely combinatorial investigations [Sta96, Sta98, AL99, AR12, LRW14]. Although many different authors have found many different bijections between Shi regions and parking words, this direction of research culminates in work of Gorsky, Mazin, and Vazirani [GMV16], who expand upon and generalize Armstrong’s work in [Arm13] from the Fuss to the rational level of generality. In this section, we prove several of their conjectures.

We first review the basic combinatorics of $\tilde{\mathfrak{S}}_n$ in Section 6.1. We state the simple relationship between parking (m, n) -filter tuples and the affine symmetric group in Theorems 6.6 and 6.10 and Proposition 6.7. This relationship allows us to define two maps from a generalization of Shi regions (alcoves in the Sommers region) to parking words, which are a restatement of Definitions 5.1 and 5.3.

6.1. The Affine Symmetric Group. The affine symmetric group $\tilde{\mathfrak{S}}_n$ is the group of bijections $w : \mathbb{Z} \rightarrow \mathbb{Z}$ such that

$$w(i+n) = w(i) + n \text{ and}$$

$$\sum_{i=1}^n w(i) = \binom{n+1}{2}.$$

We often represent elements of $\tilde{\mathfrak{S}}_n$ in (short) one-line notation

$$w = [w(1), w(2), \dots, w(n)].$$

A *dominant permutation* is an affine permutation w whose one-line notation increases, so that $w(1) < w(2) < \dots < w(n)$. An *inversion* of w is a pair (i, j) with $1 \leq i \leq n$ and $i < j$ such that $w(i) > w(j)$. We refer the reader to [Lus83, GMV16] for more details.

The one-line notation of affine permutations bijectively corresponds to the alcoves in the affine $\tilde{\mathfrak{S}}_n$ hyperplane arrangement, introduced in Section 2.2.⁷

⁷But note that we are now working with $\tilde{\mathfrak{S}}_n$ not $\tilde{\mathfrak{S}}_m$.

Theorem 6.1 ([GMV16, Lemma 2.9]). *Each alcove of $\mathbb{R}^n \setminus \mathcal{H}$ contains a unique point (x_1, \dots, x_n) that is the one-line notation of an element of $\tilde{\mathfrak{S}}_n$. Conversely, each element of $\tilde{\mathfrak{S}}_n$ occurs as such a point.*

The alcove labeled by the identity permutation $(1, 2, \dots, n)$ is called the *fundamental alcove* \mathcal{A}_0 . An inversion (i, j) of $w \in \tilde{\mathfrak{S}}_n$ corresponds to the hyperplane $\mathcal{H}_{i,j}^k$ that separates the alcove containing the one-line notation for w from \mathcal{A}_0 , where $j' = \begin{cases} j \bmod n & \text{if } j \neq 0 \bmod n \\ n & \text{otherwise} \end{cases}$ and $k = \frac{1}{n}(j - j')$. The bijection of [Theorem 6.1](#) between $\tilde{\mathfrak{S}}_n$ and the alcoves of $\mathbb{R}^n \setminus \mathcal{H}$ is illustrated for $n = 3$ in [Figure 15](#). On the other hand, [Figure 16](#) depicts the the labeling of an alcove by the *inverse* of the corresponding permutation.

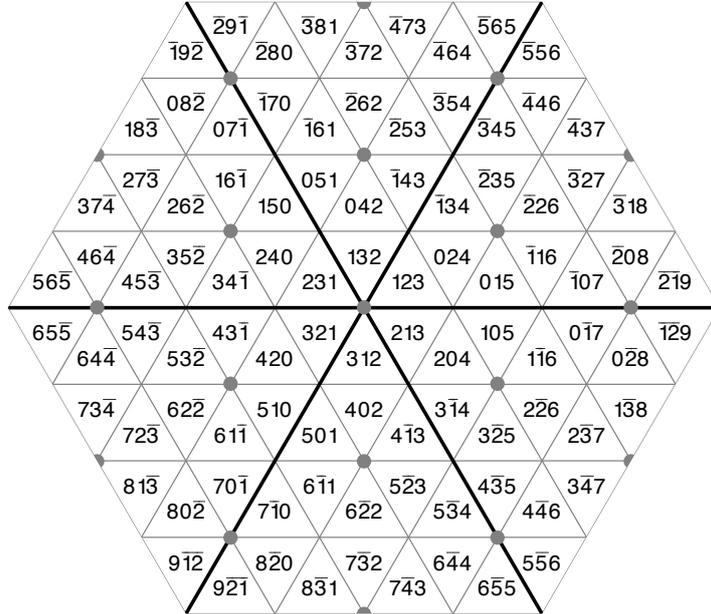


FIGURE 15. The labeling of alcoves in $\mathbb{R}^n \setminus \mathcal{H}$ by $\tilde{\mathfrak{S}}_3$. The three solid black lines are the hyperplanes $\mathcal{H}_{1,2}^0, \mathcal{H}_{1,3}^0$, and $\mathcal{H}_{2,3}^0$.

