COHERENCE, LOCAL QUASICONVEXITY, AND THE PERIMETER OF 2-COMPLEXES

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Abstract. A group is coherent if all its finitely generated subgroups are finitely presented. In this article we provide a criterion for positively determining the coherence of a group. This criterion is based upon the notion of the perimeter of a map between two finite 2-complexes which is introduced here. In the groups to which this theory applies, a presentation for a finitely generated subgroup can be computed in quadratic time relative to the sum of the lengths of the generators. For many of these groups we can show in addition that they are locally quasiconvex.

As an application of these results we prove that one-relator groups with sufficient torsion are coherent and locally quasiconvex and we give an alternative proof of the coherence and local quasiconvexity of certain 3-manifold groups. The main application is to establish the coherence and local quasiconvexity of many small cancellation groups.

Contents

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1. Introduction

1.1. **Coherence.** A group is *coherent* if all its finitely generated subgroups are finitely presented. The best known examples of coherent groups are free groups, surface groups, polycyclic groups, and 3-manifold groups. Outside of these examples, few criteria for determining the coherence or incoherence of an arbitrary group presentation are known. Free groups are easily proven to be coherent by observing that subgroups of free groups are free and hence finitely presented if they are finitely generated. Similarly, surface groups and polycyclic groups are easily shown to be coherent. The coherence of the fundamental groups of 3-manifolds is a deeper result proved independently by Scott [20] and Shalen (unpublished).

Theoretical interest in the coherence of various groups has been prompted in part by a desire to perform calculations. Groups in which all of the finitely generated subgroups have a computable finite presentation are especially amenable to computer investigation. The range of possible positive results is limited by the existence of various counterexamples. Rips has produced examples of incoherent word-hyperbolic groups [18], Bestvina and Brady have produced examples of incoherent right-angled Artin groups [3], and Wise has produced examples of compact negatively curved 2-complexes with incoherent fundamental groups [28].

Recently, Feighn and Handel proved the remarkable positive result that the mapping torus of any injective endomorphism of a free group is coherent [7]. Their theorem is related to the coherence of 3-manifolds in the following sense. Many 3-manifolds arise as surface bundles over a circle, and their fundamental groups are thus isomorphic to extensions of surface groups by $\mathbb{Z}$. The result of [7] shows that extensions of free groups by $\mathbb{Z}$ are also coherent, thus extending the often successful analogy between free groups and surface groups.

1.2. **Coherence Results.** The current investigation was primarily motivated by the following open problem which has remained unresolved for over thirty years with very little forward progress:

Problem.(G.Baumslag, [2]) *Is every one-relator group coherent?*

In this article we describe a criterion which, if successful, allows one to conclude that the group under consideration is coherent. The criterion involves a new notion which we call "perimeter". Roughly speaking, given a map $Y \to X$ between 2-complexes, the perimeter of Y which we denote by $\mathbf{P}(Y)$, is a measure of how large the "boundary" of Y is relative to X. The strategy underlying the results in this paper is that if $Y \to X$ is unsatisfactory, because for instance, it is not π_1-injective, then some 2-cells can be added to Y which reduce the perimeter, and after repeating this finitely many times, we obtain a satisfactory map between 2-complexes.

We will now state several of our main results. All of the undefined terminology, such as perimeter, weighted 2-complex, and the various hypotheses will be explained in the course of the article. The main coherence result is the following:

Theorem 3.7 (Coherence theorem)**.** *Let X be a weighted 2-complex which satisfies the perimeter reduction hypothesis.*

A) If Y is a compact connected subcomplex of a cover $\widehat{X}$ of X, and the inclusion $Y \to \widehat{X}$ is not π_1-injective, then Y is contained in a compact connected subcomplex Y' such that $\mathbf{P}(Y') < \mathbf{P}(Y)$.

B) For any compact subcomplex $C \subset \widehat{X}$, there exists a compact connected subcomplex Y containing C, such that $\mathbf{P}(Y)$ is minimal among all compact connected subcomplexes containing C. Consequently $\pi_1 X$ is coherent.

A geometric consequence of Theorem 3.7 is the following:

Theorem 3.9. *Let X be a weighted aspherical 2-complex which satisfies the perimeter reduction hypothesis. If $\widehat{X} \to X$ is a covering space and $\pi_1\widehat{X}$ is finitely generated, then every compact subcomplex of $\widehat{X}$ is contained in a compact core of $\widehat{X}$.*

1.3. **Local Quasiconvexity Results.** A subspace Y of a geodesic metric space X is quasiconvex if there is an ϵ neighborhood of Y which contains all of the geodesics in X which start and end in Y. In group theory, a subgroup H of a group G generated by A is quasiconvex if the 0-cells corresponding to H form a quasiconvex subspace of the Cayley graph $\Gamma(G, A)$.

The next main result in this article is a criterion which, if satisfied, allows one to conclude that a group is locally quasiconvex, i.e. that all finitely generated subgroups are quasiconvex. As the reader will observe from some of the applications described below, most of the groups which we can show are coherent also satisfy this stronger criterion, and thus will be locally quasiconvex as well. The exact relationship between the two criteria will become clear in the course of the article. Our main quasiconvexity theorem is the following:

Theorem 12.2 (Subgroups quasi-isometrically embed)**.** *Let X be a compact weighted 2-complex. If X satisfies the straightening hypothesis, then every finitely generated subgroup of $\pi_1 X$ embeds by a quasi-isometry. Furthermore, if $\pi_1 X$ is word-hyperbolic then it is locally quasiconvex.*

1.4. **Some applications.** The statements of the perimeter reduction hypothesis Definition 5.5 and the straightening hypothesis (Definition 12.1) are rather technical, but the strength of the theorems above can be illustrated easily through some explicit consequences that we now describe. First of all, as a consequence of Theorem 3.7 we obtain the following result:

Theorem 8.3. *Let W be a cyclically reduced word and let $G = \langle a_1, \dots \mid W^n \rangle$. If $n \geq |W| - 1$, then G is coherent. In particular, for every word W, the group $G = \langle a_1, \dots \mid W^n \rangle$ is coherent provided that n is sufficiently large.*

With a slightly stronger requirement on the degree of torsion, we can obtain the following consequence of Theorem 12.2:

Theorem 13.4. *Let $G = \langle a_1, \dots | W^n \rangle$ be a one-relator group with $n \geq 3|W|$. Then G is locally quasiconvex.*

A similar result holds for multi-relator groups:

Theorem 13.7 (Power theorem)**.** *Let $\langle a_1, \dots \mid W_1, \dots \rangle$ be a finite presentation, where each W_i is a cyclically reduced word which is not a proper power. If W_i is not freely conjugate to $W_j^{\pm 1}$ for $i \neq j$, then there exists a number N such that for all choices of integers $n_i \geq N$ the group $G = \langle a_1, \dots \mid W_1^{n_1}, \dots \rangle$ is coherent. Specifically, the number*

$$N = 6 \cdot \frac{|W_{max}|}{|W_{min}|} \sum |W_i| \tag{1}$$

has this property, where W_{max} and W_{min} denote longest and shortest words among the W_i, respectively. Moreover, if $n_i > N$ for all i, then G is locally quasiconvex.

A small cancellation application with a different flavor is the following:

Theorem 13.3. *Let $G = \langle a_1, \dots \mid R_1, \dots \rangle$ be a small cancellation presentation which satisfies $C'(1/n)$. If each a_i occurs at most $n/3$ times among the R_j, then G is coherent and locally quasiconvex.*

More precise applications to additional groups which are important in geometric group theory can be found in sections 8, 9, 13, and 14. Many of the individual results derived in these sections can be summarized by the following qualitative description:

Qualitative Summary. *If a presentation has a large number of generators relative to the sum of the lengths of the relators, and the relators are relatively long and sufficiently spread out among the generators, then the group is coherent and locally quasiconvex.*

The main triumph of these ideas, is that while we have only partially solved Baumslag's problem, we have substantially answered the problem raised by C.T.C. Wall of whether small cancellation groups are coherent [26]. In a separate paper [12], we give a much more detailed application of the strongest results in this paper to small cancellation groups. Furthermore, families of examples are constructed there which show that the applications to small cancellation theory are asymptotically sharp. An application of our theory towards the local quasiconvexity of one-relator groups with torsion is given in [9], an application towards the coherence of various other one-relator groups is given in [11], and an application towards the subgroup separability of Coxeter groups is given in [19]. Finally, an introduction to the idea of perimeter reduction can be found in [4].

1.5. **Descriptions of the Sections.** We conclude this introduction with a brief section-by-section description of the article. The concept of the perimeter of a map is introduced in section 2 and it is from this concept that all of our positive results are derived. Section 3 shows how this concept leads to the notion of a perimeter reduction and it also contains our general coherence theorem. Sections 4, 5, and 7 develop the specific case where the perimeter can be reduced through the addition of a single 2-cell. The proofs lead to procedures which are completely algorithmic. These algorithmic approaches are described in section 6. In particular we show that for the groups included in the 2-cell coherence theorem there exists an algorithm to compute an explicit finite presentation for an arbitrary finitely generated subgroup. Additionally, we show that the time it takes to produce such a finite presentation is quadratic in the total length of its set of generators. Section 7 presents a more technical theorem about coherence using sequences of paths. In section 8 the theory developed in the first half of the article is applied to the class of one-relator groups with torsion. Similarly, section 9 presents some background on small cancellation theory, and gives some applications of the theory to the coherence of small cancellation groups.

As this work has evolved over the past six years, two things have become clear: First of all, a much richer collection of positive results can be obtained by attaching "fans" of 2-cells to reduce perimeter instead of attaching single 2-cells. Secondly, the most significant conclusion of the theory appears to be local quasiconvexity rather than coherence, and in fact, we know of no word-hyperbolic group which satisfies our coherence criterion but which is not locally-quasiconvex as well. The latter part of the paper introduces fans in the context of coherence theorems that utilize them. Thereafter, fans are employed in the statements and proofs of the local quasiconvexity theorems.

Fans are defined in section 10, where the general theory is extended by incorporating fans into the statements and arguments. Section 11 contains basic definitions and results about quasiconvexity which we will need, and section 12 presents our main theorem about local quasiconvexity. In section 13 we return to small cancellation groups, and give additional coherence applications as well as some local quasiconvexity applications. Section 14 uses these applications to small cancellation groups to obtain results about 3-manifold groups. While the coherence of 3-manifold groups has long been known, when successful, our method gives a different approach towards understanding the reasons behind this remarkable theorem. Finally, in section 15, we describe theorems and algorithms related to the finitely generated intersection property and the generalized word problem.

2. Perimeter

The main goal of this section is to introduce the notion of the perimeter of a 2-complex Y relative to a particular map $\phi\colon Y \to X$. We begin with a number of basic definitions and the definition of the unit perimeter. In the second half of the section we broaden the definition to allow for the introduction of weights. The weighted perimeter of a 2-complex is a measure of the complexity of a map which will be used to prove coherence and local quasiconvexity theorems throughout the article.

Definition 2.1 (Combinatorial maps and complexes)**.** A map $Y \to X$ between CW complexes is *combinatorial* if its restriction to each open cell of Y is a homeomorphism onto an open cell of X. A CW complex X is *combinatorial* provided that the attaching map of each open cell of X is combinatorial for a suitable subdivision.

It will be convenient to be explicit about the cells in a combinatorial 2-complex.

Definition 2.2 (Polygon)**.** A *polygon* is a 2-dimensional disc whose cell structure has n 0-cells, n 1-cells, and one 2-cell where $n \geq 1$ is a natural number. If X is a combinatorial 2-complex then for each open 2-cell $C \hookrightarrow X$ there is a polygon R, a combinatorial map $R \to X$ and a map $C \to R$ such that the diagram

$$\begin{array}{ccc} C & \hookrightarrow & X \\ \downarrow & \nearrow & \\ R & & \end{array}$$

commutes, and the restriction $\partial R \to X$ is the attaching map of C. In this article the term 2-*cell* will always mean a combinatorial map $R \to X$ where R is a polygon. The corresponding *open* 2-*cell* is the image of the interior of R.

A similar convention applies to 1-cells. Let e denote the graph with two 0-cells and one 1-cell connecting them. Since combinatorial maps from e to X are in one-to-one correspondence with the characteristic maps of 1-cells of X, we will often refer to a map $e \to X$ as a 1-*cell* of X.

Definition 2.3 (Standard 2-complex)**.** In the study of infinite groups, the most commonly considered combinatorial 2-complexes correspond to presentations. Recall that the *standard* 2-*complex* of a presentation is formed by taking a unique 0-cell, adding a labeled oriented 1-cell for each generator, and then attaching a 2-cell along the closed combinatorial path corresponding to each relator.

Convention 2.4. Unless noted otherwise, all complexes in this article are combinatorial 2-complexes, and all maps between complexes are combinatorial maps. In addition, we will avoid certain technical difficulties by always assuming that all of the attaching maps for the 2-cells are immersions. For

2-complexes with a unique 0-cell, this is equivalent to allowing only cyclically reduced relators in the corresponding presentation.

Definition 2.5 (Basic definitions)**.** A local injection between topological spaces is an *immersion.* If $\phi\colon Y \to X$ is an immersion on $Y \setminus Y^{(0)}$, then ϕ is a *near-immersion.* If $\phi\colon Y \to X$ is an immersion on $Y^{(1)}$ then ϕ is a 1-*immersion*. Let $\phi_*\colon \pi_1 Y \to \pi_1 X$ be the induced homomorphism between fundamental groups. The map ϕ is π_1-*injective* [respectively π_1-*surjective*] if ϕ_* is injective [surjective]. Finally, if $\phi\colon Y \to X$ and $\psi : Z \to X$ are fixed maps, then a map $\rho : Z \to Y$ is a *lift of* ψ or *a lift of* Z *to* Y whenever the composition $\phi \circ \rho = \psi$.

Definition 2.6 (Path and cycle)**.** A *path* is a map $P \to X$ where P is a subdivided interval or a single 0-cell. In the latter case, P is a *trivial path.* A *cycle* is a map $C \to X$ where C is a subdivided circle. Given two paths $P \to X$ and $Q \to X$ such that the terminal point of P and the initial point of Q map to the same 0-cell of X, their concatenation $PQ \to X$ is the obvious path whose domain is the union of P and Q along these points. The path $P \to X$ is a *closed path* provided that the endpoints of P map to the same 0-cell of X. A path or cycle is *simple* if the map is injective on 0-cells. The *length* of the path P or cycle C is the number of 1-cells in the domain and it is denoted by $|P|$ or $|C|$. The *interior* of a path is the path minus its endpoints. In particular, the 0-cells in the interior of a path are the 0-cells other than the endpoints. A *subpath* Q of a path P [or a cycle C] is given by a path $Q \to P \to X$ [$Q \to C \to X$] in which distinct 1-cells of Q are sent to distinct 1-cells of P [C]. Notice that the length of a subpath is at most that of the path [cycle] which contains it. Finally, note that any nontrivial closed path determines a cycle in the obvious way. Finally, when the target space is understood we will often just refer to $P \to X$ as the path P.

Convention 2.7. The letters X and Y will always refer to spaces, R will always refer to a closed 2-cell, and P will always denote a path. We follow the convention that lowercase letters (such as x, y, and r) refer to specified 1-cells in the space denoted by the corresponding uppercase letter. Thus r is a 1-cell in the boundary of the 2-cell R and x is a 1-cell in the space X.

We will be very interested in examining the behavior of maps and spaces along selected 1-cells. Accordingly, the pair (Y, y) will denote a space together with a chosen 1-cell in that space, and we will write $\rho : (R, r) \to (X, x)$ to denote a map $\rho : R \to X$ with the property that $\rho(r) = x$.

Definition 2.8 (Side)**.** Let X be a fixed 2-complex, and let R be a 2-cell of X. Let r be a 1-cell in ∂R and let x be the image of r in X. The pair (R, r) will then be called a *side of a* 2-*cell of* X *which is present at* x. The collection of all sides of X which are present at x will be denoted by $\mathsf{Sides}_X(x)$, and the full collection of sides of 2-cells of X which are present at 1-cells of X will be denoted by Sides_X. Notice that saying a side (R, r) is present at x is equivalent to saying that the map $R \to X$ extends to a map $(R, r) \to (X, x)$.

Notice also that if r and r' are distinct 1-cells of R which are mapped to the same 1-cell x of X, then $(R,r) \to (X,x)$ and $(R,r') \to (X,x)$ are distinct sides at x, even though r and r' come from the same 2-cell R and are mapped to the same 1-cell x. Thus a 2-cell R whose boundary has length n will have exactly n distinct sides in Sides_X. Alternatively, the elements of $\mathsf{Sides}_X(x)$ can be viewed as the connected components in $(X-x) \cap B$, where B is a small open ball around a point in the interior of x.

Next, let $\phi\colon Y \to X$ be a map, let (R,r) be a side of X which is present at x, and let y be a 1-cell of Y with $\phi(y) = x$. We say that the side $(R,r) \to (X,x)$ is *present at* y if the map $(R,r) \to (X,x)$ factors through a map $(R,r) \to (Y,y)$ as indicated in the following commutative diagram:

$$\begin{array}{ccc} & & (Y,y) \\ & \nearrow & \downarrow \\ (R,r) & \to & (X,x) \end{array}$$

Specifically, there must exist a map $\rho : (R,r) \to (Y,y)$ such that $\phi \circ \rho$ is the map $(R,r) \to (X,x)$. If the map $(R,r) \to (X,x)$ does not factor through ϕ then (R,r) is said to be *missing at* y. The set of all sides of X which are present at y will be denoted by $\mathsf{Sides}_X(y)$, while the set of all sides of X which are missing at y will be denoted $\mathsf{Missing}_X(y)$.

Remark 2.9. It is important to notice that the definitions of the sets $\mathsf{Sides}_X(y)$ and $\mathsf{Sides}_X(x)$ both refer to the sides of 2-cells of the complex X. In particular, if $\phi(y) = x$ then $\mathsf{Sides}_X(y) \subset \mathsf{Sides}_X(x)$, and $\mathsf{Sides}_Y(y)$ is not comparable with either of these since it is a subset of Sides_Y. Moreover, $\mathsf{Sides}_X(y)$ can be smaller than $\mathsf{Sides}_Y(y)$ if the map ϕ is not a near-immersion. In fact, ϕ is a near-immersion if and only if $|\mathsf{Sides}_X(y)| = |\mathsf{Sides}_Y(y)|$ for all 1-cells $y \in Y$.

Definition 2.10 (Unit perimeter)**.** Let $\phi\colon Y \to X$ be a combinatorial map between 2-complexes. We define the *unit perimeter* of ϕ to be

$$\mathbf{P}(\phi) = \sum_{y \in \mathsf{Edges}(Y)} |\mathsf{Missing}_X(y)| = \sum_{y \in \mathsf{Edges}(Y)} |\mathsf{Sides}_X(\phi(y))| - |\mathsf{Sides}_X(y)| \tag{2}$$

For each 1-cell y of Y, we can either count the sides of X at $x = \phi(y)$ which are missing at y, or else we can count the number of sides of X that are present at x and then subtract off those which are also present at y. From the first description it is clear that the perimeter of ϕ is nonnegative.

The following examples will illustrate these distinctions. In particular, they will illustrate the significance of the maps $\phi\colon Y \to X$ and $R \to X$, respectively.