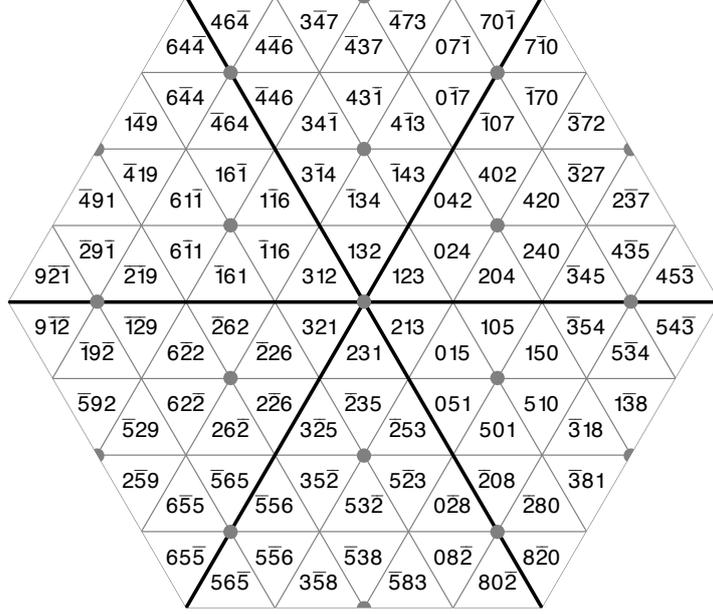
6.2. The Sommers Region.

Definition 6.2. For m coprime to n , the *Sommers region* $\mathcal{S}_m^n \subset \mathbb{R}^{n-1}$ is the region bounded by the n affine hyperplanes in $\tilde{\mathfrak{S}}_n$ of height m .

Note that when m is not coprime to n , the hyperplanes of height m do not bound a finite region. By abuse of notation, we shall write $w \in \mathcal{S}_m^n$ if w is an affine permutation labeling an alcove inside \mathcal{S}_m^n . We can detect such affine permutations with the following simple proposition.

Proposition 6.3 ([GMV16, Definition 2.14]). *An affine permutation $w^{-1} \in \tilde{\mathfrak{S}}_n$ labels an alcove in the region \mathcal{S}_m^n iff $w(i) - w(j) \neq m$ for all $i < j$.*

6.2.1. *History of the Sommers Region.* The Sommers region originated in Shi's study of Kazhdan-Lusztig cells of affine Weyl groups [Shi86], as we now outline.

FIGURE 16. The labeling of alcoves in $\mathbb{R}^n \setminus \mathcal{H}$ by inverse permutations.

The collection of affine hyperplanes

$$\bigcup_{1 \leq i < j \leq n} (\mathcal{H}_{i,j}^0 \cup \mathcal{H}_{j,i}^1)$$

is called the *Shi arrangement*, and these hyperplanes cut out connected regions called *Shi regions*. Each Kazhdan-Lusztig cell is a union of Shi regions. Following a suggestion of Carter, Shi gave an elegant geometric proof that there are $(n+1)^{n-1}$ Shi regions by showing that the inverses of the permutations labeling the minimal alcoves in the Shi regions coalesce into what has become known as the Sommers region \mathcal{S}_{n+1}^n [Shi87, Som05].⁸

There is a Fuss analogue of the Shi arrangement, defined as the hyperplanes

$$\bigcup_{\substack{1 \leq i < j \leq n \\ -k \leq s \leq k-1}} \mathcal{H}_{i,j}^s.$$

This arrangement has $(kn+1)^{n-1}$ connected regions—again, the inverses of the minimal alcoves coalesce into the Sommers region \mathcal{S}_{kn+1}^n .

The *fundamental alcove* \mathcal{A}_0 in $\tilde{\mathfrak{S}}_n$ is the simplex bounded by the affine simple hyperplanes. It turns out that \mathcal{S}_m^n is congruent to the m -fold dilation of the fundamental alcove $m\mathcal{A}_0$ —this may be realized by multiplication by the element [GMV16, Lemma 2.16],[TW17, Theorem 4.2]

$$(13) \quad w_m^n := [\ell, \ell + m, \dots, \ell + (n-1)m] \in \tilde{\mathfrak{S}}_n, \text{ where } \ell = \frac{1 + m + n - mn}{2}.$$

Variations on subarrangements of affine Weyl hyperplane arrangements has led to interesting and surprisingly difficult combinatorics [Sta96, Sta98, Ath98, PS00, AR12, LRW14, TW14], but outside of $m = kn+1$ there are no hyperplane arrangements whose regions have minimal alcoves given by the inverses of the elements in

⁸Eric Sommers was surprised to learn that the region has recently been named after him.

\mathcal{S}_m^n [GMV16, Example 9.2]. Suggestive results exist for $m = kn - 1$ using Zaslavsky's theorem enumerating bounded regions of a hyperplane arrangement (or Ehrhart duality) [FV10, FTV13], and some work has been done when m and n are not coprime [GMV17].

The regions \mathcal{S}_4^3 and \mathcal{S}_5^3 are illustrated in Figure 17.

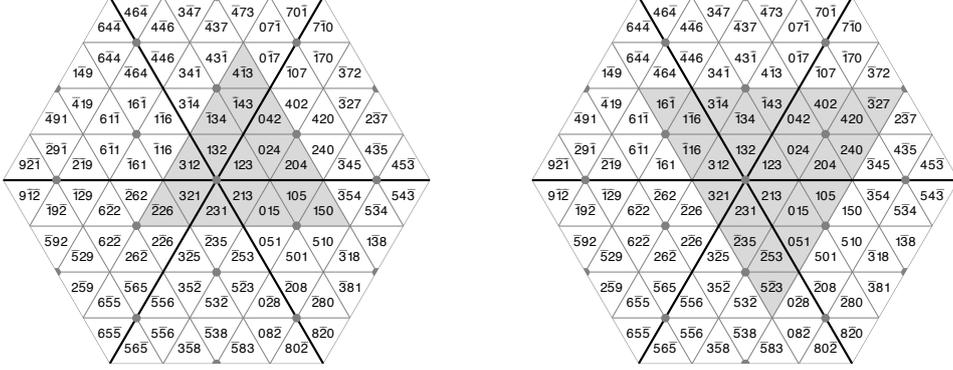


FIGURE 17. The Sommers regions \mathcal{S}_4^3 and \mathcal{S}_5^3 , with alcoves labeled by inverse permutations.

6.2.2. *Filters and the Sommers Region.* To connect (m, n) -filters and affine permutations, we define the analogue of the directed graph \mathfrak{F}_m^n in Definition 4.3.

Fix $w^{-1} \in \mathcal{S}_m^n \subset \tilde{\mathfrak{S}}_n$. An m -minimal element of w is an element from the set $\{w(i) : i \in \mathbb{N}\}$ that is minimal in its residue class modulo m . We say that an m -minimal element of w is *removable* if it is in the short one-line notation of w —that is, if it is $w(i)$ for some $1 \leq i \leq n$.

Definition 6.4. Define a directed graph \mathfrak{P}_m^n with vertex set

$$\{w \text{ dominant} : w^{-1} \in \mathcal{S}_m^n\}$$

and a directed edge between w and w' iff the short one-line notation of w' can be obtained from the short one-line notation of w by adding n to a removable m -minimal element of w , subtracting one from every element, and then resorting.

Lemma 6.5. *Acting as described in Definition 6.4 on a removable m -minimal element of a dominant w with $w^{-1} \in \mathcal{S}_m^n$ produces another dominant element whose inverse is in \mathcal{S}_m^n .*

Proof. Suppose that $w(i)$ is a removable m -minimal element, and let w' be produced as above starting from that element. Clearly w' is dominant. We now apply the condition of Proposition 6.3 to w' . The only way a problem could arise would be if there were some $j > n$ with $w(j) = w(i) + n - m$. But if $j - n < i$, the fact that $w(j - n)$ is congruent modulo m to $w(i)$ would violate the m -minimality of i , while $j - n > i$ would violate the condition of Proposition 6.3 for w . \square

We now relate (m, n) -filters and the Sommers region, using the balanced representatives of (m, n) -filters. We first use (m, n) -filters to understand *dominant* affine permutations whose inverses lie in the Sommers region.

Theorem 6.6. *A dominant affine permutation w satisfies $w^{-1} \in \mathcal{S}_m^n$ if and only if*

$$[w(1), w(2), \dots, w(n)] = n(\mathbf{b}_w)$$

for some balanced (m, n) -filter $\mathbf{b}_w \in \mathcal{BF}_m^n$.