Example 2.11. Let X be the complex formed by attaching two squares along a common 1-cell x, and let Y be another complex which is isomorphic to X with common 1-cell y. Let $\phi\colon Y \to X$ be an isomorphism, and let $\psi : Y \to X$ be a map which sends y to x but which folds the two squares

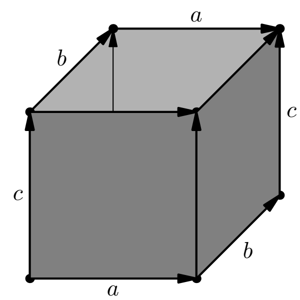

FIGURE 1. The space Y of Example 2.13

of Y to the same square of X. Observe that $\mathbf{P}(\phi\colon Y \to X) = 0$ but $\mathbf{P}(\psi : Y \to X) = 1$.

Example 2.12. Let X be the standard 2-complex of the presentation $\langle a, b \mid (aab)^3 \rangle$ and let $R \to X$ be the unique 2-cell of X. Notice that ∂R wraps three times around the path aab in X, and that there are exactly six sides present at the 1-cell labeled a and exactly three sides present at the 1-cell labeled b.

Inside the universal cover $\widetilde{X}$ of X, one can find three distinct 2-cells which share the same boundary cycle. Let $Y \subset X$ be the union of two of these three 2-cells and define $\phi\colon Y \to X$ to be the composition $Y \hookrightarrow \widetilde{X} \to X$. Observe that Y is a sphere and ϕ is an immersion. If y is a 1-cell labeled b in Y and x is its image under ϕ, then $|\mathsf{Sides}_Y(y)| = 2$, $|\mathsf{Sides}_X(y)| = 2$, and $|\mathsf{Sides}_X(x)| = 3$. Thus $|\mathsf{Missing}_X(y)| = 1$. To see the importance of the map $R \to X$, let r be a 1-cell labeled b in R and let $(R, r) \to (X, x)$ be the corresponding map of pairs. There is a 1-cell y in Y such that (R, r) is missing at y even though there are two distinct maps from (R, r) to Y which send r to y and which when composed with ϕ agree with the map $R \to X$ on ∂R. The side (R, r) is missing from y because neither of these maps agree with $R \to X$ on the interior of R.

Example 2.13 ($\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$)**.** Let $G = \langle a, b, c \mid [a, b] = [a, c] = [b, c] = 1 \rangle$ be the standard presentation of the free abelian group on three generators and let X be the standard 2-complex corresponding to this presentation. The universal cover $\widetilde{X}$ of X is usually thought of as the points of $\mathbb{R}^3$ with $x \in \mathbb{Z}$ or $y \in \mathbb{Z}$ or $z \in \mathbb{Z}$. That is, $\widetilde{X}$ is isomorphic to the union of the integer translates of the x-y, y-z, and x-z planes. The 1-cells and 2-cells of $\widetilde{X}$ are unit intervals and unit squares. Moreover, $\widetilde{X}$ is labeled so that 1-cells parallel to the x-axis are labeled by the generator a and directed in the positive x direction. Similarly, 1-cells parallel to the y-axis and z-axis are labeled by b and c directed in the positive y-direction and z-direction respectively.

If Y is a 1-by-1-by-1 box with four walls, a bottom and no top (see Figure 1), and ϕ is the obvious embedding of Y into $\widetilde{X}$, then the perimeter of Y is as follows. Each 1-cell e along the top of the open box contributes a perimeter of 3 corresponding to the sides of the three squares which are incident at $\phi(e)$ in $\widetilde{X}$ but which do not lift to Y. The vertical 1-cells along the sides of the box contribute a perimeter of 2 each, as do the 1-cells on the bottom of the box. The total perimeter of Y is 28 sides of squares.

Next, consider a more typical example of a map which is not an embedding. Let $Y \to X$ denote the composition map $Y \to \widetilde{X} \to X$. The exact same count shows that the perimeter of $Y \to X$ is also 28.

We will now define a more flexible notion of perimeter which employs a weighting on the sides of the 2-cells in X. The weighted perimeter of $\phi\colon Y \to X$ is intuitively just the sum of the weights of the corresponding missing sides. We will now make this notion more precise.

Definition 2.14 (Weighted perimeter)**.** A *weight function* on a 2-complex X is a function of the form $\mathbf{Wt} : \mathsf{Sides}(X) \to \mathbb{R}$. For most of our applications we will require that the weight of a side be nonnegative. Let $\mathbf{Wt} : \mathsf{Sides}(X) \to \mathbb{R}$ be a weight function on X, and let $\phi\colon Y \to X$ be a combinatorial map of 2-complexes. The *weighted perimeter* of $\phi\colon Y \to X$ is the sum of the weights of the sides of X which are missing at 1-cells of Y. More precisely, the weighted perimeter is defined to be the following double sum:

$$\mathbf{P}(\phi\colon Y \to X) = \sum_{y \in \mathsf{Edges}(Y)} \sum_{(R,r) \in \mathsf{Missing}_X(y)} \mathbf{Wt}\big((R,r)\big) \tag{3}$$

Notice that the weighted perimeter is equivalent to the unit perimeter when each side is assigned a weight of 1. This weight function will be called the *unit weighting*. Note also that if the assigned weights are nonnegative, then the weighted perimeter of ϕ will be nonnegative. When the map ϕ is understood we will write $\mathbf{P}(Y)$ or $\mathbf{P}(Y \to X)$ for $\mathbf{P}(\phi\colon Y \to X)$.

Definition 2.15 (Weighted 2-complex)**.** A 2-complex X is a *weighted 2-complex* if each of the sides of X has been assigned a nonnegative integer weight, the perimeter of each 1-cell is finite, and the weight of each 2-cell is positive.

Although the definition of a weighted 2-complex adopts the requirement of an integer weighting, it is sufficient, and often quite natural, to use a finite set of rationals. In the theorems which follow, a successful real-valued weight function can always be approximated by a successful rational weight function. After clearing the denominators, we then would obtain a successful integer-valued weight function. Thus there is no real loss of generality in assuming that the values of the weights are integers.

Perimeters of weighted 2-complexes satisfy the following useful property:

Lemma 2.16. *Let X be a weighted 2-complex and consider maps $\rho : Z \to Y$, $\phi\colon Y \to X$, and $\psi = \phi \circ \rho : Z \to X$. If ρ is surjective then $\mathbf{P}(Z \to X) \geq \mathbf{P}(Y \to X)$.*

Proof. First notice that $\mathsf{Missing}_X(z) \supset \mathsf{Missing}_X(y)$ whenever $\rho(z) = y$ (or equivalently that $\mathsf{Sides}_X(z) \subset \mathsf{Sides}_X(y)$). Since ρ is surjective, for every 1-cell y in Y we can select a 1-cell z in Z with $\rho(z) = y$. It is also clear that the z's chosen for distinct y's are themselves distinct. Thus the terms in the sum for $\mathbf{P}(\phi)$ can be identified with distinct terms in the sum for $\mathbf{P}(\psi)$. Finally, since the weights are nonnegative it follows that $\mathbf{P}(\psi) \geq \mathbf{P}(\phi)$. □

Definition 2.17 (Induced weights)**.** Given a weight function on X there is also an induced value assigned to each of the 1-cells and 2-cells of X. We define the *perimeter of a* 1*-cell* x in X to be the sum of the weights assigned to the sides in $\mathsf{Sides}_X(x)$. This agrees with our earlier definition of perimeter in the sense that it is the weighted perimeter of the map $\phi\colon x \to X$ which sends the single 1-cell to x in X. In particular, it measures the weights of the sides which are *not* present when the 1-cell x is considered in isolation.

We define the *weight of a* 2*-cell* R in X to be the sum of the weights assigned to the sides of the form (R, r) for some r in ∂R. The sum of the weights of the sides of a 2-cell, on the other hand, is called a weight since it is the sum of the weights of sides which are present in the 2-cell itself and it ignores the weights of the other sides which are incident at 1-cells in its boundary. Formally, we have the equations

$$\mathbf{P}(x) = \sum_{(R,r)\in \mathsf{Sides}_X(x)} \mathbf{Wt}((R,r)) \tag{4}$$

$$\mathbf{Wt}(R) = \sum_{r\in \mathrm{Edges}(\partial R)} \mathbf{Wt}((R,r)) \tag{5}$$

If Y is compact and the map $\phi\colon Y \to X$ is a near-immersion, then the perimeter of Y can be calculated from the perimeters of its 1-cells and the weights of its 2-cells. Specifically we have the following result.

Lemma 2.18. *If X is a weighted 2-complex, Y is compact, and the map $\phi\colon Y \to X$ is a near-immersion, then*

$$\mathbf{P}(\phi) = \sum_{y\in \mathsf{Edges}(Y)} \mathbf{P}(\phi(y)) - \sum_{S\in \mathsf{Cells}(Y)} \mathbf{Wt}(\phi(S)) \tag{6}$$

Proof. The second summation is the one which requires the immersion hypothesis. By Remark 2.9, the restriction on ϕ implies that the sides of X which are present at y are in one-to-one correspondence with the sides of Y which are present at y. These sides of Y can then be collected together according to the 2-cell in Y to which they belong, and then the sum of the weights of the sides of a particular 2-cell S in Y can be rewritten as the weight of the 2-cell in X which is the image of S under ϕ. □

The following example illustrates these types of calculations.

Example 2.19 (Weighted $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$)**.** Let X, Y, and $\phi\colon Y\to X$ be the spaces and maps described in Example 2.13. Specifically, let X denote the standard 2-complex of the presentation

$$\langle a,b,c \mid aba^{-1}b^{-1}, aca^{-1}c^{-1}, bcb^{-1}c^{-1}\rangle$$

Denote the three 2-cells of X by R_1, R_2 and R_3, and observe that each of them has four sides. Since the four letters of each defining word are in one-to-one correspondence with the four sides of the 2-cell, the weights of the sides of the 2-cell can be indicated by a sequence of four numbers. If we assign weights to the sides of 2-cells of X via the sequences $(1,2,3,4),(1,2,0,0)$, and $(1,3,5,0)$, then the reader can verify that $\mathbf{Wt}(R_1)=10$, $\mathbf{Wt}(R_2)=3$, $\mathbf{Wt}(R_3)=9$, $\mathbf{P}(a)=5$, $\mathbf{P}(b)=12$, and $\mathbf{P}(c)=5$. Since the map $Y\to X$ is an immersion, by Lemma 2.18 the weighted perimeter of the map $\phi\colon Y\to X$ can be calculated as follows:

$$\mathbf{P}(Y)=4\mathbf{P}(a)+4\mathbf{P}(b)+4\mathbf{P}(c)-\mathbf{Wt}(R_1)-2\mathbf{Wt}(R_2)-2\mathbf{Wt}(R_3)=54$$

3. Coherence theorem

In this section we describe a general framework for showing that groups are coherent and then employ the notion of perimeter to state our main hypothesis and to prove our main coherence theorem.

Definition 3.1 (Complexity function)**.** Let X be a fixed 2-complex, let $(N,<)$ be a well-ordered set, and let $\mathcal{C}$ be a function which assigns an element of N to each map $\phi\colon Y\to X$ with a compact domain. The function $\mathcal{C}$ is a *complexity function* for X, and the value $\mathcal{C}(\phi)$ is the *complexity of the map* ϕ. In practice, $(N,<)$ will either be $\mathbb{R}^+$ with the usual ordering, or N will be $\mathbb{R}^+\times\mathbb{R}^+$ and $<$ is defined so that $(a,b)<(c,d)$ if either $a<c$ or $a=c$ and $b<d$. This is the usual lexicographic ordering on ordered pairs.

Definition 3.2 (Reduction method)**.** Let X be a fixed 2-complex and let $\mathcal{C}$ be a complexity function for X. If for all compact spaces Y and maps $\phi\colon Y\to X$ such that ϕ is not already π_1-injective, there is a "procedure" (in any sense of the word) which produces a compact space Z and a map $\rho: Z\to X$ such that $\rho_*(\pi_1 Z)=\phi_*(\pi_1 Y)$ and such that $\mathcal{C}(\rho)<\mathcal{C}(\phi)$, then this procedure will be called a *reduction method* for $\mathcal{C}$.

Remark 3.3. Notice that for every finitely generated subgroup H in $\pi_1 X$ there exists a compact space Y and map $\phi\colon Y\to X$ such that the image of $\pi_1 Y$ under ϕ_* is exactly H. One procedure for creating Y and ϕ goes as follows: Suppose that H is generated by n elements of $\pi_1 X$ and represent each of these generators by a closed path in the 1-skeleton of X starting at the basepoint. Next let Y be a bouquet of n circles, and after subdividing Y, define $\phi\colon Y\to X$ so that the restriction of ϕ to the i-th subdivided circle is identical to the i-th closed path.

Alternatively, we could let $\widehat{X}$ be the based covering space of X corresponding to the subgroup H, and let Y be the union of the based lifts to

$\widehat{X}$ of a finite set of closed based paths representing the generators of H in $\pi_1 X$. It is clear that both of these constructions yield π_1-surjective maps. The latter has the advantage of being an immersion.

The following theorem is the philosophical basis for the coherence results in this paper.

Theorem 3.4. *Let X be a fixed space and let $\mathcal{C}$ be a complexity function for X. If there is a reduction method for $\mathcal{C}$ then $\pi_1 X$ is coherent.*

Proof. Let H be an arbitrary finitely generated subgroup of $\pi_1 X$. By Remark 3.3, there is at least one combinatorial map $\phi\colon Y \to X$ such that Y is compact and $\phi_*(\pi_1 Y) = H$. If ϕ is not π_1-injective then there is another combinatorial map with the same properties which has a strictly lower complexity. Since $(N, <)$ is well ordered, there cannot be an infinite sequences of reductions. Hence the process of replacing one combinatorial map with another must terminate at a π_1-injective combinatorial map $\rho : Z \to X$ where Z is compact and $\rho_*(\pi_1 Z) = H$. Since ρ is π_1-injective, $\pi_1 Z$ is itself isomorphic to H. Since Z is compact, H is finitely presented. In particular, a standard 2-complex for a finite presentation of H can be obtained by contracting a maximal tree in $Z^{(1)}$. □

Note that if the reduction method for $\mathcal{C}$ is constructive, then the proof of Theorem 3.4 can be used as an algorithm to effectively compute finite presentations for finitely generated subgroups. Many of the reduction methods we introduce are in fact constructive and in Section 6 we explicitly describe a resulting algorithm.

Remark 3.5. The converse of Theorem 3.4 is also true in the following sense. Given a space X with a coherent fundamental group, we define the complexity of a map $\phi\colon Y \to X$ where Y is compact to be the minimum number of 2-cells which must be added to Y to yield a π_1-injection. It is easy to see that this is indeed a complexity function, that the "procedure" of adding one of the necessary 2-cells is a method of reducing the complexity, and that there can be no infinite sequences of reductions.

We will now specialize to the case where weighted perimeter is used to measure the complexity of a map.

Definition 3.6 (Reduction hypothesis)**.** Let X be a weighted 2-complex. It will satisfy the *perimeter reduction hypothesis* if for any compact and connected space Y and for any based 1-immersion $\phi\colon Y \to X$ which is not π_1-injective, there exists a based map $\phi^+ : Y^+ \to X$ and a commutative diagram

$$\begin{array}{ccc} Y & \to & X \\ \downarrow & \nearrow & \\ Y^+ & & \end{array}$$

such that Y^+ is compact and connected, $\mathbf{P}(Y^+) < \mathbf{P}(Y)$ and $Y^+ \to X$ has the same π_1-image as $Y \to X$. Typically, the final requirement that

$\phi_*^+(\pi_1 Y^+) = \phi_*(\pi_1 Y)$ is deduced from a more stringent requirement that $Y \to Y^+$ is π_1-surjective.

Theorem 3.7 (Coherence theorem)**.** *Let X be a weighted 2-complex which satisfies the perimeter reduction hypothesis.*

A) If Y is a compact connected subcomplex of a cover $\widehat{X}$ of X, and the inclusion $Y \to \widehat{X}$ is not π_1-injective, then Y is contained in a compact connected subcomplex Y' such that $\mathbf{P}(Y') < \mathbf{P}(Y)$.

B) For any compact subcomplex $C \subset \widehat{X}$, there exists a compact connected subcomplex Y containing C, such that $\mathbf{P}(Y)$ is minimal among all such compact connected subcomplexes containing C. Consequently $\pi_1 X$ is coherent.

Proof. To prove Statement A, suppose that Y is connected and compact but the inclusion map $Y \to \widehat{X}$ is not π_1-injective. Then by the perimeter reduction hypothesis, there exists a commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \\ Y^+ \arrow[ur] & \end{tikzcd}$$

such that Y^+ is compact and connected, such that $\mathbf{P}(Y^+) < \mathbf{P}(Y)$, and such that $Y^+ \to X$ has the same π_1-image as $Y \to X$. Observe that $Y^+ \to X$ lifts to a map $Y^+ \to \widehat{X}$ which extends the lift of Y to $\widehat{X}$. Let Y' denote the image of Y^+ in $\widehat{X}$. By Lemma 2.16, $\mathbf{P}(Y') \leq \mathbf{P}(Y^+) < \mathbf{P}(Y)$.

Statement B follows immediately from the fact that the perimeters of compact subcomplexes containing C are nonnegative integers. To see that $\pi_1 X$ is coherent, let $\widehat{X}$ be a based cover of X such that $\pi_1 \widehat{X}$ is finitely generated. Observe that there exists a based compact connected subspace $C \subset \widehat{X}$ whose inclusion induces a π_1-surjection. Let $Y \subset \widehat{X}$ denote a compact connected subspace containing C such that $\mathbf{P}(Y)$ is minimal among all such compact connected subspaces. Then $Y \to \widehat{X}$ is π_1-surjective because $C \subset Y$, and $Y \to \widehat{X}$ is π_1-injective by Statement A. □

In the remainder of the article we will provide three conditions which will imply the perimeter reduction hypothesis: the 2-cell reduction hypothesis, the path reduction hypothesis, and the fan reduction hypothesis. These three hypotheses are more concrete than the perimeter reduction hypothesis and thus tend to be more useful in establishing the coherence of specific presentations.

We conclude this section by noting that Theorem 3.7 very nearly shows that finitely generated covers of complexes satisfying the perimeter reduction hypothesis have compact cores.

Definition 3.8 (Core)**.** A subcomplex Y of the complex Z is a *core* of Z if the inclusion map $Y \hookrightarrow Z$ is a homotopy equivalence. Since Y and Z are CW-complexes, Y is a core of Z if and only if there is a strong deformation retraction from Z to Y, which is true if and only if the map $Y \to Z$ induces an isomorphism on all of the homotopy groups ([27]). Note that when Z is an aspherical 2-complex, Y will be a core for Z if and only if the inclusion of Y induces a π_1-isomorphism. Indeed, if $Y \hookrightarrow Z$ is π_1-injective, then the based component of the preimage of Y in the universal cover $\widetilde{Z}$ is clearly isomorphic to the universal cover $\widetilde{Y}$ of Y. But then $\pi_2(\widetilde{Y}) = H_2(\widetilde{Y}) \subset H_2(\widetilde{Z}) = 0$, and so we see that $\widetilde{Y}$ and thus Y is aspherical.

Theorem 3.9. *Let X be a weighted aspherical 2-complex which satisfies the perimeter reduction hypothesis. If $\widehat{X} \to X$ is a covering space and $\pi_1\widehat{X}$ is finitely generated, then every compact subcomplex of $\widehat{X}$ is contained in a compact core of $\widehat{X}$.*

Proof. This follows immediately from Theorem 3.7 and Definition 3.8. □

The existence of a compact core in a 2-complex is a nontrivial fact. For example, there exists a covering space $\widehat{X}$ of a 2-complex X with a single 2-cell, such that $\pi_1\widehat{X}$ is finitely generated, but $\widehat{X}$ has no compact core. See [29] for details.