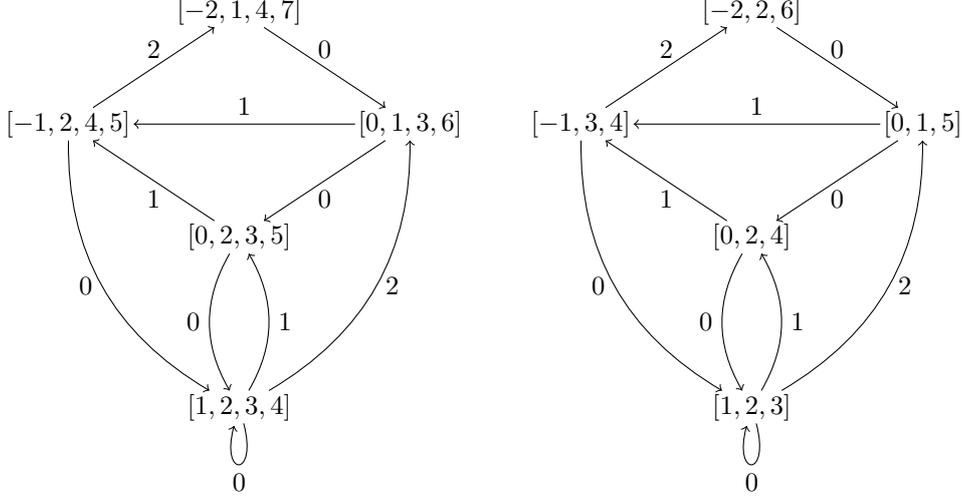


FIGURE 18. The five dominant permutations whose inverses lie in \mathcal{S}_3^4 and \mathcal{S}_4^3 , arranged in the directed graphs $\mathfrak{P}_3^4 \cong \mathfrak{P}_4^3$. The edge labels record the *position* of the removable m -minimal element chosen. Each parking word in \mathcal{PW}_4^3 occurs as a unique directed cycle of length 3 in \mathfrak{P}_3^4 , while each parking word in \mathcal{PW}_3^4 occurs as a unique directed cycle of length 4 in \mathfrak{P}_4^3 . Compare with Figure 5.

Proof. Note that the one-line notation of the element w_m^n defined in Equation (13) is $n(\mathbf{b}_m^n)$, where \mathbf{b}_m^n is the balanced (m, n) -filter generated by the points with level ℓ (see Definition 4.9). If we have $w = n(\mathbf{b})$ for some balanced (m, n) -filter, then the corresponding notion of minimal elements coincide, and acting on a minimal element of w mirrors removing the corresponding minimal element of \mathbf{b} . The result now follows from Definitions 4.3 and 6.4. \square

Of course, Theorem 6.6 applies equally well with the roles of m and n switched, and so we obtain an $(m \leftrightarrow n)$ -bijection and a version of Proposition 4.8 for dominant affine permutations whose inverses lie in the Sommers region.

Proposition 6.7. *There is a bijection*

$$\{w \text{ dominant} : w^{-1} \in \mathcal{S}_m^n\} \leftrightarrow \{w \text{ dominant} : w^{-1} \in \mathcal{S}_n^m\}.$$

Furthermore, both sets have cardinality

$$\frac{1}{n+m} \binom{n+m}{n}.$$

Proof. The enumeration follows from Theorem 6.6, and the bijection is induced by the map $n(\mathbf{b}) \leftrightarrow m(\mathbf{b})$. \square

Remark 6.8. For example, looking at the balanced (m, n) -filter on the righthand side of Figure 9, and disregarding the labels on the horizontal steps, the sorted list of the left-most level in each row gives $m(\mathbf{b}) = [-1, 3, 4]$, while the sorted list of the bottom level in each column gives $n(\mathbf{b}) = [-1, 2, 3, 5, 6]$.

Proposition 6.7 is well-known in the language of simultaneous (n, m) -cores using the bijection between n -cores and m -cores and the coroot lattices of $\tilde{\mathfrak{S}}_n$ and $\tilde{\mathfrak{S}}_m$. This bijection of Proposition 6.7 takes an element in $\tilde{\mathfrak{S}}_n$ associated to a particular