The restriction in Theorem 3.9 that X be aspherical is not particularly stringent since one of our main sources of applications will be small cancellation complexes, and small cancellation complexes in which none of the 2-cells are attached by proper powers are known to be aspherical [10, §III.11]. Roughly speaking, a 2-cell R is attached by a proper power if $\partial R \to X$ is obtained by traversing a closed path in X two or more. See Definition 4.4.

Problem 3.10 (Asphericity)**.** It appears likely that if X is a compact 2-complex which satisfies the perimeter reduction hypothesis then $\pi_1 X$ acts properly discontinuously on a contractible 2-complex. We have been unable to decide whether this is the case.

4. Attachments

By Theorem 3.7, the fundamental group of a weighted 2-complex X is coherent if there is a method for reducing the perimeter of the maps $Y \to X$ which are not π_1-injections. One of the simplest possibilities is where the perimeter is reduced through the attachment of a single 2-cell, a possibility which will be examined in detail in Section 5. In this section, we provide the definitions and results about paths, 2-cells and attachments which will be needed.

We will now describe the two elementary ways of changing a path $P \to X$: to remove a backtrack and to push across a 2-cell.

Definition 4.1 (Removing backtracks)**.** If $P \to X$ contains a subpath of the form ee^{-1} where e is a 1-cell of X, then there is another path $P' \to X$ obtained by simply removing these two 1-cells from the path. Such a change is called *removing a backtrack*. Notice that the paths $P \to X$ and $P' \to X$ are homotopic relative to their endpoints, that a path P is immersed if and only if it has no backtracks to remove, and that removing a backtrack reduces the length of the path.

Definition 4.2 (Complement)**.** Let $R \to X$ be a 2-cell, and let Q be a subpath of ∂R. There exists a unique subpath S of ∂R, called the *complement of Q in R*, such that the concatenation QS^{-1} is a closed path which corresponds to the boundary cycle ∂R. Note that if $|Q| = |\partial R|$, then S is a trivial path.

Definition 4.3 (Pushing across a 2-cell)**.** Let $P \to X$ be a path, let $R \to X$ be a 2-cell, and let Q be a subpath of both P and ∂R, so that we have the following commutative diagram:

$$\begin{array}{ccc} Q & \to & P \\ \downarrow & & \downarrow \\ R & \to & X \end{array}$$

Let S be the complement of Q in R, and observe that since S and Q have the same endpoints in X, we can form a new path P' by substituting S for the subpath Q of P. In particular, if the path $P \to X$ is the concatenation of a path P_1 followed by the path Q followed by a path P_2, then the modified path $P' \to X$ is the concatenation P_1SP_2. The replacement of $Q \to X$ by $S \to X$ is called *pushing across the 2-cell $R \to X$*. It is clear that if P' is obtained from P by pushing across a 2-cell, then P and P' will be homotopic relative to their endpoints. Notice also that $|P| > |P'|$ whenever $|Q| > |\partial R|/2$.

Definition 4.4 (Exponent of a 2-cell)**.** Let X be a 2-complex, and let $R \to X$ be one of its 2-cells. Let n be the largest number such that the map $\partial R \to X$ can be expressed as a path W^n in X, where W is a closed path in X. This number n, which measures the periodicity of the map of $\partial R \to X$, is the *exponent* of R, and a path such as W is a *period* for ∂R. Notice that any other closed path which determines the same cycle as W will also be a period of ∂R. If the exponent n is greater than 1, then the $\partial R \to X$ is called a *proper power*.

Definition 4.5 (Packet)**.** Let R be a 2-cell in X of exponent n and let W be a period of ∂R. The attaching map $\partial R \to X$ can be expressed as a path $W^n \to X$. Consider a circle subdivided into $|W|$ 1-cells, and attach a copy of R by wrapping ∂R around the circle n times. We call the resulting 2-complex $\bar{R}$. Note that there is a map $\bar{R} \to X$ such that $R \to X$ factors as $R \to \bar{R} \to X$. Observe that $\pi_1\bar{R} \cong \mathbb{Z}/n\mathbb{Z}$ and that the universal cover of $\bar{R}$ has a 1-skeleton which is identical to that of R together with n distinct copies of R attached by embeddings. The universal cover of $\bar{R}$ is the *packet of R* and is denoted by $\widetilde{R}$. Technically we should write $\widetilde{\bar{R}}$ but we will use the notation of $\widetilde{R}$ since R is its own universal cover and thus there is no danger of confusion. Notice that if the exponent of R is 1 then the packet

$\widetilde{R}$ is the same as R itself. Notice also that the map $\widetilde{R} \to X$ can be viewed as an extension of the map $R \to X$.

Let $\phi\colon Y \to X$ be a fixed map. The map ϕ will be called *packed* if whenever there is a lift of a 2-cell $R \to X$ to a 2-cell $R \to Y$, there is also a lift of $\widetilde{R} \to X$ to a map $\widetilde{R} \to Y$ which extends the map $R \to Y$. Since we will treat the packets $\widetilde{R}$ as the basic building blocks of our 2-complexes, almost all of the maps under discussion will be packed.

Definition 4.6 (2-cell attachment)**.** Let $\phi\colon Y \to X$ be an arbitrary packed map and let $R \to X$ be a 2-cell in X. The pair of paths $R \leftarrow Q \to Y$ will be called a 2-*cell attachment site* if they satisfy the following conditions:

(1) the path $Q \to R$ is a subpath of ∂R

(2) the diagram $\begin{array}{ccc} Q & \to & Y \\ \downarrow & & \downarrow \\ R & \to & X \end{array}$ commutes

(3) there does not exist a map $R \to \widetilde{R} \to Y$ which is a lift of the map $R \to \widetilde{R} \to X$ such that the composition $Q \to R \to \widetilde{R} \to Y$ equals the path $Q \to Y$.

Intuitively, a 2-cell attachment site is a portion of the boundary of R which is found in the complex Y at a location where the packet $\widetilde{R}$ does not already exist. In other words, it is a place at which attaching a copy of $\widetilde{R}$ will have an effect on the perimeter of the map. Notice that when the length of Q is equal to the length of ∂R, the path $Q \to Y$ may have distinct endpoints even though the endpoints of the path $Q \to \partial R$ are identical.

Definition 4.7 (Maximal attachment)**.** A 2-cell attachment site is *maximal* if there does not exist another pair of maps $R \leftarrow Q' \to Y$ where $Q \to R$ is a proper subpath of $Q' \to R$ and $Q \to Y$ is a proper subpath of $Q' \to Y$. Technically, we require that there does not exist a proper inclusion $Q \to Q'$ such that $Q \to Q' \to R$ is the map $Q \to R$ and $Q \to Q' \to Y$ is the map $Q \to Y$. This forces Q to appear as a proper subpath of Q' in the same manner in both cases. If $|Q| < |\partial R|$ and the 2-cell attachment site is maximal we will call it an *incomplete* 2-*cell attachment*. When $|Q| = |\partial R|$, we will call this a *complete* 2-*cell attachment*. Notice that complete attachments are automatically maximal.

Definition 4.8 (2-cell reduction)**.** Let X be a weighted 2-complex and let $Y \to X$ be a packed map. A 2-cell attachment $R \leftarrow Q \to Y$ will be called a 2-*cell perimeter reduction* if $\mathbf{P}(\widetilde{R}) < \mathbf{P}(Q)$. If $\mathbf{P}(\widetilde{R}) \leq \mathbf{P}(Q)$ it will be called a *weak* 2-*cell perimeter reduction*.

Remark 4.9 (The main idea)**.** A 2-cell perimeter reduction $R \leftarrow Q \to Y$ is so named because it can be used to reduce the perimeter of the map $\phi\colon Y \to X$ (Lemma 5.3). The main idea is as follows: Simply attach the packet $\widetilde{R}$ to Y along the path Q. Technically, the identification space $Y \cup_Q \widetilde{R}$ is formed by identifying the image of the 1-cells of Q in $\widetilde{R}$ with their image in Y. For simplicity, we write $Y^+ = Y \cup_Q \widetilde{R}$ for the resulting complex, and we call the extended map $\phi^+ : Y^+ \to X$. Since $\mathbf{P}(\widetilde{R}) < \mathbf{P}(Q)$, the cells which are in $\widetilde{R}$ and not in Q must make a net negative contribution to the perimeter and consequently $\mathbf{P}(Y^+) < \mathbf{P}(Y)$. The details and the qualifications which are necessary to justify this calculation are contained in Lemma 5.1 and Lemma 5.2. These two technical lemmas will be the key ingredients in the proof of Theorem 5.6.

The following lemma shows how the relationship between the perimeter of Q and the perimeter of the packet $\widetilde{R}$ can be reformulated as a relationship between the exponent of R, the weight of R, and the perimeter of the complement of Q. This alternative form makes it easier to verify that a specific reduction is a perimeter reduction. The original form is easier to understand conceptually.

Lemma 4.10. *Let X be a weighted 2-complex, let $R \to X$ be a 2-cell, and let $Q \to R$ be a subpath of ∂R. If n is the exponent of R and S is the complement of Q in R, then $\mathbf{P}(\widetilde{R}) = \mathbf{P}(Q) + \mathbf{P}(S) - n \cdot \mathbf{Wt}(R)$. Consequently, $\mathbf{P}(\widetilde{R}) \leq \mathbf{P}(Q)$ if and only if $\mathbf{P}(S) \leq n \cdot \mathbf{Wt}(R)$, and the first inequality is strict if and only if the second one is strict.*

Proof. Since the map $\widetilde{R} \to X$ is an immersion, Lemma 2.18 can be used to yield the first equation. The inequalities then follow as simple rearrangements of this basic equation. ☐

We conclude this section with the notion of a redundant 2-cell.

Definition 4.11 (Redundant 2-cell)**.** Let $\phi\colon Y \to X$ be a fixed map and let $R_1 \to Y$ and $R_2 \to Y$ be 2-cells in Y. We say that $R_2 \to Y$ is *redundant (relative to R_1 and the map $Y \to X$)* provided that R_1 and R_2 are distinct 2-cells in Y which have the same boundary cycle, but R_1 and R_2 project to the same 2-cell in X. More precisely, their interiors in Y are disjoint, but there exists a map $R_1 \to R_2$ which restricts to $\partial R_1 \to \partial R_2$, such that the following two diagrams commute:

$$\begin{array}{ccc} \partial R_1 & \rightarrow & \partial R_2 \\ & \searrow & \downarrow \\ & & Y \end{array} \qquad\qquad \begin{array}{ccc} R_1 & \rightarrow & R_2 \\ & \searrow & \downarrow \\ & & X \end{array}$$

Because of the way that perimeter is calculated, redundant 2-cells have no effect on the perimeter of $Y \to X$. This is made precise below and will be used in the proofs in Section 5.

Lemma 4.12. *Let X be a weighted 2-complex, let $Y \to X$ be a map, and let R_1 and R_2 be redundant 2-cells of Y. If Y' is Y minus the interior of R_1 and ϕ' is the restriction of ϕ to Y', then $\mathbf{P}(Y') = \mathbf{P}(Y)$. More generally, if Y and Y' differ by the addition or removal of redundant 2-cells, then $\mathbf{P}(Y) = \mathbf{P}(Y')$.*

Proof. Since the 1-skeletons are identical and $Y' \subset Y$, it is clear that each side that is missing at y in Y is also missing at y in Y'. To see the reverse implication, let $(R, r) \to (X, x)$ be a side of X which is present at y in Y. If $(R, r) \to (X, x)$ lifts to a side of y which is a side of the 2-cell $R_1 \to Y$, then by the definition of redundant 2-cells, it also lifts to a side of the 2-cell $R_2 \to Y$ at y. Thus every side at x which is present at y in Y is also present at y in Y'. The final assertion is now immediate. □

Finally, we relate the lack of redundant 2-cells to immersions in the following lemma whose proof is immediate.

Lemma 4.13. *If $\phi\colon Y \to X$ is a 1-immersion and Y has no redundant 2-cells, then ϕ is an immersion.*

5. 2-cell coherence theorem

In this section we show how 2-cell perimeter reductions can be used to lower the perimeter of a map $Y \to X$. At the end of the section we use this to prove a 2-cell version of our coherence theorem.

Lemma 5.1 (Complete attachment)**.** *Let X be a weighted 2-complex, let $\phi\colon Y \to X$ be a packed 1-immersion, and suppose that $\mathbf{P}(Y)$ is finite. If $R \leftarrow Q \to Y$ is a complete 2-cell attachment, then the perimeter of the induced map $\phi^+ : Y^+ \to X$ satisfies the equation*

$$\mathbf{P}(Y^+) \leq \mathbf{P}(Y) - \mathbf{Wt}(R) < \mathbf{P}(Y) \tag{7}$$

Proof. Since by assumption $|Q| = |\partial R|$, the space $Y^+ = Y \cup_Q \widetilde{R}$ can be formed by first identifying the endpoints of Q in Y, if they are not already identical, and then attaching the packet $\widetilde{R}$ along its boundary.

Next, since the space Y, with the two endpoints of Q identified, is a subcomplex of Y^+ with an identical 1-skeleton, any side of X which is missing at y in Y^+ is also missing at y in Y. This shows that the terms in the sum defining $\mathbf{P}(Y^+)$ are contained as distinct terms in the sum defining $\mathbf{P}(Y)$.

Let (R, r) be a side of X, and let y be the image of this 1-cell r under the map $Q \to Y$. If the side (R, r) was already present at y, then, using the fact that $\phi\colon Y \to X$ is a packed 1-immersion, we find that there already existed a lift of $\widetilde{R} \to X$ to Y for which the composition $Q \to R \to \widetilde{R} \to Y$ is the given map $Q \to Y$. Since this contradicts our assumption that $R \leftarrow Q \to Y$ is a 2-cell attachment site, we have shown that the side (R, r) was missing at y in Y, even though it is clearly present at y in Y^+. If we repeat this argument for each of the sides of R we can conclude that $\mathbf{P}(Y^+) \leq \mathbf{P}(Y) - \mathbf{Wt}(R)$, which is less than $\mathbf{P}(Y)$ since $\mathbf{Wt}(R) > 0$. □

A careful argument would show that $\mathbf{P}(Y^+) = \mathbf{P}(Y) - \frac{n}{d}\mathbf{Wt}(R)$ where d is the exponent of the 2-cell $R \to Y^+$.

Lemma 5.2 (Incomplete attachment)**.** *Let X be a weighted 2-complex, let $\phi\colon Y \to X$ be a packed 1-immersion and suppose that $\mathbf{P}(Y)$ is finite. If*

$R \leftarrow Q \rightarrow Y$ is an incomplete 2-cell attachment then the perimeter of the induced map $\phi^+ : Y^+ \rightarrow X$ satisfies the equation

$$\mathbf{P}(Y^+) = \mathbf{P}(Y) + \mathbf{P}(\widetilde{R}) - \mathbf{P}(Q) \tag{8}$$

Proof. Since the perimeter of $\phi\colon Y \rightarrow X$ is unaffected by the addition or removal of redundant 2-cells from Y (Lemma 4.12), we might as well assume that Y has no redundancies. By Lemma 4.13 this means that we may assume that ϕ is an immersion. We will now show that the map $Y^+ \rightarrow X$ is a near-immersion.

Since the maps $Y \rightarrow X$ and $\widetilde{R} \rightarrow X$ are immersions we only need to show that this is true when y lies in the image of Q under the map $Q \rightarrow Y^+$. Let (R, r) be a side of X which is present at y in Y^+. If this side was already present at y in Y, then, using the fact that $\phi\colon Y \rightarrow X$ is a packed immersion, we find that there already existed a lift of $\widetilde{R} \rightarrow X$ to Y for which the composition $Q \rightarrow R \rightarrow \widetilde{R} \rightarrow Y$ is the map $Q \rightarrow Y$. Since this contradicts our assumption that $R \leftarrow Q \rightarrow Y$ is a 2-cell attachment, we conclude that (R, r) must be missing at the 1-cell y in Y.

Next, suppose that (R, r) is a side of X which is present in $\widetilde{R}$ at r_1 and present in $\widetilde{R}$ at r_2. Suppose further that both r_1 and r_2 lie in Q and that they are sent to the same 1-cell y in Y^+. Since all of the sides of R are distinct, the only way in which this could happen is if the exponent of R is nontrivial, these two copies of R in $\widetilde{R}$ are distinct, and the 1-cells r_1 and r_2 differ by a path which is a multiple of the period W of ∂R. As a consequence we find that the path from r_1 to r_2 in $\widetilde{R}$ is sent to a closed path in Y which is a multiple of a period of ∂R, and it is possible to extend the path $Q \rightarrow Y$ to the entire boundary of R, thereby contradicting the maximality assumption on Q. We thus conclude that distinct sides of 2-cells in $\widetilde{R}$ are sent to distinct sides of 2-cells in Y^+. Since we also showed that these sides are disjoint from the sides of X which are present at y in Y, we now know that the map from Y^+ to X is an immersion in a small neighborhood of a point in the interior of each 1-cell.

If we assume for the moment that Y is compact, then we can calculate the perimeter of Y^+ using Equation (6) of Lemma 2.18. According to Equation (6), the perimeter of Y^+ equals the weight of its 1-cells minus the weight of its 2-cells. If we apply Equation (6) to $\widetilde{R}$ and Y separately then we add the weight of their 1-skeletons and subtract the weights of their 2-cells. The difference between these counts is precisely the 1-cells of Q in $\widetilde{R}$ which get identified to 1-cells in Y in the space Y^+. This proves Equation (8).

In the general case where we assume that $\mathbf{P}(Y)$ is finite but not that Y is compact, then we cannot use Equation (6) as we did above. Instead we argue as follows: Let S be the complement of Q in ∂R. The change in perimeter from Y to Y^+ can be computed by first adding $\mathbf{P}(S)$ corresponding to the new 1-cells in Y^+ and then subtracting $n \cdot \mathbf{Wt}(R)$ corresponding to the new sides. The resulting change in perimeter is $\mathbf{P}(S) - n \cdot \mathbf{Wt}(R)$, which is equal to $\mathbf{P}(\widetilde{R}) - \mathbf{P}(Q)$ by Lemma 4.10. $\square$

Combining Lemma 5.1 and Lemma 5.2, we have the following.

Lemma 5.3 (2-cell attachment)**.** *Let X be a weighted 2-complex, let $\phi\colon Y \to X$ be a packed 1-immersion and suppose that $\mathbf{P}(Y)$ is finite. If $R \leftarrow Q \to Y$ is a 2-cell perimeter reduction then the perimeter of the induced map $\phi^+ : Y^+ \to X$ is strictly less than $\mathbf{P}(Y)$. If it is a weak 2-cell perimeter reduction, $\mathbf{P}(Y^+) \leq \mathbf{P}(Y)$.*

Proof. Without loss of generality we may assume that the 2-cell perimeter reduction $R \leftarrow Q \to Y$ is maximal. Let $Y^+ = Y \cup_Q \widetilde{R}$. That $\mathbf{P}(Y^+) < \mathbf{P}(Y)$ [$\mathbf{P}(Y^+) \leq \mathbf{P}(Y)$] now follows immediately from either Lemma 5.1 or Lemma 5.2 depending on whether the reduction is complete or incomplete. $\square$

In addition to the process of attaching 2-cells, we will also need a second operation called folding.