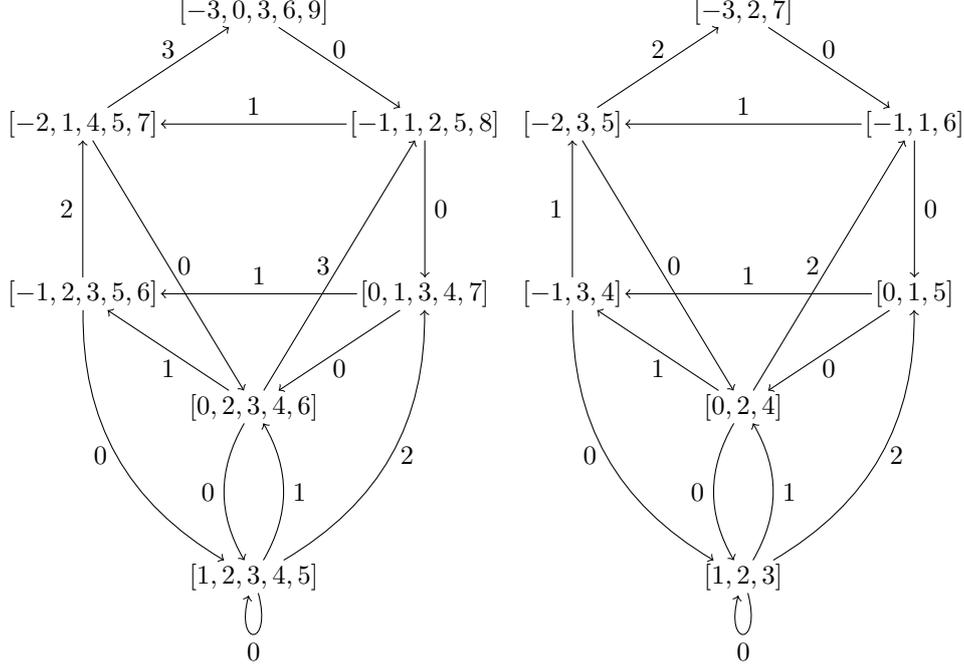


FIGURE 19. The seven dominant permutations whose inverses lie in \mathcal{S}_3^5 and \mathcal{S}_5^3 , arranged in the directed graphs $\mathfrak{P}_3^5 \cong \mathfrak{P}_5^3$. The edge labels record the *position* of the minimal element chosen. Each parking word in \mathcal{PW}_5^3 occurs as a unique directed cycle of length 3 in \mathfrak{P}_3^5 , while each parking word in \mathcal{PW}_3^5 occurs as a unique directed cycle of length 5 in \mathfrak{P}_5^3 . Compare with Figure 6.

simultaneous (m, n) -core and produces the corresponding element in $\tilde{\mathfrak{S}}_m$ associated to the same (m, n) -core. We refer the reader to [LM05], [AHJ14, Section 4] and [CDH16] for more details.

Remark 6.9. We can compute the bijection of Proposition 6.7 directly on the one-line notation of an affine permutation w by recording the m -minimal elements of w . The sequence $w(1), w(2), \dots$ is obtained by recording the lowest entry of each column of \mathfrak{b}_w , in order, then the second-lowest entry of each column, and continuing in this way. The first time an entry in a given congruence class is recorded is when we come to the leftmost entry of the corresponding row (i.e., an element of $m(\mathfrak{b}_w)$). Thus, the 3-minimal elements of $w = [-1, 2, 3, 5, 6]$ are $[-1, 3, 4]$:

$$\begin{array}{c|cccc|cc} i & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ w(i) & -\mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{5} & \mathbf{6} & \mathbf{4} & \dots \\ w(i) \bmod 3 & \mathbf{2} & \mathbf{2} & \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{1} & \dots \end{array}$$

Similarly, the 5-minimal elements of $w = [-1, 3, 4]$ are $[-1, 2, 3, 5, 6]$:

$$\begin{array}{c|ccc|ccc|cc} i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots \\ w(i) & -\mathbf{1} & \mathbf{3} & \mathbf{4} & \mathbf{2} & \mathbf{6} & \mathbf{7} & \mathbf{5} & \dots \\ w(i) \bmod 5 & \mathbf{4} & \mathbf{3} & \mathbf{4} & \mathbf{2} & \mathbf{1} & \mathbf{2} & \mathbf{0} & \dots \end{array}$$

In fact, Theorem 6.6 can be extended to the whole Sommers region if we pass from balanced (m, n) -filters to balanced (m, n) -filter tuples.

Theorem 6.10. *An affine permutation w satisfies $w^{-1} \in \mathcal{S}_m^n$ if and only if*

$$[w(1), w(2), \dots, w(n)] = n(\mathfrak{p}_w)$$

for some balanced (m, n) -filter tuple $\mathfrak{p}_w \in \mathcal{BT}_m^n$.

Proof. Choose $\mathfrak{p} \in \mathcal{BT}_m^n$. Now $n(\mathfrak{p})$ is the short one-line notation of an affine permutation w since $n(\mathfrak{p})$ is a permutation of $n(\mathfrak{p}^{(0)})$ and $\mathfrak{p}^{(0)}$ is balanced. We can think of the sequence $w(1), w(2), \dots$ as being obtained by recording the levels removed from $n(\mathfrak{p}^{(0)})$ by repeatedly removing boxes in the order specified by \mathfrak{p} . (In this way, $w(1)$ through $w(n)$ are the levels removed on the first pass, $w(n+1), \dots, w(2n)$ are the levels removed on the second pass, and so on.) Since levels that differ by m lie in the same row, the smaller is necessarily removed before the larger, guaranteeing that the condition of [Proposition 6.3](#) is satisfied, so $w \in \mathcal{S}_m^n$.