Definition 5.4 (Folding along a path)**.** Let $Y \to X$ be a map between 2-complexes and let $P \to Y$ be a length 2 path whose projection to X is of the form ee^{-1} (i.e. a backtrack). If the 1-cells of P are distinct in Y, then the map $Y \to X$ can be factored as $Y \to Y' \to X$ where the complex Y' is obtained from Y by identifying the endpoints of P (if they are not already identical) and then identifying the 1-cells in the image of $P \to Y$ in the obvious way. The complex Y' is said to be obtained from Y by *folding along the path P*. If Y can be folded along some path $P \to Y$, then $Y \to X$ *admits a fold*.

Definition 5.5 (2-cell reduction hypothesis)**.** A space X is said to satisfy the *2-cell reduction hypothesis* if for any map $\phi\colon Y \to X$ which is a packed 1-immersion which is not a π_1-injection, there exists a 2-cell $R \to X$ and a 2-cell perimeter reduction $R \leftarrow Q \to Y$. Notice that if X satisfies the 2-cell reduction hypothesis and $Y \to X$ is a packed map which does not admit a fold or a 2-cell perimeter reduction, then the induced map $\pi_1 Y \to \pi_1 X$ is injective.

Theorem 5.6 (2-cell coherence)**.** *If X is a weighted 2-complex that satisfies the 2-cell reduction hypothesis, then it satisfies the perimeter reduction hypothesis, and thus $\pi_1 X$ is coherent.*

Proof. Let $Y \to X$ be a 1-immersion which is not π_1-injective. Since adding the 2-cells necessary to make $Y \to X$ a packed map does not increase perimeter, we may assume it is packed without loss of generality. By hypothesis, there is a 2-cell perimeter reduction and by Lemma 5.3 the perimeter of Y^+ will be smaller. The fact that Y and Y^+ have the same π_1 image in X is obvious. That $\pi_1 X$ is coherent now follows from Theorem 3.7. $\square$

6. Algorithms

The 2-cell coherence theorem (Theorem 5.6) can also be presented as an algorithm for constructing finite presentations from a given finite set of generators. The algorithm may be viewed as a generalization of Stallings' algorithm for graphs [22].

Theorem 6.1 (Algorithm)**.** *If X is a compact weighted 2-complex which satisfies the 2-cell reduction hypothesis, then there is an algorithm which produces a finite presentation for any subgroup of $\pi_1 X$ given by a finite set of generators.*

Proof. To help clarify that the algorithm terminates, we will use a complexity function other than the usual perimeter. We define the complexity of a map $\phi\colon Y \to X$ to be the ordered pair $\big(\mathbf{P}(Y), |Y|\big)$ where $\mathbf{P}(Y)$ is the perimeter of the map, $|Y|$ is the number of 1-cells in Y, and the ordering is the dictionary ordering. Let H be a subgroup of $\pi_1 X$ generated by a set of r elements represented by closed based paths. We let Y_1 be a based bouquet of r circles corresponding to these paths, and we define $\phi_1 : Y_1 \to X$ so that ϕ_1 takes each circle of Y_1 to the closed based path that it corresponds to. We then subdivide Y_1 so that ϕ_1 is combinatorial. Clearly the image of $\pi_1 Y_1$ equals H. Observe that since Y_1 is compact and $\mathbf{P}(x)$ is finite for each 1-cell x of X, both $\mathbf{P}(Y_1)$ and $|Y_1|$ are finite. Finally, note that Y_1 is packed.

Beginning with Y_1, the algorithm produces a sequence of maps $\phi_i : Y_i \to X$ such that for each i, $\pi_1(Y_i)$ is mapped onto H. For each i, Y_{i+1} is obtained from Y_i by either folding along a path in Y_i or by adding a copy of $\widetilde{R}$ along a path Q in Y_i such that $R \leftarrow Q \to Y_i$ is a 2-cell perimeter reduction. We will give a detailed description of these procedures below. Each of these procedures will decrease the complexity and so we know that the sequence must terminate at a 1-immersion $\phi_t : Y_t \to X$ such that Y_t does not admit a 2-cell perimeter reduction. Since X satisfies the 2-cell reduction hypothesis, we conclude that ϕ_t induces a π_1-injection, and therefore maps $\pi_1 Y_t$ isomorphically onto H, thus yielding a finite presentation for H. As will be seen from the descriptions given below, each of these procedures can be implemented algorithmically. Assume inductively that Y_i is compact and packed and that $\phi_i : Y_i \to X$ maps $\pi_1 Y_i$ onto H.

Folding along a path: If ϕ_i is not an immersion on $Y_i^{(1)}$, then there exists a map $\rho_i : Y_i \to Y_{i+1}$ which is obtained by folding along a path. There is also a map $\phi_{i+1} : Y_{i+1} \to X$ such that ϕ_i factors as $Y_i \to Y_{i+1} \to X$. Because $\rho_i : Y_i \to Y_{i+1}$ is π_1-surjective, we see that ϕ_{i+1} maps $\pi_1 Y_{i+1}$ onto H. Thus by Lemma 2.16 we have $\mathbf{P}(\phi_{i+1}) \leq \mathbf{P}(\phi_i)$. Since we also have $|Y_{i+1}| < |Y_i|$, we see that the complexity of ϕ_{i+1} is strictly less than the complexity of ϕ_i.

We can continue folding along paths until we reach a map $\phi_j : Y_j \to X$ where the restriction of ϕ_j to $Y_j^{(1)}$ is an immersion. At this point we begin looking for a 2-cell perimeter reduction.

Adding 2-cells: Suppose that the restriction of ϕ_i to $Y_i^{(1)}$ is an immersion, but that there exists a 2-cell perimeter reduction, $R \leftarrow Q \rightarrow Y_i$. The reduction can be chosen to be maximal, and the result is a 2-cell reduction which is either complete or incomplete. In both cases we define Y_{i+1} to be the identification space $Y_i \cup_Q \widetilde{R}$ obtained by identifying the 1-cells of Q in $\widetilde{R}$ with their images in Y_i under the map $Q \rightarrow Y_i$. The map $\phi_{i+1} : Y_{i+1} \rightarrow X$ is well defined since $\phi_i : Y_i \rightarrow X$ and $\widetilde{R} \rightarrow X$ agree on the respective images of the 1-cells of Q which were identified to form Y_{i+1}. Also it is easy to see that the natural map $\psi_i : Y_i \rightarrow Y_{i+1}$ is π_1-surjective, and therefore since $\phi_i = \phi_{i+1} \circ \psi_i$ we conclude that the π_1-image of ϕ_{i+1} is H. It is again clear that Y_{i+1} is packed and compact.

Finally, the complexity of ϕ_{i+1} is strictly less than the complexity of ϕ_i since by Lemma 5.3 $\mathbf{P}(Y_{i+1}) < \mathbf{P}(Y_i)$. □

The sequence of spaces described above is very similar to the sequence of spaces which would be constructed by Theorem 3.7 when X satisfies the 2-cell reduction hypothesis. The main difference between the two is that the sequence of spaces in the proof of Theorem 3.7 are subcomplexes of the covering space $\widehat{X}$. In an algorithmic approach, the structure of this covering space is unavailable and the spaces described above have been constructed without reference to the space $\widehat{X}$. In fact, these spaces may not embed or even immerse into $\widehat{X}$ throughout the course of the proof.

We also note the following features of the algorithm:

(1) The algorithm gives an alternate proof of Theorem 5.6.
(2) The compactness assumption can be replaced by an appropriate recursiveness hypothesis and the algorithm is still effective.

The algorithm can be used to prove coherence when the hypothesis is weakened to allow for weak 2-cell perimeter reductions. In this more general context, however, one will not know when to stop running the algorithm and the algorithm as stated cannot be used effectively, even when X is compact. We note, however, that Oliver Payne [16] has developed a variation of our algorithm which is effective for weak 2-cell perimeter reductions so long as all of the sides have positive weights.

We conclude this section with an estimate of the efficiency of the algorithm for finding finite presentations.

Corollary 6.2. *Let X be a compact weighted 2-complex which satisfies the perimeter reduction hypothesis and let Y_1 be the based bouquet of r circles corresponding to a set of generators of a subgroup of $\pi_1 X$. There exist constants C_1 and C_2 depending only on X such that the algorithm described above terminates in fewer than $C_1|Y_1|$ steps and the time it takes to complete each step is bounded by $C_2|Y_1|$, where $|Y_1|$ denotes the number of 1-cells in Y_1. In particular, the algorithm to calculate a finite presentation for the subgroup with these r generators is $O(|Y_1|^2)$.*

Proof. Since X is compact, there is a bound C on the perimeter of any 1-cell in X and a bound C' on the length of the boundary of a 2-cell in X. Next, notice that both folds and perimeter reductions will decrease the integer $C'\mathbf{P}(Y_i) + |Y_i|$. This is because a fold will decrease the number of 1-cells without increasing the perimeter, while a perimeter reduction will decrease $\mathbf{P}(Y_i)$ by 1 while the number of 1-cells is increased by at most $C' - 1$. Since the perimeter remains nonnegative, the number of steps will be bounded by

$$C'\mathbf{P}(Y_1) + |Y_1| \leq C'C|Y_1| + |Y_1|$$

Thus we can choose $C_1 = C'C + 1$. Notice that since the number of steps is $O(|Y_1|)$ and since each step adds at most a bounded number of 1-cells, the number of 1-cells in Y_i is also $O(|Y_1|)$. And since $Y_i^{(1)}$ is connected, the number of 0-cells in Y_i is also $O(|Y_1|)$.

Next we show that the time it takes to complete each step is $O(|Y_1|)$. Let $|X|$ be the number of 1-cells in X and let the 1-skeleton of Y_i be represented as an adjacency list. To check for the existence of a fold in Y_i only requires an examination of the links (the adjacency lists) of each 0-cell. In each link we only need to check $2|X| + 1$ 1-cells before we either find a fold or exhaust the link. Thus a single link can be checked in constant time. Since it is well-known that the time it takes to implement a breadth-first search of a connected graph represented by adjacency lists is $O(|E|)$ (see [5, Section 23.2]), the time it takes to visit each 0-cell in Y_i is $O(|Y_1|)$, and thus checking for a fold in Y_i is $O(|Y_1|)$.

Next, suppose that $Y_i^{(1)}$ is immersed into X. Since X is compact, there is a finite list of paths $Q \to R \to X$ which can lead to 2-cell perimeter reductions. Given one of these paths and a 0-cell v in Y_i it takes a finite amount of time to check whether there is a lift of Q which starts at v. The constant nature of this search depends on the fact that the 1-skeleton of Y_i is immersed into X. This guarantees that the links of the 0-cells are bounded in size and that at each point there is at most one extension of the lift which is a viable candidate. Thus the search for a 2-cell perimeter reduction in this type of complex is also $O(|Y_1|)$. Since the final complex can easily be converted into a finite presentation in quadratic time, the proof is complete. $\square$

7. Path coherence theorem

In this section we provide a second, more technical application of Theorem 3.7. Our new hypothesis will imply that any immersion which is not π_1-injective admits a 2-cell attachment which does not increase the perimeter, but with additional restrictions. The new hypothesis will involve sequences of closed paths in the space X. We begin with an example which shows why these technicalities might be desirable.

Example 7.1 (Infinite reductions)**.** Let X be the standard 2-complex of $\langle a, b \mid ab = ba\rangle \cong \mathbb{Z}^2$, and give X the unit weighting. Let $H = \langle ab^{-1}\rangle$,

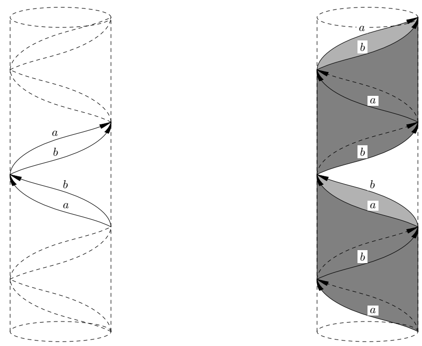

FIGURE 2. The spaces described in Example 7.1

and let $\widehat{X}$ be the based cover of X corresponding to H. As illustrated in Figure 2, $\widehat{X}$ is an infinite cylinder. Observe that every proper π_1-surjective subcomplex of $\widehat{X}$ admits a weak 2-cell perimeter reduction. This example will show that hypothesizing weak 2-cell perimeter reductions is insufficient to guarantee that the process of successively attaching 2-cells will stop.

Consider the situation where we begin with the subcomplex Y which is the image of the closed path $a^{-1}b^{-1}ab$ in $\widehat{X}$. This subcomplex is shown on the left side of Figure 2. The four 1-cells determine a length 4 closed path which is the boundary path of a 2-cell in the cylinder. Note that the inclusion map $Y \to \widehat{X}$ is not π_1-injective. Although there is an obvious complete 2-cell attachment which will make the inclusion a π_1-injection, it is also possible to apply an infinite sequence of 2-cell attachments which are *weak* 2-cell perimeter reductions, but at each stage the inclusion map will still fail to be π_1-injective. These 2-cell attachments are formed by adding squares above or below the square hole bounded by the original closed path. The right side of Figure 2 shows the subcomplex obtained by adding two squares above the original closed path and two squares below it. The perimeter is 8, which is the same as $\mathbf{P}(Y)$. Clearly, the operation of adding squares which do not change the perimeter can continue indefinitely. We conclude that a weak version of the 2-cell reduction hypothesis, in and of itself, is insufficient to guarantee that a π_1-injective subcomplex will be obtained after a finite number of steps.

The reason why we never reach a π_1-injective subcomplex in Example 7.1 is that attached 2-cells were not linked in any way to the failure of the π_1-injectivity. Our plan will be that the order in which the 2-cell attachments are applied will be tied to the existence of curves which are essential in Y and null-homotopic in $\widehat{X}$. Such precision was not needed for the 2-cell reduction hypothesis because the process automatically stopped after a finite number of steps at a π_1-injective subcomplex.

Definition 7.2 (Pushing across perimeter-reducing 2-cell)**.** Suppose that the path $P' \to X$ is obtained from $P \to X$ by pushing across the 2-cell

$R \to X$. According to Definition 4.3, this means there is a certain subpath Q of P which is also a subpath of ∂R, and P' is obtained from P by replacing Q with its complement S in R. We will now augment this definition with certain perimeter requirements. If, in addition, $\mathbf{P}(\widetilde{R}) < \mathbf{P}(Q)$, then P' is obtained from P by *pushing across a perimeter-reducing* 2*-cell*. Similarly, if $\mathbf{P}(\widetilde{R}) \leq \mathbf{P}(Q)$, then it is obtained by *pushing across a weakly perimeter-reducing* 2*-cell*.

Definition 7.3 (Path reduction hypothesis)**.** We say that a weighted 2-complex X satisfies the *path reduction hypothesis* provided the following condition holds:

For every nontrivial closed null-homotopic path $P \to X$, there is a sequence of closed paths $\{P_1, \ldots, P_t\}$ which starts at the path $P = P_1$, ends at trivial path P_t, and for each i, P_{i+1} is obtained from P_i by either the removal of a backtrack or a weakly perimeter-reducing push across a 2-cell.

Two elementary conditions which imply the path reduction hypothesis are a decrease in length and a descrease in area. In order to make the second condition precise we recall the defintion of area of a disc diagram.

Definition 7.4 (Disc diagram)**.** A *disc diagram* D is a compact contractible 2-complex with a fixed embedding in the plane. A *boundary cycle* P of D is a closed path in ∂D which travels entirely around D (in a manner respecting the planar embedding of D).

Let $P \to X$ be a closed null-homotopic path. A *disc diagram in* X *for* P is a disc diagram D together with a map $D \to X$ such that the closed path $P \to X$ factors as $P \to D \to X$ where $P \to D$ is the boundary cycle of D. The van Kampen's lemma [25] essentially states that every null-homotopic path $P \to X$ is the boundary cycle of a disc diagram. We define $\mathsf{Area}(D)$ to be the number of 2-cells in D. For a null-homotopic path $P \to X$, we define $\mathsf{Area}(P)$ to equal the minimal number of 2-cells in a disc diagram $D \to X$ that has boundary cycle P. The disc diagram $D \to X$ will then be referred to as a *minimal area disc diagram* for P.

Lemma 7.5. *Each of the following implies the path reduction hypothesis:*

(1) *Every immersed nontrivial null-homotopic path* $P \to X$ *admits a push across a weakly perimeter-reducing* 2*-cell which yields a strictly shorter path* $P' \to X$*.*
(2) *Every immersed nontrivial null-homotopic path* $P \to X$ *admits a push across a weakly perimeter-reducing* 2*-cell which yields a path* $P' \to X$ *satisfying* $\mathsf{Area}(P) > \mathsf{Area}(P')$*.*

Proof. In either case, there is an obvious procedure for creating the sequence of paths $P_i \to X$ which starts at a given closed null-homotopic path $P \to X$ and ends at the trivial path. We first remove backtracks repeatedly until we obtain an immersed path, then use the condition to find a weakly perimeter-reducing push across a 2-cell, and then repeat. In each case, the process must

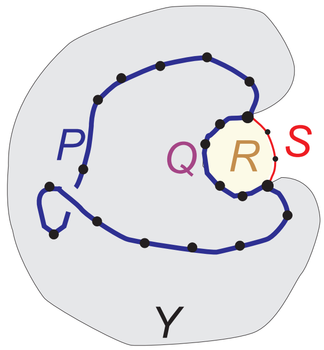

FIGURE 3. A 2-cell attachment to Y along a portion of the essential path P

terminate at a trivial path after finitely many steps because the removal of backtracks does not increase either length or area. □

We will now show that the path reduction hypothesis implies coherence.

Theorem 7.6 (Path coherence)**.** *If X is a weighted 2-complex which satisfies the path reduction hypothesis, then X satisfies the perimeter reduction hypothesis, and thus $\pi_1 X$ is coherent.*

Proof. We will assume that the map $Y \to X$ is packed, for otherwise we could attach 2-cells to form a packed map $Y^+ \to X$ with $\mathbf{P}(Y^+) < \mathbf{P}(Y)$ and with the inclusion $Y \to Y^+$ a π_1-surjection.

If $Y \to X$ is not π_1-injective then there is a closed essential path $P \to Y$ such that the composition $P \to Y \to X$ is a null-homotopic path in X, and by the path reduction hypothesis there exists a sequence of paths $P_i \to X$ for $1 \le i \le t$ which starts at $P_1 = P \to X$, ends at the trivial path $P_t \to X$, and for each i, P_{i+1} is obtained from P_i by either removing a backtrack or a weakly perimeter-reducing push across a 2-cell. We will use this sequence of paths to create a sequence of compact spaces $Y_1 \to Y_2 \to \cdots \to Y_t$ and maps $Y_i \to X$ and a sequence of paths $P_i \to Y_i$ which are lifts of the paths $P_i \to X$.

Let $Y_1 = Y$ and let $P_1 \to Y_1$ equal $P \to Y$ and assume that Y_i and $P_i \to Y_i$ have been defined for some i. The space Y_{i+1} is obtained from Y_i as follows. If the operation transforming P_i into P_{i+1} is either the removal of a backtrack or a weakly perimeter-reducing push across a 2-cell R where the map $\widetilde{R} \to X$ already lifts to Y at the appropriate point, then $Y_{i+1} = Y_i$. The exact requirement in the latter case is that $\widetilde{R} \to X$ lift to a map $\widetilde{R} \to Y_i$ such that the composition $Q \to R \to \widetilde{R} \to Y_i$ is the map $Q \to Y_i$ obtained by restricting the path $P_i \to Y$. Since $Y_i = Y_{i+1}$ it is clear that $\mathbf{P}(Y_i) = \mathbf{P}(Y_{i+1})$. The path $P_{i+1} \to Y_{i+1}$ is defined to be the obvious modification of the path $P_i \to Y_i$.