Now \mathcal{S}_m^n is an m -fold dilation of the fundamental alcove in \mathbb{R}^{n-1} , and so contains m^n affine permutations. Since $\mathfrak{p} \mapsto n(\mathfrak{p})$ is a bijection and $|\mathcal{BT}_m^n| = m^n$ by [Proposition 4.14](#), we conclude the result. \square

Remark 6.11. Since $\mathfrak{P}_m^n \cong \mathfrak{F}_m^n$, by [Definition 4.11](#) we can interpret affine elements w with $w^{-1} \in \mathcal{S}_m^n$ as cycles with n vertices in the directed graph \mathfrak{P}_m^n . The short one-line notation of w is given by reading the minimal element chosen for the edge (undoing the rebalancing that occurs at each step). For example, reproducing [Example 4.12](#), the cycle in \mathfrak{P}_3^3 with removable 3-minimal elements in bold

$$[-1, \mathbf{3}, 4] \rightarrow [-\mathbf{2}, 3, 5] \rightarrow [\mathbf{0}, 2, 4] \rightarrow [1, \mathbf{2}, 3] \rightarrow [0, \mathbf{2}, 4] \rightarrow [-1, 3, 4] \text{ (rebalanced)}$$

$$+0 \quad +1 \quad +2 \quad +3 \quad +4 \quad +5$$

$$[-1, \mathbf{3}, 4] \rightarrow [-1, 4, 6] \rightarrow [\mathbf{2}, 4, 6] \rightarrow [4, \mathbf{5}, 6] \rightarrow [4, \mathbf{6}, 8] \rightarrow [4, 8, 9] \text{ (not rebalanced)}$$

produces the one-line notation of the affine Weyl group element

$$w = [3, -1, 2, 5, 6] \in \tilde{\mathfrak{S}}_5.$$

6.3. Parking Words from the Sommers Region. Using [Theorem 6.10](#), we can easily restate the maps A and B from [Sections 5.2.1](#) and [5.2.2](#)—originally defined on parking (m, n) -tuple filters—in the language of affine permutations. These maps originally appeared in this form in [\[GMV16\]](#).

Remark 6.12. There are *many* statistics one can define on Dyck paths and parking functions (in their various combinatorial manifestations). In [\[Arm13\]](#) for $(m, n) = (n+1, n)$, Armstrong introduced statistics on the affine symmetric group that corresponded to what Haglund and Loehr called *area'* and *bounce* in [\[HL05\]](#). Armstrong suggested that his statistics would recover work in the $(kn+1, n)$ case, previously considered by Loehr and Remmel in [\[LR04\]](#). By using the relationship between Shi arrangements and Sommers regions, Gorsky, Mazin, and Vazirani generalized Armstrong's constructions to general coprime (m, n) —and called the statistics *dinv* and *area* (see [Section 5.1](#)). Finally, we note that the paper [\[ALW16\]](#) also defines statistics for general coprime (m, n) , but doesn't define a zeta map on (m, n) -parking paths or words.

6.3.1. The Map A : the Anderson Labeling. Translating [Definition 5.1](#) using the bijection [Theorem 6.10](#) gives the following definition (compare with [\[GMV16, Section 3.1\]](#)).

Definition 6.13. For $w^{-1} \in \mathcal{S}_m^n$, let $k = \min\{w(1), w(2), \dots, w(n)\}$ and write $w'(i) = w(i) - k$. Then $A(w)$ is defined by

$$w = [w(1), w(2), \dots, w(n)] \xrightarrow{A} [a \cdot w'(1), a \cdot w'(2), \dots, a \cdot w'(n)] \bmod m,$$

where $an = -1 \bmod m$.

Example 6.14. As in [Example 5.2](#), for $w = [3, -1, 2, 5, 6]$ with $w^{-1} \in \mathcal{PT}_3^5$, we compute $A(w)$ by

$$w = [3, -1, 2, 5, 6] \mapsto [1 \cdot 4, 1 \cdot 0, 1 \cdot 3, 1 \cdot 6, 1 \cdot 7] \bmod 3 = [1, 0, 0, 0, 1],$$

since $1 \cdot 5 = -1 \bmod 3$.

We also find $A(w)$ for $w = [5, -2, 3]$ with $w^{-1} \in \mathcal{PT}_5^3$:

$$w = [5, -2, 3] \mapsto [3 \cdot 7, 3 \cdot 0, 3 \cdot 5] \bmod 5 = [1, 0, 0],$$

since $3 \cdot 3 = -1 \bmod 5$.

If the short one-line notations of w_1 and w_2 are permutations of each other, then so are $A(w_1)$ and $A(w_2)$, so that elements in the same coset of $\tilde{\mathfrak{S}}_n/\mathfrak{S}_n$ are assigned to the same (m, n) -parking word, up to a permutation. It follows from [Section 5.2.1](#) and [Theorem 6.10](#) that A is a bijection; this is illustrated for $(m, n) = (4, 3)$ and $(5, 3)$ in [Figure 20](#).