If the operation is a weakly perimeter-reducing push across a 2-cell R and the map $\widetilde{R} \to X$ does not lift to Y at the appropriate point, then

Y_{i+1} is defined to be $Y_i \cup_Q \widetilde{R}$ and P_{i+1} is again the obvious modification of the path $P_i \to Y_i \to Y_{i+1}$. Figure 3 illustrates a 2-cell attachment which arises in this way. The technical condition is that there does not exist a lift to a map $\widetilde{R} \to Y_i$ such that the composition $Q \to R \to \widetilde{R} \to Y_i$ is the map $Q \to Y_i$ obtained by restricting the path $P_i \to Y$. Notice that in this case $R \leftarrow Q \to Y_i$ is a 2-cell attachment which is a weak 2-cell perimeter reduction. If this attachment is complete, then by Lemma 5.1, $\mathbf{P}(Y_{i+1}) < \mathbf{P}(Y_i)$. If the attachment is incomplete, then by Lemma 5.2, $\mathbf{P}(Y_{i+1}) = \mathbf{P}(Y_i) + \mathbf{P}(\widetilde{R}) - \mathbf{P}(Q) \leq \mathbf{P}(Y_i)$.

In each instance the path $P_{i+1} \to Y_{i+1}$ is obtained from $P_i \to Y_i$ by lifting the operation which occurred in X to Y_{i+1}. Combining the sequence of perimeter inequalities we see that $\mathbf{P}(Y) \geq \mathbf{P}(Y_t)$ with a strict inequality if any of the 2-cell attachments were complete attachments. It only remains to show that at least one of the attachments was complete. Notice that at each stage the closed path $P_i \to Y_i \subset Y_{i+1}$ is homotopic to the closed path $P_{i+1} \to Y_{i+1}$. The crucial observation is that the removal of a backtrack or an incomplete attachment can never change an essential path into a null-homotopic one. Since the final path P_t is null-homotopic, at least one of the attachments must have been complete. Thus $\mathbf{P}(Y) > \mathbf{P}(Y_t)$.

To complete the proof, let $Y^+ = Y_t$. We note $Y_1 \to Y^+$ is π_1-surjective because it is the composition $Y = Y_1 \to Y_2 \to \cdots \to Y_t$ and for each i the map $Y_i \to Y_{i+1}$ is either a homeomorphism, a complete attachment or an incomplete attachment and thus always a π_1-surjection. □

Remark 7.7. Let X be a weighted 2-complex which satisfies the path reduction hypothesis. If Y is a compact connected subcomplex of a cover $\widehat{X}$ of X, then according to Theorem 3.7 and Theorem 7.6, there exists a sequence of subcomplexes $Y = Y_1 \subset Y_2 \subset \cdots \subset Y_f = Z$ such that for $i \geq 1$, the space Y_{i+1} is the image of $Y_i \cup_Q R$ in $\widehat{X}$, where $R \leftarrow Q \to Y_i$ is a 2-cell attachment which is a weak perimeter reduction and Z is a π_1-isomorphic subcomplex of $\widehat{X}$.

Theorem 7.6 and Remark 7.7 lead to a pair of interesting corollaries. The first corollary is a bound on the number of relators needed in the presentation for a finitely generated subgroup.

Corollary 7.8. *Let X be a weighted 2-complex which satisfies the path reduction hypothesis and assume that each 2-cell of X is attached along a simple cycle in $X^{(1)}$. If $\{W_i \to X\}$ is a finite set of closed based paths in X, then the finitely generated subgroup $H \subset \pi_1 X$ generated by the closed based paths $\{W_i \to X\}$ has a finite presentation with at most $\sum_i \mathbf{P}(W_i)$ relations. Similarly, for any π_1-surjective subcomplex $Y \subset \widehat{X}$ whose perimeter is finite and whose fundamental group is free, there is a finite presentation for $\pi_1 \widehat{X}$ where the number of relators is at most $\mathbf{P}(Y \to X)$.*

Proof. We first note that the second assertion includes the first assertion as a special case. In particular, given closed paths $W_i \to X$ we can take the based cover $\widehat{X}$ which corresponds to H and lift the closed paths $W_i \to X$ to closed paths $W_i \to \widehat{X}$ with a common basepoint. The union of the images of these closed paths is a finite graph Y in $\widehat{X}$ which satisfies the description in the second half of the corollary. Thus it suffices to prove the second assertion.

According to Remark 7.7, there is a sequence of subspaces $Y = Y_1 \subset Y_2 \subset \cdots \subset Y_f = Z$ such that for $i \geq 1$, the space Y_{i+1} is the image of $Y_i \cup_Q R$ in $\widehat{X}$, where $Y_i \leftarrow Q \to R$ is a 2-cell attachment which is a weak perimeter reduction, and such that Z is a π_1-isomorphic subcomplex of $\widehat{X}$.

If all of the 2-cell attachments are perimeter reductions, then the argument is easy, because then $f \leq \mathbf{P}(Y)$ and so Z can be obtained from Y by the addition of fewer than $\mathbf{P}(Y)$ 2-cells. Consequently, $\pi_1 Z$ has a presentation with fewer than $\mathbf{P}(Y)$ 2-cells and we are done.

When some of the 2-cell attachments are weak perimeter reductions, we argue as follows: Let $\mathbf{R}(Y_i)$ be the minimum number of relators which are needed to give a finite presentation of $\pi_1 Y_i$. We will show inductively that $\mathbf{R}(Y_i) + \mathbf{P}(Y_i) \leq \mathbf{P}(Y_1)$. This is true for $i = 1$ since the fundamental group of $Y_1 = Y$ is free. Suppose that Y_{i+1} is obtained from Y_i by an incomplete attachment. In this case there is a new 1-cell which is added to Y_{i+1} and this 1-cell appears exactly once in the attaching map of the new 2-cell. This is where we use the additional hypothesis that the attaching map of each 2-cell embeds in X. Now we can collapse the new 2-cell across this new 1-cell to see that no new relations have been added, although this 2-cell attachment may have added new generators. Since the perimeter has not increased, the inequality is still true. If, on the other hand, Y_{i+1} is obtained from Y_i by a complete attachment, then $\mathbf{R}(Y_{i+1}) \leq \mathbf{R}(Y_i) + 1$, but $\mathbf{P}(Y_{i+1}) \leq \mathbf{P}(Y_i) - 1$. Thus the inequality holds in this case. When the process stops, the perimeter is still nonnegative and thus $\mathbf{R}(Y_f)$ is bounded by $\mathbf{P}(Y_1)$, which is the assertion. □

Note that a similar result (using essentially the same proof) can be proved under the assumption that each 2-cell is attached along a (possibly trivial) power of some simple cycle. The same type of proof can also be used to provide an upper bound on the Euler characteristic of a subgroup.

Corollary 7.9. *Let X be a weighted 2-complex which satisfies the path reduction hypothesis and assume that no 2-cell of X is attached by a proper power. For any π_1-surjective compact subcomplex $Y \subset \widehat{X}$ there is a compact π_1-isomorphic subcomplex $Z \subset \widehat{X}$ such that $\chi(Z) + \mathbf{P}(Z) \leq \chi(Y) + \mathbf{P}(Y)$.*

Proof. According to Remark 7.7, there is a sequence of subspaces $Y = Y_1 \subset Y_2 \subset \cdots \subset Y_f = Z$ such that for $i \geq 1$, the space Y_{i+1} is the image of $Y_i \cup_Q R$ in $\widehat{X}$, where $Y_i \leftarrow Q \to R$ is a 2-cell attachment which is a weak perimeter reduction, and such that Z is a π_1-isomorphic subcomplex of $\widehat{X}$.

We will deduce that $\chi(Z) + \mathbf{P}(Z) \leq \chi(Y) + \mathbf{P}(Y)$ by showing that for each i we have $\chi(Y_{i+1}) + \mathbf{P}(Y_{i+1}) \leq \chi(Y_i) + \mathbf{P}(Y_i)$. For each i, Y_{i+1} is the union of Y_i and the closure of some 2-cell. First suppose that $Y_{i+1}^{(1)} = Y_i^{(1)}$. In this case, Y_{i+1} is obtained from Y_i by the addition of a single 2-cell and so $\chi(Y_{i+1}) = \chi(Y_i) + 1$ but $\mathbf{P}(Y_{i+1}) \leq \mathbf{P}(Y_i) - 1$, so the inequality holds. Next suppose that $Y_{i+1}^{(1)} \neq Y_i^{(1)}$, in which case $\chi(Y_{i+1}) \leq \chi(Y_i)$ because while a 2-cell has been added, at least one nontrivial arc of 1-cells is added to Y_i along its endpoints. Since $\mathbf{P}(Y_{i+1}) \leq \mathbf{P}(Y_i \cup_Q R) \leq \mathbf{P}(Y_i)$, we see that $\chi(Y_{i+1}) + \mathbf{P}(Y_{i+1}) \leq \chi(Y_i) + \mathbf{P}(Y_i)$. $\square$

We note that a similar statement can be proved in case some of the 2-cells are attached by proper powers. We close the section with the following problem.

Problem 7.10. Let X be a compact weighted 2-complex. Suppose that $\mathbf{P}(\Delta) < \mathbf{P}(\partial\Delta)$ for every minimal area disc diagram $\Delta \to X$. Does it follow that $\pi_1 X$ is coherent?

We conjecture that the answer is yes, but it is not clear how to proceed. The problem which arises is that maps $\Delta \to X$ which do not send the sides of the boundary of Δ injectively to the sides of X are, in a fairly strong sense, unavoidable.

8. One-relator groups with torsion

In this section we present a criterion for the coherence of one-relator groups with torsion, followed by some applications. Additional criteria for the coherence of other types of one-relator groups are developed in [11] and a similar criterion will be described for small cancellation groups in Section 9. The coherence criterion for one-relator groups is a combination of Theorem 7.6 and the "spelling theorem" of B.B. Newman. (The original reference is [15]; see [10] and [9] for combinatorial and geometric proofs.) Here is the theorem as it is usually formulated.

Theorem 8.1 (B.B. Newman)**.** *Let $G = \langle a_1, \dots \mid W^n \rangle$ where W is a cyclically reduced word and $n > 1$. Let U and V be words in $\{a_1^{\pm 1}, a_2^{\pm 1}, \dots\}$ which are equivalent in G. If U is freely reduced and V omits a generator which occurs in U, then U contains a subword W' which is also a subword of W^n and $|W'| > |W^{n-1}|$. In particular, if U is a nontrivial word which represents the identity in G, then U contains such a subword W'.*

Because of the correspondence between presentations and their standard 2-complexes, we will express our main theorem about one-relator groups in the language of 2-complexes. Recall that by Convention 2.4 the 2-complexes under consideration will be those which correspond to presentations whose defining relators are cyclically reduced.

Theorem 8.2 (Coherence criterion for one-relator groups)**.** *Let X be a weighted 2-complex with a unique 2-cell $R \to X$ and a unique 0-cell. Let $W \to X$ be the period and let $n > 1$ be the exponent of $\partial R \to X$. If the* *inequality* $\mathbf{P}(S) \leq n \cdot \mathbf{Wt}(R)$ *holds for every subpath* S *of* ∂R *satisfying* $|S| < |W|$*, then* $\pi_1 X$ *is coherent.*

Proof. Let P be a closed immersed null-homotopic path in X and let U be the word corresponding to P in the generators of the presentation corresponding to X. Since P is immersed, U is freely reduced, and so by Theorem 8.1 there exists a subpath Q in P such that Q is a subpath of ∂R and $|Q| > (n-1)|W|$. Note that we are applying the spelling theorem in the special case where V is the trivial word. Since the complement of Q is a path $S \to \partial R$ with $|S| < |W|$, we know by assumption that $\mathbf{P}(S) \leq n \cdot \mathbf{Wt}(R)$. By Lemma 4.10, it follows that $\mathbf{P}(\widetilde{R}) \leq \mathbf{P}(Q)$. Therefore P can be pushed across a weakly perimeter-reducing 2-cell. Moreover, the new path obtained by replacing Q with S, is strictly shorter than P because $n > 1$. Thus by Lemma 7.5, X satisfies the path reduction hypothesis, and so $\pi_1 X$ is coherent by Theorem 7.6. □

As an application of Theorem 8.2, we obtain the following:

Theorem 8.3. *Let* W *be a cyclically reduced word and let* $G = \langle a_1, \dots \mid W^n \rangle$*. If* $n \geq |W| - 1$*, then* G *is coherent. In particular, for every word* W*, the group* $G = \langle a_1, \dots \mid W^n \rangle$ *is coherent provided that* n *is sufficiently large.*

Proof. Let X be the standard 2-complex of the presentation with the unit weighting. We can assume that $n > 1$ since otherwise G is virtually free, and hence obviously coherent. Without loss of generality we can also assume that W is not a proper power since this would only serve to make the hypothesis more stringent.

Let R denote the unique 2-cell of X, and regard the word W as a path $W \to X$. Then $\partial R \to X$ has period $W \to X$ and has exponent n. Since the perimeter of a 1-cell e in X will be the number of times its associated generator occurs in $\partial R = W^n$ in either orientation, we can estimate that $\mathbf{P}(e) \leq n \cdot |W|$, and that $\mathbf{P}(S) \leq n \cdot |W| \cdot (|W| - 1)$ for any word $S \in \partial R$ with $|S| < |W|$. On the other hand, the weight of the 2-cell R is exactly $n \cdot |W|$. The coherence criterion of Theorem 8.2 will be satisfied so long as

$$n \cdot |W| \cdot (|W| - 1) \leq n \cdot n \cdot |W|$$

In particular, if $|W| - 1 \leq n$ then Theorem 8.2 shows that the group G is coherent. □

The next theorem lowers the bound on the exponent by choosing a more appropriate weight function.

Theorem 8.4. *Let* W *be a cyclically reduced word, let* $G = \langle a_1, \dots, a_r \mid W^n \rangle$*, and let* a_1 *occurs exactly* k *times* $(k > 0)$ *in the word* W*. If* $n \geq k$*, then the group* G *is coherent. In particular, if every* a_i $(1 \leq i \leq r)$ *occurs in* W*, then* G *is coherent for all* $n \geq \frac{|W|}{r}$*.*

Proof. The proof is nearly identical to the previous one, except that the weight function on X has changed. Assign a weight of 1 to any side labeled by the generator a_1 and assign a weight of 0 otherwise. The perimeter of the 1-cell labeled by a_1 is exactly $n \cdot k$ (since this is the number of occurrences of a_1 in W^n), and the perimeter of any other 1-cell is 0. Since any word $S \in \partial R$ with $|S| < |R|$ contains at most k 1-cells labeled by a_1, we estimate that $\mathbf{P}(S) \leq n \cdot k \cdot k$. On the other hand, $n \cdot \mathbf{Wt}(R)$ is exactly $n \cdot n \cdot k$. Thus whenever $n \geq k$, the criterion of Theorem 8.2 will be satisfied, and the group $G = \pi_1 X$ will be coherent. The final assertion is immediate since the word W contains at least r letters and thus one of them occurs at most $\frac{|W|}{r}$ times. □

9. Small cancellation I

In this section we apply our coherence results to small cancellation groups. We begin with a brief review of the basic notions of small cancellation theory. The reader is referred to [13] for a rigorous development of these notions that is consistent with their use here.

Definition 9.1 (Piece)**.** Let X be a combinatorial 2-complex. Intuitively, a piece of X is a path which is contained in the boundaries of the 2-cells of X in at least two distinct ways. More precisely, a nontrivial path $P \to X$ is a *piece* of X if there are 2-cells R_1 and R_2 such that $P \to X$ factors as $P \to R_1 \to X$ and as $P \to R_2 \to X$ but there does not exist a homeomorphism $\partial R_1 \to \partial R_2$ such that there is a commutative diagram

$$\begin{array}{ccc} P & \to & \partial R_2 \\ \downarrow & \nearrow & \downarrow \\ \partial R_1 & \to & X \end{array}$$

Excluding commutative diagrams of this form ensures that P occurs in ∂R_1 and ∂R_2 in essentially distinct ways.

Definition 9.2 ($C(p)$-$T(q)$-complex)**.** An *arc* in a diagram is a path whose internal vertices have valence 2 and whose initial and terminal vertices have valence ≥ 3 The arc is *internal* if it its interior lies in the interior of D, and it is a *boundary arc* if it lies entirely in ∂D.

A 2-complex X satisfies the $T(q)$ condition if for every minimal area disc diagram $D \to X$, each internal 0-cell of D has valence 2 or valence $\geq q$. Similarly, X satisfies the $C(p)$ condition if the boundary path of each 2-cell in D either contains a nontrivial boundary arc, or is the concatenation of at least p nontrivial internal arcs. Finally, for a fixed positive real number α, the complex X satisfies $C'(\alpha)$ provided that for each 2-cell $R \to X$, and each piece $P \to X$ which factors as $P \to R \to X$, we have $|P| < \alpha|\partial R|$. Note that if X satisfies $C'(\alpha)$ and $n \leq \frac{1}{\alpha} + 1$ then X satisfies $C(n)$.

It is a fact that if $D \to X$ is minimal area then each nontrivial arc in the interior of D is a piece in the sense of Definition 9.1. Although the rough definition given above is not quite technically correct (for instance, it uses minimal area diagrams instead of reduced diagrams), it should give the

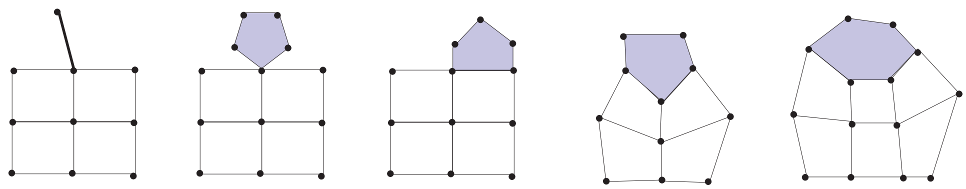

FIGURE 4. Spurs and i-shells

reader unfamiliar with small cancellation complexes an approximate idea of their properties. We refer the interested reader to [13] for precise definitions.

When p and q are sufficiently large, minimal area diagrams over X will always contain either spurs or i-shells.

Definition 9.3 (i-shells and spurs)**.** Let D be a diagram. An *i-shell* of D is a 2-cell $R \hookrightarrow D$ whose boundary cycle ∂R is the concatenation $P_0 P_1 \cdots P_i$ where $P_0 \to D$ is a boundary arc, the interior of $P_1 \cdots P_i$ maps to the interior of D, and $P_j \to D$ is a nontrivial interior arc of D for all $j > 0$. The path P_0 is the *outer path* of the i-shell.

A 1-cell e in ∂D which is incident with a valence 1 0-cell v is a *spur*. In analogy with the outer path of an i-shell, we will refer to the length 2 path (either ee^{-1} or $e^{-1}e$) that passes through v as the *outer path* of the spur.

Illustrated from left to right in Figure 4 are disc diagrams containing a spur, a 0-shell, a 1-shell, a 2-shell, and a 3-shell. In each case, the 2-cell R is shaded, and the boundary arc P_0 is $\partial R \cap \partial D$.

The classical result which forms the basis of small cancellation theory is called Greendlinger's Lemma (see [10, Thm V.4.5]). The following strengthening of Greendlinger's Lemma was proven in [13, Thm 9.4]. While the results of this section only require Greendlinger's lemma itself, we will require the full strength of the following theorem in Section 13.

Theorem 9.4. *If D is a $C(4)$-$T(4)$ [$C(6)$-$T(3)$] disc diagram, then one of the following holds:*

(1) *D contains at least three spurs and/or i-shells with $i \leq 2$ [$i \leq 3$].*
(2) *D is a ladder of width ≤ 1, and hence has a spur, 0-shell or 1-shell at each end.*
(3) *D consists of a single 0-cell or a single 2-cell.*

Moreover, if D is nontrivial and v is a 0-cell in ∂D, then D contains a spur or an i-shell with $i \leq 2$ [$i \leq 3$] which avoids v, and if the cut-tree of D has ℓ leaves, then D contains at least ℓ separate such spurs and i-shells.

See [13] for details. In the present article we will only need the following immediate corollary.

Corollary 9.5. *Let D be a $C(4)$-$T(4)$ [$C(6)$-$T(3)$] disc diagram and let P and Q be immersed paths such that PQ^{-1} is the boundary cycle of D. If* *neither path contains the outer path of an i-shell in D with $i \leq 2$ [$i \leq 3$], then every 2-cell of D contains an edge in P and an edge in Q.*

Theorem 9.6 (Coherence using i-shells)**.** *Let X be a weighted 2-complex which satisfies $C(6)$-$T(3)$ [$C(4)$-$T(4)$]. Suppose $\mathbf{P}(S) \leq n\mathbf{Wt}(R)$ for each 2-cell $R \to X$ and path $S \to \partial R$ which is the concatenation at most three [two] consecutive pieces in the boundary of R. Then $\pi_1 X$ is coherent.*

Proof. We will prove the $C(6)$-$T(3)$ case; the $C(4)$-$T(4)$ case is handled similarly. By Theorem 7.6, it is sufficient to show that X satisfies the path reduction hypothesis. Let $P \to X$ be a closed immersed nontrivial null-homotopic path. Let $D \to X$ be a minimal area disc diagram with boundary cycle P. According to Theorem 9.4, there exists an i-shell of D ($i \leq 3$) which avoids the basepoint of P. By hypothesis, the new boundary path, obtained by removing the boundary arc and the 2-cell of this i-shell from the diagram, is a path P' which can be obtained from P by a weakly perimeter-reducing push across a 2-cell. Since P' is a path satisfying $\mathsf{Area}(P) > \mathsf{Area}(P')$, Lemma 7.5 shows that X satisfies the path reduction hypothesis and the proof is complete. □

Theorem 9.6 can be improved by using more complicated weight functions, by using more complicated reductions, or by altering the presentation substantially before a weight function is applied. The following example is an illustration of the latter possibility. Additional examples can be found in Section 13 and in [12].

Example 9.7. Consider the following one-relator group.

$$G = \langle a, b, c, d, e \mid (abcde)a(abcde)b(abcde)c(abcde)d(abcde)e\rangle.$$

Since the relator is not a proper power, the theorems in Section 8 do not apply. If we alter the presentation of G by introducing a new generator $f = abcde$, then $G = \langle a, b, c, d, e, f \mid abcdef^{-1}, fafbfcfdfe\rangle$ and the new presentation satisfies certain small cancellation conditions. This can be seen from the link of the 0-cell of the standard 2-complex X of the modified presentation. As illustrated in Figure 5, the link is simplicial and so all pieces are of length 1, and since both relators have length at least 6, X is a $C(6)$ presentation. Because the shortest circuit in the link has length 4, X satisfies $T(4)$. If we assign a weight of 1 to each of the sides in the relations which are labeled by a, b, c, d or e, and assign a weight of 0 to the sides labeled f, then the corresponding 1-cell perimeters and 2-cell weights are as follows. The 1-cells labeled a, b, c, d, and e have a perimeter of 2, and both 2-cells have a weight of 5. Since the presentation satisfies $C(6)$-$T(4)$ we can use the coherence criterion for $C(4)$-$T(4)$-complexes (Theorem 9.6). The criterion is satisfied since $\mathbf{Wt}(\widetilde{R}) = 5$ and $\mathbf{P}(Q) \leq 4$ for all appropriate R and Q. Consequently this group is coherent.

The reader may have noticed that although a different weight is allowed for each side of each 2-cell of X, in all of the examples we have given so far,

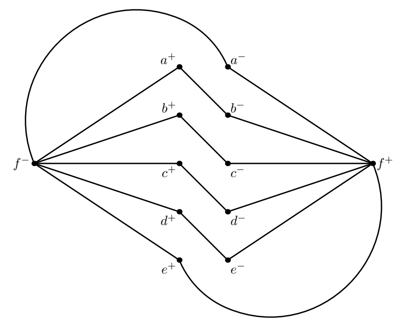

FIGURE 5. The link of 0-cell of the standard 2-complex for the presentation in Example 9.7

we have always chosen the weights to be equal on all of the sides incident at any particular 1-cell in X. Since it is clear that the perimeter of a 1-cell e in X is unaffected by the distribution of the weights among the sides present at e so long as their total is left invariant, this raises the question of whether the added flexibility we have allowed will ever be needed. In our final example we show that the weights of the sides at e sometimes do need to be different.

Example 9.8. Consider a presentation of the form $\langle a_1, \dots \mid U, V\rangle$. Suppose that for each i, the generator a_i appears exactly the same number of times in U as in V, so that in particular $|U| = |V|$. And suppose further that the pieces of V are longer than the pieces of U. This is the situation in which it makes sense that a side at a_i in V will need more weight than a side at a_i in U. The following is a concrete example. Consider the two-relator presentation:

$$\left\langle 1,2,3,4,5,6,7,8 \;\middle|\; \begin{array}{l} (1437)(2548)(3651)(4762)(5873)(6184)(7215)(8326), \\ (1111)(2222)(3333)(4444)(5555)(6666)(7777)(8888) \end{array} \right\rangle$$

The parentheses are included for emphasis only. We will call the first relator U and the second relator V. Observe that the presentation is invariant under a cyclic shift of the generators. Notice also that the presentation satisfies $T(4)$ and $C(16)$, that every piece in U has length 1, and that V has pieces of length at most 3. Finally it is clear that the subpath 111222 in V is a union of two consecutive pieces.

If we assign the sides of U weight 1 and we assign the sides of V weight 3 then the perimeter of each 1-cell is 16 and the weights of the 2-cells corresponding to U and V are 32 and 96 respectively. Consequently the coherence criterion of Theorem 9.6 is satisfied and so the group is coherent. On the other hand, if we used the unit weighting, then the perimeter of each 1-cell is 8. Observe that the path 111222 has perimeter 48 which is greater than the weight 32 of the 2-cell corresponding to V, and so the criterion of Theorem 9.6 fails.

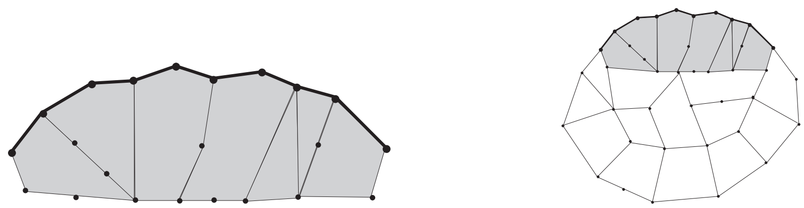

Figure 6. On the left is a fan F whose outer path Q is the bold path on its boundary. The disc diagram on the right contains the fan F as a subcomplex. Note that Q is a subpath of the boundary path of D.

We will now show that more is true. For this presentation, there does not exist a way to assign weights to the sides of the 2-cells so that (1) all of the sides labeled by a given generator receive the same weight, and (2) the coherence criterion of Theorem 9.6 is satisfied. A set of weights which satisfies the coherence criterion of Theorem 9.6 will be called *satisfactory*. The argument now goes as follows: observe that the sum of any two sets of weights which are satisfactory will also be satisfactory, and that a cyclic shift of a set of weights which are satisfactory will remain satisfactory. Next suppose that a set of weights existed which satisfied conditions (1) and (2). By the above observations we could add this set of weights to all of its cyclic shifts to show that a scalar multiple of the unit perimeter is satisfactory. But since we know that the unit perimeter fails the weight criterion, this contradiction shows that no such set of weights can exist.

10. Fan coherence theorems

In this section we introduce our final coherence hypotheses and our final coherence theorems which employ fans instead of single 2-cells. Many of the definitions, statements, and proofs will be analogous to those in previous sections.

Definition 10.1 (Fan)**.** A *fan* F is a 2-complex homeomorphic to a closed disc, which is the union of closed 2-cells $\cup_{1\leq i\leq n} R_i$, with the property that for each i, $F - R_i$ is the disjoint union of the connected sets $\cup_{j<i} R_j$ and $\cup_{j>i} R_j$ (note that when $i = 1$ or $i = n$ one of these sets is empty.) The *outer path* Q of F is a concatenation $Q = Q_1 Q_2 \dots Q_n$ where each Q_i is a subpath of ∂R_i. We refer the reader to Figure 6 for a picture of a typical fan. The unique path S such that QS^{-1} is the boundary cycle of F will be called the *inner path* of F.

Given a map $F \to X$ there is a unique extension to a packed map (Definition 4.5) where the 1-skeleton of the domain is unchanged. We will denote this extended domain by $\widetilde{F}$ in analogy with $\widetilde{R}$.

We will only be interested in fans equipped with a map $F \to X$ such that $\widetilde{F} \to X$ is a near-immersion, and we will refer to such a mapped fan as a *fan in* X. In this case, we will also regard the outer path $Q \to F$ of F as a path $Q \to X$.

The disc diagram $D \to X$ *contains the fan* $F \to X$, provided that $F \to X$ factors as $F \to D \to X$, where the outer path Q of F maps to ∂D, and the inner path S of F is an internal path in D.

Example 10.2. The simplest fans are i-shells (Definition 9.3). In this case ∂R is the concatenation QS^{-1} where Q is the outer path of the i-shell, and S is the concatenation of i-pieces in X.

We will often be interested in a collection $\mathcal{T}$ of fans in a 2-complex X which satisfy additional properties. The next three definitions are technical conditions which will enable us to perform a perimeter calculation for fan attachments parallel to the calculation in Lemma 5.3.

Definition 10.3 (Perimeter-reducing fan)**.** Let X be a weighted 2-complex. The fan $F \to X$ is *perimeter-reducing* provided that the perimeter of $\widetilde{F} \to X$ is less than the perimeter of its outer path $Q \to X$. In other words, $\mathbf{P}(\widetilde{F}) < \mathbf{P}(Q)$. Similarly, it is *weakly perimeter-reducing* if $\mathbf{P}(\widetilde{F}) \leq \mathbf{P}(Q)$.

Definition 10.4 (Missing along outer path)**.** Let $Y \to X$ be a packed 1-immersion, let $F \to X$ be a fan, and let $Q \to Y$ be a lift of the outer path of F to Y. We say F is *missing in* Y *along* $Q \to Y$ provided that the lift of $Q \to \widetilde{F} \to X$ to the path $Q \to Y$ does not extend to a lift of any 2-cell R of $\widetilde{F}$ that intersects Q in a nontrivial path. Specifically, if R is a 2-cell of $\widetilde{F}$ and $Q' = \partial R \cap Q$ is a nontrivial path, then there should not exist a lift of $R \cup_{Q'} Q$ to Y which extends the path $Q \to Y$. Equivalently $F \to X$ is missing in Y along $Q \to Y$ provided that for each 1-cell q in Q, no side of a 2-cell of X at x is present at both y and f, where f, y, and x are the images of q in $\widetilde{F}$, Y, and X.

Definition 10.5 (Spread-out)**.** A fan $F \to X$ is *spread-out* provided that the sides of 2-cells of $\widetilde{F}$ along 1-cells in the outer path $Q \to F$ project to distinct sides of 2-cells along 1-cells in X. This condition is certainly satisfied when the outer path $Q \to \widetilde{F}$ projects to a path $Q \to X$ which does not pass through any 1-cell of X more than once. For instance, $F \to X$ is spread-out when $\widetilde{F} \to X$ is an embedding, and it is spread-out when $Q \to X$ is a (possibly closed) simple path.

The following lemma calculates the perimeter of $Y \cup_Q \widetilde{F}$ in terms of the perimeters of its constituents.

Lemma 10.6 (Fan attachment)**.** *Let X be a weighted 2-complex, let $\phi\colon Y \to X$ be a packed 1-immersion with $\mathbf{P}(Y) < \infty$, and let $Q \to Y$ be a lift of the outer path of a fan $F \to X$. If $F \to X$ is spread-out and F is missing along $Q \to Y$, then, letting $Y^+ = Y \cup_Q \widetilde{F}$, the perimeter of the induced map $\phi^+ : Y^+ \to X$ satisfies:*

$$\mathbf{P}(Y^+) = \mathbf{P}(Y) + \mathbf{P}(\widetilde{F}) - \mathbf{P}(Q) \tag{9}$$

Thus, if $F \to X$ is perimeter-reducing then $\mathbf{P}(Y^+) < \mathbf{P}(Y)$ and if $F \to X$ is weakly perimeter-reducing then $\mathbf{P}(Y^+) \leq \mathbf{P}(Y)$.

Proof. The proof is similar to that of Lemma 5.2 where it is obvious that a fan consisting of a single 2-cell is spread-out. Since the perimeter of $\phi\colon Y \to X$ is unaffected by the addition or removal of redundant 2-cells from Y (Lemma 4.12), we may assume that Y has no redundancies. By Lemma 4.13 this means that we may assume that ϕ is an immersion.

Next, we show that the map $Y^+ \to X$ is a near-immersion. By the definition of a fan, distinct 1-cells of Q are sent to distinct 1-cells in $\widetilde{F}$ under the map $Q \to \widetilde{F}$ and thus distinct 1-cells of Y are mapped to distinct 1-cells of Y^+ under the map $Y \to Y^+ = Y \cup_Q \widetilde{F}$. This shows that the induced map $\mathsf{Sides}_Y \to \mathsf{Sides}_{Y^+}$ is an injection. On the other hand, the map $\widetilde{F} \to X$ is a near-immersion by definition. Combined with the fact that $F \to X$ is spread-out, this shows that the induced map $\mathsf{Sides}_{\widetilde{F}} \to \mathsf{Sides}_{Y^+}$ is also an injection. Thus, if $Y^+ \to X$ fails to be a near-immersion, it must fail along the path $Q \to Y^+$. More precisely, to show that $Y^+ \to X$ is a near-immersion, it only remains to be shown that a side of X which lifts to a side of Q in $\widetilde{F}$ could not also lift to a side of Q in Y. This is impossible because of the assumption that F is missing along $Q \to Y$.

If we assume for the moment that Y is compact, then we can calculate the perimeter of Y^+ using Equation (6) of Lemma 2.18. According to Equation (6), the perimeter of Y^+ equals the sum of the perimeters of its 1-cells minus the weights of its 2-cells. If we apply Equation (6) to $\widetilde{F}$ and Y separately then we would add the perimeters of their 1-skeletons and subtract the weights of their 2-cells. The difference between these counts arises from the 1-cells of Q in $\widetilde{F}$ which get identified with 1-cells of Y in the space Y^+. This proves Equation (9). In the general case where $\mathbf{P}(Y)$ is finite but Y is not compact, the proof proceeds as in Lemma 5.2, except that Lemma 2.18 is applied to the packed fan $\widetilde{F}$ instead of the packet $\widetilde{R}$. □

Having established conditions under which we can control the change in perimeter, it is now relatively easy to define a hypothesis and prove a coherence theorem.

Definition 10.7 (Fan reduction hypothesis)**.** A packed 1-immersion $\phi\colon Y \to X$ admits a *fan perimeter reduction* provided there is a perimeter-reducing spread-out fan $F \to X$ and there exists a lift of its outer path to Y such that F is missing along $Q \to Y$. A weighted 2-complex X satisfies the *fan reduction hypothesis* if each packed 1-immersion $\phi\colon Y \to X$ which is not π_1-injective, admits a fan perimeter reduction.

Theorem 10.8 (Fan coherence)**.** *Let X be a weighted 2-complex. If X satisfies the fan reduction hypothesis then X satisfies the perimeter reduction hypothesis, and thus $\pi_1 X$ is coherent.*

Proof. Let $Y \to X$ be a 1-immersion which is not π_1-injective. Since adding the 2-cells necessary to make $Y \to X$ a packed map does not increase perimeter, we may assume it is packed without loss of generality. By hypothesis, there is a perimeter-reducing fan $F \to X$ which is spread-out and a lift of its outer path to Y such that F is missing along $Q \to Y$. This can be used to create a complex Y^+ whose perimeter is smaller, by Lemma 10.6. The fact that Y and Y^+ have the same π_1 image in X is obvious. Thus X satisfies the perimeter reduction hypothesis. $\square$

In most applications, we will only use a special case of Theorem 10.8 which can be formulated in terms of disc diagrams.

Theorem 10.9 (Diagram fan coherence)**.** *Let X be a weighted 2-complex and let $\mathcal{T}$ be a collection of perimeter-reducing spread-out fans in X. If each nontrivial minimal area disc diagram $D \to X$ contains a spur or a fan from $\mathcal{T}$, then X satisfies the perimeter reduction hypothesis, and thus $\pi_1 X$ is coherent.*

Proof. By Theorem 10.8, it is sufficient to show that X satisfies the fan reduction hypothesis. Let Y be a compact 2-complex and let $Y \to X$ be a packed 1-immersion which is not π_1-injective. There exists at least one essential immersed closed path $P \to Y$ whose image in X is a null-homotopic closed immersed path. We assume that $P \to Y$ has been chosen so that $\mathsf{Area}(P)$ is as small as possible, and such that the disc diagram $D \to X$ realizes this minimum area. By hypothesis, D contains a fan $F \to D \to X$ which is perimeter-reducing and spread-out. Let $Q \to Y$ be the restriction of $P \to Y$ to the outer path of F. Since P was chosen to have minimal area, the fan F is missing along $Q \to Y$. Consequently X satisfies the fan reduction hypothesis. Indeed, if some some side of a 2-cell R in $\widetilde{F}$ is already present along some edge q of the path P, then P is homotopic in Y to a path P' which travels around the boundary of $D - \big(\text{Interior}(R) \cup q\big)$. But $\mathsf{Area}(P') < \mathsf{Area}(P)$, so we can find an immersed essential path in Y which bounds a smaller area diagram in X, and this is impossible. $\square$

We conclude this section with two further generalizations of results from the previous sections.

Theorem 10.10 (Fan algorithm)**.** *If X is a compact weighted 2-complex which satisfies the fan reduction hypothesis, then there is an algorithm which produces a finite presentation for any subgroup of $\pi_1 X$ given by a finite set of generators.*

Proof. The proof is analogous to the proof of Theorem 6.1 and we leave the details to the reader. $\square$

Theorem 10.11 (Weak fan coherence)**.** *Let X be a compact weighted 2-complex. Let $\mathcal{T}$ be a collection of spread-out weakly perimeter-reducing fans, and suppose that for each fan $F \in \mathcal{T}$, we have $\mathbf{P}(\widetilde{F}) < \mathbf{P}(\partial F)$. If every* *nontrivial minimal area disc diagram contains a spur or a fan in $\mathcal{T}$ then $\pi_1 X$ is coherent.*

Proof. The proof is essentially a generalization of the proof of Theorem 7.6 that uses fans instead of 2-cells. Let Y_1 be a compact π_1-surjective packed subcomplex of a cover $\widehat{X}$. If Y_1 is not π_1-injective, then there is a minimal area disc diagram $D \to \widehat{X}$ whose boundary cycle is an essential immersed path in Y_1. As in the proof of Theorem 10.9, a minimal area disc diagram for an essential immersed path in Y yields a sequence of weakly perimeter reducing spread-out fans that can be attached.