Theorem 6.15. *The map*

$$\begin{aligned} A : \mathcal{S}_m^n &\rightarrow \mathcal{PW}_m^n \\ w &\mapsto A(w^{-1}) \end{aligned}$$

is a bijection.

There is a more geometric way to recover the parking word $A(w)$, which we quickly sketch. There is a natural bijection between dominant permutations of $\tilde{\mathfrak{S}}_n$ and the coroot lattice $\tilde{Q} := \{\mathbf{x} \in \mathbb{Z}^n : \sum_{i=1}^n x_i = 0\}$:

$$w \in \tilde{\mathfrak{S}}_n \mapsto w^{-1}(0).$$

The restriction of this bijection to the m -fold dilation of the fundamental alcove gives a set of representatives for $\tilde{Q}/m\tilde{Q}$, which are in bijection with (m, n) -parking words using natural coordinates and the cycle lemma. We refer the reader to [\[Hai94, GMV16, Thi16\]](#) for more details relating to this construction.

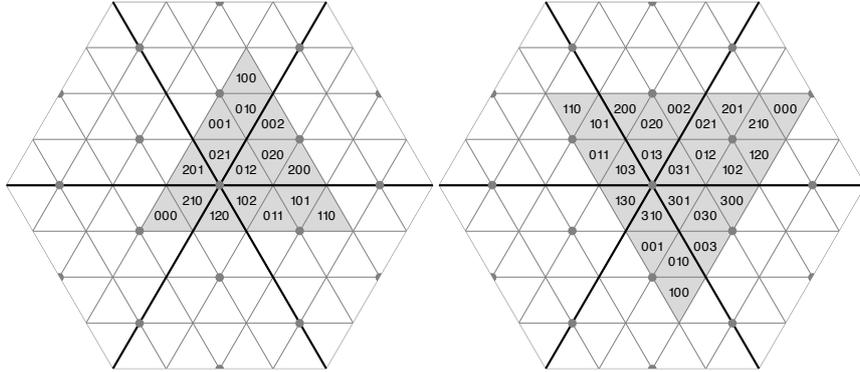


FIGURE 20. The Sommers regions \mathcal{S}_4^3 and \mathcal{S}_5^3 , with alcoves labeled by parking words under the Anderson bijection.

6.3.2. The Map B : the Pak-Stanley Labeling. It is natural to ask for a *bijective* proof for the number of Shi regions—for example, via a bijection between Shi regions and $(n+1, n)$ -parking words. Pak and Stanley found such a labeling of the Shi regions [\[Sta96, Theorem 5.1\]](#), which Stanley later extended to the Fuss level of generality [\[Sta98\]](#). Using the correspondence between the minimal alcoves of the Shi arrangement and the Sommers region, the Pak-Stanley labeling was finally

extended to the rational level in [GMV16] as an affine analogue of the *code* of a permutation.

Definition 6.16. For $w^{-1} \in \mathcal{S}_m^n$, $B(w)$ is

$$B : \tilde{\mathfrak{S}}_n \rightarrow [m]^n \\ w \mapsto \mathfrak{p}_1 \dots \mathfrak{p}_n,$$

where for $1 \leq i \leq n$,

$$\mathfrak{p}_i = \left| \left\{ j > i \in \mathbb{N} : \begin{array}{l} w(j) < w(i), \\ w(i) - w(j) < m \end{array} \right\} \right|.$$

Using the correspondence between inversions and hyperplanes, \mathfrak{p}_i counts the number of hyperplanes of the form $\mathcal{H}_{i,j}^k$ of height less than m separating the alcove corresponding to w^{-1} from the fundamental alcove. The Pak-Stanley labeling of the Sommers region is illustrated in the cases $(n, m) = (3, 4)$ and $(3, 5)$ in Figure 21.

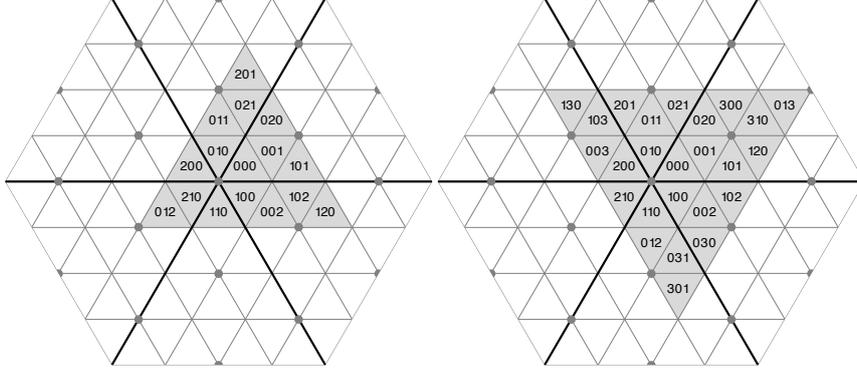


FIGURE 21. The Sommers regions \mathcal{S}_4^3 and \mathcal{S}_5^3 , with alcoves labeled by parking functions under the Pak-Stanley bijection.