We claim that in the appropriate sense $\widetilde{F}$ is missing along the 1-cells in ∂F that map to 1-cells of Y_i in $\widehat{X}$. Indeed, if some 2-cell of $\widetilde{F}$ was already contained in Y_i then a corresponding 2-cell R of F is contained in Y_i. Let ∂R be the concatenation $Q_1 Q_2^{-1}$ where Q_1 is the part of ∂R that is the subpath of the outer path Q of F. Now P_i is homotopic in D_i and Y_i to a path P_i' which is identical to P except that Q_2 is substituted for Q_1. Since P_i' doesn't go around $R \subset D_i$, we see that $\mathsf{Area}(P_i') \leq \mathsf{Area}(P_i) - 1$ and therefore after removing spurs from a disc diagram for P_i' (and identifying some 1-cells on the boundary), we obtain an immersed path homotopic to P_i in Y_i whose area is strictly less than the area of P_i which is impossible.

We let Y_{i+1} be the union of Y_i with (the image of) $\widetilde{F}$ be a new compact subcomplex in $\widehat{X}$. Now the outer path Q of F extends to a path $\partial F \to \widehat{X}$, and the argument breaks down according to whether ∂F is contained in Y_i. If $\partial F \subset Y_i$ then our hypothesis that $\mathbf{P}(\widetilde{F}) < \mathbf{P}(\partial F)$ implies that $\mathbf{P}(Y_{i+1}) < \mathbf{P}(Y_i)$. If ∂F is not a path in Y_i, then P is not null-homotopic in Y_{i+1} since $\pi_1(Y_i \cup \widetilde{F}) = \pi_1(Y_i \cup F)$ and $Y_i \cup F$ collapses onto the union of Y_i and some nontrivial arcs. Now our hypothesis that F is weakly perimeter reducing implies that $\mathbf{P}(Y_{i+1}) \leq \mathbf{P}(Y_i)$, and Y_{i+1} contains the essential immersed path P_{i+1} with $\mathsf{Area}(P_{i+1}) < \mathsf{Area}(P_i)$, where P_{i+1} is defined as follows: First remove the interiors of F and Q from D_i to obtain a diagram D_i', and then fold $\partial D_i'$ until the boundary is immersed. Note that we cannot obtain a sphere in this way, because otherwise ∂F is the same as ∂D_i and so we could have used F instead of D_i to begin with, contradicting that D_i is minimal area.

This process can only be repeated finitely many times without the perimeter strictly decreasing and hence $\widehat{X}$ satisfies the perimeter reduction hypothesis and so $\pi_1 X$ is coherent. $\square$

11. Quasi-isometries and quasiconvexity

In this section we review the interconnections between quasi-isometries, quasiconvexity, and word-hyperbolicity. Since these results are well-known, we simply state the definitions and lemmas we will need and refer the interested reader to [1], [6], and [21] for more detailed accounts.

Definition 11.1 (Geodesic metric space)**.** Let (X, d) and (X', d') be metric spaces. A map $\phi\colon X' \to X$ which preserves distances is called an *isometric embedding* of X' into X, and an isometric embedding of an interval $[a, b]$ of the real line is called a *geodesic* from $\phi(a)$ to $\phi(b)$. If any two points in X can be connected by a geodesic, then X is a *geodesic metric space*.

A fundamental example of a geodesic metric space is a connected graph with the path metric. Note that by a 'graph' we mean a 1-dimensional CW-complex, so that loops and multiple edges are allowed.

Definition 11.2 (Path metric)**.** The *path metric* on a connected graph Γ makes each 1-cell of Γ locally isometric to the unit interval, and then defines the distance between two arbitrary points of Γ to be the length of the shortest path between them. It is easy to see that such a minimal path always exists, and that it will be a geodesic. Thus connected graphs are geodesic metric spaces using the path metric.

Definition 11.3 (Cayley graph)**.** Let X be a connected 2-complex, and let $\widetilde{X}$ be its universal cover. Since the 1-skeleton $\widetilde{X}^{(1)}$ is a connected graph it is a geodesic metric space with the path metric. If X has a unique 0-cell, then X is the standard 2-complex of some group presentation $G = \langle A | \mathcal{R} \rangle$, and the graph $\widetilde{X}^{(1)}$ is the *Cayley graph of the presentation*. Alternatively, the Cayley graph, often denoted $\Gamma(G, A)$, can be defined as follows. Begin with a 0-cell set corresponding to the elements of G and an edge set labeled by the elements of $G \times A$. Then attach the edges to the 0-cells so that the edge labeled (g, a) begins at the 0-cell g and ends at the 0-cell ga. We endow $\Gamma(G, A)$ with the path metric. Since G can be identified with the 0-skeleton of $\Gamma(G, A)$, we can metrize G by giving it the subspace metric. Since this metric on G depends on the generating set A, we will denote the resulting metric space by G_A.

Although distinct generating sets for G will produce distinct metrics using this procedure, all of the metrics on a finitely generated group will be roughly equivalent. We will now make this precise.

Definition 11.4 (Quasi-isometry)**.** Let (X, d) and (X', d') be metric spaces and let $\phi\colon X' \to X$ be a map between them. If there exist constants $K > 0$ and $\epsilon \geq 0$ such that for all $x, y \in X'$,

$$K d(x, y) + \epsilon > d(\phi(x), \phi(y)) > \frac{1}{K} d(x, y) - \epsilon$$

then ϕ is a (K, ϵ)*-quasi-isometric embedding* of X' into X. The special case of a (K, ϵ)-quasi-isometric embedding of an interval of the real line into X is a (K, ϵ)*-quasigeodesic*. If every point in X is within a uniformly bounded distance of a point in the image of ϕ, then ϕ is a (K, ϵ)*-quasi-isometry* between X' and X. A map will be called a *quasi-isometry* if it is a (K, ϵ)-quasi-isometry for some choice of K and ϵ, and the spaces involved will be said to be quasi-isometric. The notion of quasi-isometry is an equivalence relation on spaces in the following sense. If there is a quasi-isometry from X to Y then there also exists a quasi-isometry from Y to X, and if $\rho : X \to Y$ and $\phi\colon Y \to Z$ are quasi-isometries, then the composition $\phi \circ \rho$ is also a quasi-isometry.

If a finitely generated group G acts in a reasonably nice way on a reasonably nice space, then the group, using the metric derived from its Cayley graph, will be quasi-isometric to the space it acts on. The following theorem contains the precise statement of this fact.

Lemma 11.5 (Theorem 3.3.6 of [6])**.** *Let X be a locally compact, connected, geodesic metric space. Let G be a finitely generated group which acts on X properly discontinuously and cocompactly by isometries. Then for any point $x \in X$, and for any finite set of generators $A \subset G$, the map $G \to X$ defined by $g \to gx$ is a quasi-isometry, where we give G its Cayley graph metric relative to the generating set A.*

As a corollary, we see that changing generating sets induces a quasi-isometry.

Corollary 11.6. *If A and B are finite generating sets for a group G, then the metric spaces G_A, G_B, $\Gamma(G, A)$, and $\Gamma(G, B)$ are all quasi-isometric. Furthermore, the quasi-isometry $G_A \to G_B$ is induced by the identity map $G \to G$.*

A second fundamental notion is that of a quasiconvex subspace of a metric space.

Definition 11.7 (Quasiconvexity)**.** A subspace Y of a geodesic metric space X is *K-quasiconvex* if there is a K-neighborhood of Y which contains all of the geodesics of X that begin and end in Y. A subspace is *quasiconvex* provided that it is K-quasiconvex for some K. The notion of quasiconvexity can be extended to groups and subgroups via Cayley graphs. Specifically, a subgroup H of a group G generated by A is *quasiconvex* if the 0-cells corresponding to H form a quasiconvex subspace of $\Gamma(G, A)$. The group G generated by A is *locally quasiconvex* if every finitely generated subgroup is quasiconvex.

We record the following two properties of quasiconvex subgroups. See [21] and the references therein for details.

Lemma 11.8 (Proposition 1 of [21])**.** *If H is a quasiconvex subgroup of a group G generated by finite set A, then H itself is finitely generated.*

Lemma 11.9. *Let G be a group with finite generating set A and let H be a subgroup of G with finite generating set B. If H is a quasiconvex subspace of $\Gamma(G, A)$, then H_B is quasi-isometrically embedded in $\Gamma(G, A)$.*

Although the various metrics which have been defined for a group G are all equivalent up to quasi-isometry (Corollary 11.6), the generating set A

does need to be specified in Definition 11.7. This is because the notion of quasiconvexity is not well-behaved under quasi-isometries. In particular, the group $\mathbb{Z} \times \mathbb{Z}$ shows that the converse of Lemma 11.9 is false. The dependence of quasiconvexity on generating sets and the distinction between quasiconvex subgroups and quasi-isometrically embedded subgroups disappears once we restrict our attention to word-hyperbolic groups.

Definition 11.10 (Hyperbolic spaces and groups)**.** Let x, y, and z be points in a geodesic metric space X and let Δ be a triangle of geodesics connecting x to y, y to z and x to z. This geodesic triangle is *δ-thin* if each of the sides is contained in a δ-neighborhood of the union of the other two. If there is a uniform δ such that every geodesic triangle in X is δ-thin, then X is a *δ-hyperbolic space.* A group G generated by a finite set A is *word-hyperbolic* if its Cayley graph $\Gamma(G, A)$ is δ-hyperbolic.

One of the key properties of δ-hyperbolic spaces is that geodesics and quasigeodesics stay uniformly close in the following sense:

Lemma 11.11 (Proposition 3.3 of [1])**.** *Let x and y be points in the δ-hyperbolic metric space X. Then there are integers $L(\lambda, \epsilon)$ and $M(\lambda, \epsilon)$ such that if α is a (λ, ϵ)-quasigeodesic between the points x, y and γ is a geodesic $[xy]$, then γ is contained in an L-neighborhood of α and α is contained in an M-neighborhood of γ.*

It is easy to deduce from Lemma 11.11 that the property of being δ-hyperbolic for some δ is preserved by quasi-isometries between geodesic metric spaces, even though the specific value of δ may have to be changed. Combined with Corollary 11.6, this shows that the property of a group being word-hyperbolic is independent of the choice of a finite generating set.

Corollary 11.12. *If X and X' are geodesic metric spaces, X is δ-hyperbolic, and $\phi\colon X \to X'$ is a quasi-isometry, then a subspace Y is quasiconvex in X if and only if $\phi(Y)$ is quasiconvex in X'.*

As a consequence, the quasiconvexity of a subgroup in a word-hyperbolic group does not depend on the generating set.

Corollary 11.13. *Let H be a subgroup of the word-hyperbolic group G and let A and B be finite generating sets for G. The subgroup H will be quasiconvex in $\Gamma(G, A)$ if and only if H is quasiconvex in $\Gamma(G, B)$. In particular, when G is word-hyperbolic, the quasiconvexity of a subgroup is independent of the choice of finite generating set for G.*

Thus, for word-hyperbolic groups there is the following partial converse to Lemma 11.9.

Lemma 11.14. *Let G be a word-hyperbolic group with finite generating set A and let H be a subgroup of G with finite generating set B. If H_B is quasi-isometrically embedded in G_A, then H is a quasiconvex subgroup of G.*

12. Fan quasiconvexity theorems

In this section we prove our main technical results about local quasiconvexity. Since the reader has already seen arguments utilizing perimeter-reducing fans in Section 10, we will treat only the general fan case. The reader not yet completely comfortable with the language of fans should keep in mind the special case of a fan consisting of a single 2-cell R with outer path Q in ∂R.

Definition 12.1 (Straightening)**.** Let X be a weighted 2-complex, let $\mathcal{T}$ be a collection of fans in X, and let K and ϵ be constants. A path $P \to X$ can be (K, ϵ)-*straightened* if there exists a sequence of paths $\{P = P_1, P_2, \ldots, P_t\}$ such that for each i, P_{i+1} is obtained from P_i by either removing a backtrack, or by pushing across a fan in $\mathcal{T}$. In addition, the final path P_t must satisfy the following condition: Consider the lift of P_t to $\widetilde{X}$ and let d denote the length of a geodesic in $\widetilde{X}^{(1)}$ with the same endpoints. There must exist a path $P' \to \widetilde{X}$ with the same endpoints as $P_t \to \widetilde{X}$ such that P' lies in a K-neighborhood of P_t and such that

$$K \cdot d + \epsilon > |P'|. \tag{10}$$

If every fan in $\mathcal{T}$ is spread-out and perimeter-reducing, and if for some fixed choice of K and ϵ, every path $P \to X$ can be (K, ϵ)-straightened, we say that X satisfies the *straightening hypothesis*.

The following is our main technical result about the straightening hypothesis.

Theorem 12.2 (Subgroups quasi-isometrically embed)**.** *Let X be a compact weighted 2-complex. If X satisfies the straightening hypothesis, then every finitely generated subgroup of $\pi_1 X$ embeds by a quasi-isometry. Furthermore, if $\pi_1 X$ is word-hyperbolic then it is locally quasiconvex.*

Proof. Let $G = \pi_1 X$, let H be a subgroup of G which is generated by a finite set B of elements, and let $\widehat{X}$ be the based covering space of X corresponding to the inclusion $H \subset G$. Next, let C be a wedge of finitely many circles, one for each generator in B, and let $\phi \colon C \to X$ be a map which sends each circle to a based path in $X^{(1)}$ representing its corresponding generator. This map lifts to a map $C \to \widehat{X}$, and we let Y_0 denote the image of C in $\widehat{X}$. Since the weights on the sides of the 2-cells are nonnegative integers, the perimeter of any compact subcomplex of $\widehat{X}$ is finite and nonnegative. In particular, there exists some compact connected subcomplex $Y \subset \widehat{X}$ which contains Y_0 and which does not admit any fan perimeter reductions. For instance, we can choose Y to be of minimal perimeter among all compact connected subcomplexes of $\widehat{X}$ containing Y_0. If $Y \to \widehat{X}$ were to admit a fan perimeter reduction, then Lemma 10.6 and Lemma 2.16 would allow us to create a slightly larger subcomplex which had a strictly smaller perimeter, contradicting the way Y was chosen.

Let Z be a K-neighborhood of Y in $\widehat{X}$, let $\widetilde{X}$ be the based universal cover of X, let $\widetilde{Y}$ denote the based component of the preimage of Y in $\widetilde{X}$, and let $\widetilde{Z}$ denote the based connected component of the preimage of Z in $\widetilde{X}$. Since $Z \supset Y \supset Y_0$, we see that Z contains a set of paths which generate $\pi_1\widehat{X} = H$ and thus the action of $H \subset G$ on $\widetilde{X}$ stabilizes $\widetilde{Z}$. In particular, the preimages of the basepoint of $\widehat{X}$ in $\widetilde{X}$ are contained in $\widetilde{Z}$ and these 0-cells are in one-to-one correspondence with the elements of H. Using this correspondence, we will consider H as a subspace of $\widetilde{X}$. Let $H_{\widetilde{Z}}$ be the metric on H defined by the 1-skeleton of $\widetilde{Z}$. Specifically, define $d(h, h')$ to be the length of the shortest path in $\widetilde{Z}$ between the appropriate 0-cells of H.

Since Y does not admit any fan perimeter reductions and Z is the K-neighborhood of Y, the straightening hypothesis allows us to conclude that every pair of points in $H \subset \widetilde{X}$ is connected by a path in $\widetilde{Z}$ which satisfies Equation (10). In particular, given a path P in $\widetilde{Y}$ connecting a pair of points in H, we can follow the sequence of alterations to obtain paths $P_1, \ldots, P_t$ without leaving the subcomplex $\widetilde{Y}$, and since the path P' lies in a K-neighborhood of P_t, the path P' does not leave the subcomplex $\widetilde{Z}^{(1)}$. Finally, since this is true for all pairs of points in $H \subset \widetilde{X}$, this shows that the map $H_{\widetilde{Z}} \to \widetilde{X}^{(1)}$ is a (K, ϵ)-quasi-isometric embedding.

Since X is compact, G has some finite generating set A. More specifically, if we select a maximal spanning tree for $X^{(1)}$, then a generator corresponding to each 1-cell not in the spanning tree is sufficient. Consider the following diagram of maps between metric spaces where the metric spaces G_A and $\Gamma(G, A)$ are the metric on the group and the metric on its Cayley graph.

$$\begin{array}{ccccc} H_B & \to & G_A & \to & \Gamma(G,A) \\ \downarrow & & \downarrow & & \\ H_{\widetilde{Z}} & \to & \widetilde{X}^{(1)} & & \end{array}$$

We have shown that the bottom map is a quasi-isometric embedding. Since G acts properly discontinuously and cocompactly on $\widetilde{X}^{(1)}$, by Lemma 11.5 the map $G_A \to \widetilde{X}^{(1)}$ is a quasi-isometry.

Next, as remarked above, the action of $H \subset G$ on $\widetilde{X}$ stabilizes $\widetilde{Z}$. The action of H on $\widetilde{Z}^{(1)}$ is clearly properly discontinuous since it is a restriction of G on $\widetilde{X}$, and it is cocompact since the quotient of $\widetilde{Z}$ by H is the compact space Z. Thus, by Lemma 11.5 the map $H_B \to \widetilde{H}_{\widetilde{Z}}$ is also a quasi-isometry. Combining these three maps we see that the map $H_B \to G_A$ is a quasi-isometric embedding. Since by Corollary 11.6, $G_A \to \Gamma(G, A)$ is also a quasi-isometry, the map from H_B to $\Gamma(G, A)$ is a quasi-isometric embedding as well. Finally, if G is word-hyperbolic then it follows from Lemma 11.14 that H is quasiconvex. $\square$

We are unable to answer the following problem about the relationship between the straightening hypothesis and word-hyperbolicity. However, one can show the answer is affirmative if one adds to Definition 12.1 the requirement that P_t lie in a K-neighborhood of P'.

Problem 12.3 (Straightening and Hyperbolicity)**.** Suppose the compact weighted 2-complex X satisfies the straightening hypothesis with respect to some finite collection $\mathcal{T}$ of fans in X. Is $\pi_1 X$ word-hyperbolic?

Two conditions which immediately imply the straightening hypothesis are a decrease in length and a decrease in area. More explicitly, if for every immersed path $P \to X$ which does not lift to a (K, ϵ)-quasigeodesic, the path P can be pushed across a perimeter-reducing fan to obtain a new path of strictly smaller length, then X will satisfy the straightening hypothesis. Indeed the sequence of reductions cannot continue indefinitely because the length decreases each time, and thus they terminate at a path $P_t = P'$ which lifts to a quasigeodesic. The most important condition that implies the straightening hypothesis will involve the following notion:

Definition 12.4 (J-thin)**.** A disc diagram D with boundary cycle PQ^{-1} is called *J-thin* for some $J \in \mathbb{N}$, if every 0-cell in P is contained in a J-neighborhood of Q and vice-versa.