We will now show that $B(w)$ in Definition 6.16 is equivalent to $B(\mathfrak{p}_w)$ in Definition 5.3 under the bijection in Theorem 6.10; in particular, this will show that $B(w)$ is a bijection from \mathcal{S}_m^n to \mathcal{PW}_m^n .

Theorem 6.17. For any w with $w^{-1} \in \mathcal{S}_m^n$,

$$B(w) = B(\mathfrak{p}_w),$$

where \mathfrak{p}_w is the (m, n) -filter tuple with $[w(1), w(2), \dots, w(n)] = n(\mathfrak{p}_w)$.

Proof. Remark 6.11 gives a bijection between n -cycles in \mathfrak{P}_n^m and affine permutations $w \in \tilde{\mathfrak{S}}_n$ with $w^{-1} \in \mathcal{S}_m^n$. Since $\mathfrak{P}_n^m \cong \mathfrak{F}_m^n$, we can see $B(\mathfrak{p}_w)$ directly on the n -cycle. Fix $1 \leq i \leq n$. At most one element from each residue class modulo m in the one-line notation of w can contribute to \mathfrak{p}_i . The number of residue classes which contribute (which equals \mathfrak{p}_i) is also the relative order of the number removed when calculating $B(\mathfrak{p}_w)$. \square

Remark 6.18. Continuing Remark 6.11, we can interpret parking words $\mathfrak{p} \in \mathcal{PW}_m^n$ as cycles with n vertices in the directed graph \mathfrak{P}_n^m . The word is obtained by reading the relative order of the minimal element chosen for the edge. For example, when $w = [3, -1, 2, 5, 6]$, recording the relative order of the element removed gives a correspondence between the 5-cycle encoding the parking $(3, 5)$ -filter tuple \mathfrak{p}_w and the parking word $B(\mathfrak{p}_w)$:

$$\begin{array}{cccccc}
 [-1, \mathbf{3}, 4] & \rightarrow & [-\mathbf{2}, 3, 5] & \rightarrow & [\mathbf{0}, 2, 4] & \rightarrow & [1, \mathbf{2}, 3] & \rightarrow & [0, \mathbf{2}, 4] & \rightarrow & [-1, 3, 4] \\
 + 0 & & + 1 & & + 2 & & + 3 & & + 4 & & + 5 \\
 [-1, \mathbf{3}, 4] & \rightarrow & [-1, 4, 6] & \rightarrow & [\mathbf{2}, 4, 6] & \rightarrow & [4, \mathbf{5}, 6] & \rightarrow & [4, \mathbf{6}, 8] & \rightarrow & [4, 8, 9] \\
 B(\mathfrak{p}_w) : & 1 & & 0 & & 0 & & 1 & & 1 &
 \end{array}$$

On the other hand, we can compute $B(w)$ by extending the short one-line notation of w . [Theorem 6.17](#) tells us that the results of these two calculations agree.

$$\begin{array}{c}
 i \\
 n(\mathfrak{p}_w) = w \\
 B(w)
 \end{array}
 \left|
 \begin{array}{cccccccccccc}
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & . \\
 \mathbf{3} & -1 & 2 & 5 & 6 & 8 & 4 & 7 & 10 & 11 & \\
 1 & 0 & 0 & 1 & 1 & & & & & &
 \end{array}
 \right.$$

The letters in the one-line notation of w that occur in $\mathfrak{p}_w^{(0)}$ are written in bold, and we have marked the inversions that count towards $B(w)$ using arrows. Note that the inversion $(i, j) = (1, 2)$ doesn't count towards $B(w)$ because $w(1) - w(2) = 3 - (-1) \geq 3$.

Now [Theorems 5.5](#) and [6.17](#) imply that B is a bijection from the S_m^n to (m, n) -parking words. This resolves [[GMV16](#), Conjecture 1.4].

Theorem 6.19 ([[GMV16](#), Conjecture 1.4]). *The map $B : S_m^n \rightarrow \mathcal{PW}_m^n$ is a bijection.*

Remark 6.20. In [[GMV16](#), Section 7.1], Gorsky, Mazin, and Vazirani provide a conjectural algorithm to invert B . Their Conjecture 7.9 (which essentially says that their algorithm succeeds) follows now from our [Theorem 5.5](#) and the convergence proved in [Lemma 3.2](#) and [Corollary 3.3](#).

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