Theorem 12.5 (Diagrammatic local quasiconvexity criterion)**.** *Let $\mathcal{T}$ be a finite collection of perimeter-reducing spread-out fans in the compact weighted 2-complex X, and let $J \in \mathbb{N}$. Suppose that for every minimal area disc diagram $D \to X$ with boundary cycle PQ^{-1}, either D is J-thin or D contains a spur or fan in $\mathcal{T}$ whose outer path is a subpath of either P or Q. Then $\pi_1 X$ is a locally quasiconvex word-hyperbolic group.*

Proof. We will first give an argument that shows that X satisfies the straightening hypothesis. We will then apply a special case of this argument to see that $\pi_1 X$ is word-hyperbolic. The result will then follow from Theorem 12.2.

Let $P \to X$ be a path, let $P \to \widetilde{X}$ be a lift of this path to the universal cover $\widetilde{X}$, and let $Q \to \widetilde{X}$ be a geodesic in $\widetilde{X}$ with the same endpoints. Since the path PQ^{-1} is a closed null-homotopic path in $\widetilde{X}$, its projection to X has the same properties. Let $D \to X$ be a minimal area disc diagram with boundary cycle PQ^{-1}.

Let D be oriented so that the path P proceeds from left to right across the top of the diagram and the path Q proceeds from left to right across the bottom. Using the diagram D, we will now construct an explicit sequence of paths P_i demonstrating the straightening hypothesis. Along the way we will need to define a sequence of paths Q_i, a sequence of diagrams D_i, and a sequence of diagrams E_i as well. The idea will be to systematically remove portions of D from the top and bottom. At each stage of this process the paths along the top and bottom will be P_i and Q_i, the diagram between P_i and Q_i will be D_i, and the diagram bounded by $Q_i Q^{-1}$ will be E_i. At the end we will reach a diagram D_t with boundary paths $P_t \to D_t$ and $Q_t \to D_t$ such that $P_t Q_t^{-1}$ is the boundary cycle of D_t, and D_t is J-thin. At each stage

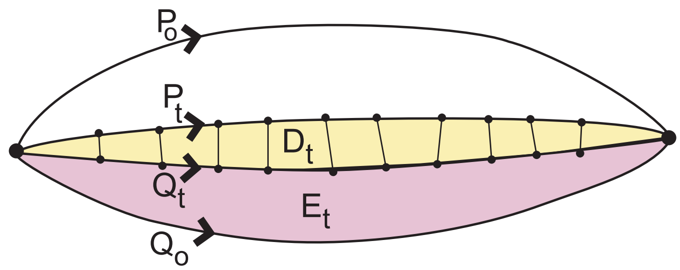

FIGURE 7. The final subdiagram D_t of D illustrated above is 1-thin.

D_i and E_i will be subdiagrams of D. The reader is referred to Figure 7 for an illustration of the diagram D as well as some of the relevant paths and subdiagrams appearing in the final situation.

To begin the process, let $P_0 = P$, $Q_0 = Q$, $D_0 = D$, and let E_0 be the diagram without 2-cells consisting of the path Q. This is a diagram since Q is a geodesic in $\widetilde{X}$ and thus simple in D. Since Definition 12.1 permits the removal of backtracks, we may assume that our initial path $P \to D$ is immersed. Now for each i, since both $P_i \to D_i$ and $Q_i \to D_i$ are immersed paths, by hypothesis the diagram D_i is either J-thin, or D_i contains a fan $F \in \mathcal{T}$ whose boundary path is a subpath of either P_i or Q_i. Removing the interior of the fan and the interior of its boundary path from D_i defines a new connected and simply-connected diagram D_{i+1} where the new boundary paths across the top and bottom of D_{i+1} are P_{i+1} and Q_{i+1}. Notice that by hypothesis, the path P_{i+1} is either identical to P_i or it is obtained from P_i by a perimeter-reducing push across a fan, and likewise, Q_{i+1} is either identical to Q_i or it is obtained from Q_i by a perimeter-reducing push across a fan. In either case, the fact that the original path is an immersion implies that the new path is an immersion as well. Since the total number of cells in the diagrams D_i is decreasing, this process must eventually terminate at a diagram D_t which is J-thin.

It only remains to show that the final path P_t lies close to a path P' which satisfies the length condition mentioned in Definition 12.1. Let $L \geq 1$ be a bound on the perimeter of a 1-cell in X (which is compact), let f be a bound on the lengths of inner paths of fans in $\mathcal{T}$, and let K be the larger of J and $fL+1$. Let $P' = Q_t$ and note that since D_t is J-thin, P' lies in a K-neighborhood of P_t.

We will now show that $|Q_t| \leq K|Q|$. Let $\widetilde{D}_i \to X$ and $\widetilde{E}_i \to X$ be the packed versions of $D_i \to X$ and $E_i \to X$. Now since Q_{i+1} is obtained from Q_i by a perimeter-reducing push across a fan, we have $\mathbf{P}(\widetilde{E}_{i+1}) < \mathbf{P}(\widetilde{E}_i)$, and consequently, $t \leq \mathbf{P}(\widetilde{E}_0) = \mathbf{P}(Q) \leq L|Q|$. Finally, since $|Q_{i+1}| \leq f + |Q_i|$ for each i, we have $|Q_t| \leq ft + |Q| \leq (fL+1)|Q| \leq K|Q|$.

We will prove that $\pi_1 X$ is word-hyperbolic by showing that $\pi_1 X$ has a linear isoperimetric function [1]. We will apply the above argument except that we will exchange the roles of P and Q, so that Q is an arbitrary immersed null-homotopic path and P is the trivial path. First observe that since no outer path of a fan can be a subpath of P, we see that $P = P_1 = P_t$. We will now use the fact that $\mathsf{Area}(Q) \leq \mathsf{Area}(Q_t) + \mathsf{Area}(E_t)$ to estimate $\mathsf{Area}(Q)$.

Let m be the maximal number of 2-cells occurring in a fan in $\mathcal{T}$. Then E_t was obtained from E_1 by adding at most $\mathbf{P}(E_0)$ fans, and so $\mathsf{Area}(E_t) \leq mt \leq mL|Q|$. Since the diagram D_t is J-thin, the path Q_t is contained in a J-neighborhood of the path $P_t = 1$. Letting C denote the ball of radius J in $\widetilde{X}$, Lemma 12.6 (established below) implies that $\mathsf{Area}(Q_t) \leq M|Q_t|$ where M is a constant that depends only on X and J. But $|Q_t| \leq K|Q|$ so $\mathsf{Area}(Q) \leq \mathsf{Area}(E_t) + \mathsf{Area}(Q_t) \leq mL|Q| + MK|Q|$, and the isoperimetric function is linear as claimed. □

Lemma 12.6. *If $\widetilde{X}$ is a simply-connected 2-complex and C is a compact, connected subspace of $\widetilde{X}$, then there is a constant M, depending only on C, such that for every closed path $Q \to C$, we have $\mathsf{Area}(Q) \leq M|P|$.*

Proof. Let $S_1, \ldots, S_j$ be the finitely many simple closed nontrivial paths in C, and for $1 \leq i \leq j$ let $M_i = \mathsf{Area}(S_i) = \mathsf{Area}(D_i)$ where $D_i \to X$ is a minimal area disc diagram with boundary cycle S_i. Such a D_i exists for each S_i since X is simply-connected. Let M be the maximum value of $\frac{M_i}{|S_i|}$. Intuitively, M measures the maximum number of 2-cells needed per 1-cell in a simple closed path. The necessary inequality is now easy to establish by breaking any closed path $Q \to C$ into simple closed paths and backtracks, creating disc diagrams for each individually, and then reassembling them into a disc diagram D for the original path. Since the maximum number of 2-cells needed per 1-cell is bounded by M for each individual portion, $\mathsf{Area}(Q) \leq \mathsf{Area}(D) \leq M|P|$ as claimed. □

We close this section with a problem analogous to Problem 7.10:

Problem 12.7. Let X be a compact weighted 2-complex such that $\pi_1 X$ is word-hyperbolic. Let K and ϵ be fixed constants, and suppose that for every path $P \to X$, there exists a (K, ϵ)-quasigeodesic Q and a disc diagram Δ with boundary cycle PQ^{-1}, such that $\mathbf{P}(\Delta) < \mathbf{P}(P)$. Is $\pi_1 X$ locally quasiconvex?

13. Small cancellation II

In this section we apply the local quasiconvexity result to various small cancellation groups.

Theorem 13.1 (Local quasiconvexity)**.** *Let X be a weighted 2-complex which satisfies $C(6)$-$T(3)$ $[C(4)$-$T(4)]$. Suppose $\mathbf{P}(S) < n\mathbf{Wt}(R)$ for each 2-cell $R \to X$ and path $S \to \partial R$ which is the concatenation at most three [two] consecutive pieces in the boundary of R. Then $\pi_1 X$ is coherent.*

Proof. This follows immediately from Corollary 9.5 and Theorem 12.5. □

Example 13.2 (Surface groups)**.** Let X be the standard 2-complex of the presentation $\langle a_1, \ldots, a_g \mid a_1^2 a_2^2 \ldots a_g^2 \rangle$. Then X is the usual cell structure for the nonorientable surface of genus g. Clearly for any $g \geq 2$, X satisfies $C(2g)$-$T(4)$, and the pieces are of length 1. Using the unit perimeter we see that the weight of any piece (1-cell) is 2 and the weight of the 2-cell is $2g$. Thus by Theorem 9.6, $\pi_1 X$ is coherent for $g \geq 2$ and by Theorem 13.1, $\pi_1 X$ is locally quasiconvex for $g > 2$.

A similar result holds if we let X be the standard 2-complex of the presentation

$$\langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \ldots [a_g, b_g] \rangle$$

so that X is an orientable surface of genus g. For $g \geq 1$ the 2-complex X satisfies $C(4g)$-$T(4)$, the pieces are of length 1, the weight of each piece is 2, and the weight of the 2-cell is $4g$. Thus by Theorem 9.6, $\pi_1 X$ is coherent for $g \geq 1$ and by Theorem 13.1, $\pi_1 X$ is locally quasiconvex for $g > 1$.

The fact that these methods can be used to prove the coherence and local quasiconvexity of surface groups is to be expected since the boundary of a 2-manifold was one of the original motivations for the notion of perimeter introduced in Section 2. Here is a more novel application of Theorem 9.6.

Theorem 13.3. *Let $G = \langle a_1, \ldots \mid R_1, \ldots \rangle$ be a presentation that satisfies $C'(1/n)$. If each a_i occurs at most $n/3$ times among the R_j, then G is coherent and locally quasiconvex.*

Proof. In case $n < 5$ this is obvious, because any generator appears at most $5/3$ times, and consequently at most once, which implies that the group is free. On the other hand, when $n \geq 5$, the complex X satisfies $C(6)$. Use the unit weighting, and let S be a subpath of the 2-cell R_j consisting of three consecutive pieces. The small cancellation assumption implies that $|S| < 3 \cdot \frac{1}{n} |\partial R_j|$. On the other hand, the bound on the number of occurrences of each generator shows that $\mathbf{P}(e) \leq \frac{n}{3}$ for each 1-cell e. Thus $\mathbf{P}(S) < 3 \frac{n}{3} \frac{1}{n} |\partial R_j| = |\partial R_j|$. By Theorem 9.6 the group is coherent and by Theorem 13.1 it is locally quasiconvex. □

In Section 8 we showed that certain one-relator groups with torsion are coherent. Since it was shown in [17] that $\langle a_1, \ldots \mid W^n \rangle$ satisfies the $C(2n)$ small cancellation condition, we can apply Theorem 13.1 to obtain a local-quasiconvexity result as well.

Theorem 13.4. *Let $G = \langle a_1, \ldots | W^n \rangle$ be a one-relator group with $n \geq 3|W|$. Then G is locally quasiconvex.*

Proof. Let X be the 2-complex of this presentation with the unit weighting. Since the weight of the unique 2-cell is $n|W|$, the weight of its packet is $n^2|W|$. On the other hand, by the spelling theorem (Theorem 8.1), the length of a piece is less than $|W|$ and we can assume that each generator appears in W fewer than $|W|$ times, for otherwise the group is virtually free and the theorem is obvious. Consequently, the perimeter of each 1-cell is at most $n(|W|-1)$ and since $n \leq |W|$, this is bounded by $(n-1)|W|$. So the perimeter of a piece is strictly less than $(n-1)|W| \cdot |W|$. Thus for $n \geq 3|W|$, G is locally quasiconvex by Theorem 13.1. □

A more detailed examination of the local quasiconvexity of one-relator groups with torsion has been carried out by Hruska and Wise in [9]. As an application of Theorem 12.2 they are able to prove the following:

Theorem 13.5. *Let $G = \langle a_1, \dots | W^n \rangle$ be a one-relator group with $n \geq |W|$. Then G is locally quasiconvex.*

Our next application is to finitely presented small cancellation groups with torsion. As in the one-relator case, small cancellation groups with sufficient torsion will always be coherent and locally quasiconvex. In the proof we will need the following classical lemma about words in the free group (see [8]).

Lemma 13.6. *Let X and Y be cyclically reduced words in the free group which are not proper powers. If X and Y are not cyclic conjugates, and a word U is both a subword of a power of X and a subword of a power of Y, then $|U| \leq |X| + |Y|$.*

Theorem 13.7 (Power theorem)**.** *Let $\langle a_1, \dots \mid W_1, \dots \rangle$ be a finite presentation, where each W_i is a cyclically reduced word which is not a proper power. If W_i is not freely conjugate to $W_j^{\pm 1}$ for $i \neq j$, then there exists a number N such that for all choices of integers $n_i \geq N$ the group $G = \langle a_1, \dots \mid W_1^{n_1}, \dots \rangle$ is coherent. Specifically, the number*

$$N = 6 \cdot \frac{|W_{max}|}{|W_{min}|} \sum |W_i| \tag{11}$$

has this property, where W_{max} and W_{min} denote longest and shortest words among the W_i, respectively. Moreover, if $n_i > N$ for all i, then G is locally quasiconvex.

Proof. We will assume that $\sum |W_i| \geq 2$. Essentially, the only case that this assumption eliminates is the presentation $\langle a_1, \dots \mid a_1 \rangle$ and the presentation $\langle a_1, \dots \mid \, \rangle$, and the theorem is trivial in these cases. Let N be the number satisfying Equation (11) and choose $n_i \geq N$ for all i. We will first show that $\langle a_1, \dots \mid W_1^{n_1}, \dots \rangle$ satisfies the $C(6)$ condition.

By Lemma 13.6 the length of the longest piece between the 2-cell labeled $W_i^{n_i}$ and the 2-cell labeled $W_j^{n_j}$ is bounded above by $|W_i| + |W_j| \leq 2 \cdot |W_{\max}|$. Notice that $2 \cdot |W_{\max}|$ is thus a uniform bound on the size of a piece which is independent of the size of the chosen n_i. Since the length of the i-th 2-cell is $n_i \cdot |W_i| \geq n_i \cdot |W_{\min}|$, and since by assumption,

$$n_i \cdot |W_{\min}| \geq N \cdot |W_{\min}| = 6 \cdot |W_{\max}| \sum |W_i| \geq 6 \cdot 2 \cdot |W_{\max}|$$

we can conclude that the $C(6)$ condition is satisfied.

Next we will choose a weighting on the sides of the 2-cells of the standard 2-complex for $\langle a_1, \dots \mid W_1^{n_1}, \dots \rangle$, and verify that the weight criterion of

Theorem 9.6 is satisfied. Let $n = \prod n_i$ and assign a weight of n/n_i to each of the sides of the 2-cell R_i corresponding to the relator $W_i^{n_i}$. If we let S denote a path in ∂R_i consisting of at most three consecutive pieces, then we must show that $n_i \mathbf{Wt}(R_i) \geq \mathbf{P}(S)$. This follows from the following string of inequalities:

$$n_i \cdot \mathbf{Wt}(R_i) \geq n_i \cdot n \cdot |W_{\min}| \geq 6 \cdot |W_{\max}| \cdot n \sum |W_i| \geq \mathbf{P}(S)$$

The first inequality is true since $\mathbf{Wt}(R_i) = \frac{n}{n_i} \cdot |W_i^{n_i}| = n \cdot |W_i|$, which is clearly greater than or equal to $n \cdot |W_{\min}|$. The middle inequality uses the restrictions assumed on n_i and N. Finally, the last inequality is a combination of two observations: (1) The perimeter of a single 1-cell will be at most the sum of the weights of all of the sides in the complex, so that $\mathbf{Wt}(e) \leq \sum \frac{n}{n_i} |W_i^{n_i}| = n \sum |W_i|$. And (2) since a piece has length at most $2 \cdot |W_{\max}|$, the perimeter of three consecutive pieces is bounded above by $6|W_{\max}| \cdot n \sum |W_i|$. Together these show the final inequality. The weight criterion of Theorem 9.6 is thus satisfied and the group G' is coherent. Similarly, if the inequalities are strict, then by Theorem 13.1, it is locally quasiconvex. □

Although there is some overlap between the groups studied in this article and those studied by Feighn and Handel in [7], the methods and the results are distinct. One indication of this is that all of the groups in [7] are indicable (i.e. admit a homomorphism onto $\mathbb{Z}$), whereas Theorem 13.7 can be used to construct coherent groups which are perfect.

Corollary 13.8. *There exist perfect groups which satisfy the perimeter reduction hypothesis and are thus coherent.*

Proof. The following example illustrates the idea: Consider the presentation

$$\langle a, b \mid a, aaba^{-1}b^{-1}, b, bbab^{-1}a^{-1} \rangle.$$

If N is chosen to satisfy Equation (11), then the following group is coherent by Theorem 13.7.

$$G = \langle a, b \mid a^N, (aaba^{-1}b^{-1})^{N+1}, b^N, (bbab^{-1}a^{-1})^{N+1} \rangle$$

But the following presentation for the abelianization of G shows that it is trivial:

$$\langle a, b \mid [a, b], a^N, a^{N+1}, b^N, b^{N+1} \rangle$$

□

14. 3-manifold groups

In this section, we use the theorems about coherence and local quasiconvexity in small cancellation groups to show that a large family of 3-manifold groups are coherent and locally quasiconvex. We begin with a theorem about branched covers of 2-complexes.

Theorem 14.1. *Let X be a compact 2-complex, and suppose that no 2-cell of X is attached by a proper power. Then there exists a constant d depending only on X such that for every branched cover $\widehat{X} \to X$ where (1) the branching is over the barycenters of 2-cells of X, and (2) all of the branching degrees are at least d, the fundamental group $\pi_1\widehat{X}$ is coherent. Similarly, there is another constant d such that the compact branched covers satisfying these conditions have a locally quasiconvex fundamental group.*

Proof. The proof is similar to the proof of Theorem 13.7, which can actually be deduced from it. First notice that it is sufficient to consider the case where no two 2-cells have the same attaching maps. It follows from Lemma 13.6 that there is a bound depending only on X on the length of a piece in the boundary of a 2-cell of $\widehat{X}$. Thus the lengths of the pieces remain bounded as the branching degrees increase, whereas the length of the boundary of each 2-cell grows linearly with d. It follows that for large d the presentation satisfies small cancellation conditions.

Using the unit perimeter, the maximum sum of the weight of three consecutive pieces remains constant, whereas the weight of each 2-cell grows linearly with d. Thus, for sufficiently large d the weight criteria for coherence (Theorem 9.6) and for local quasiconvexity (Theorem 13.1) will be satisfied. □

Remark 14.2. Observe that finite branched covers with high branching degrees correspond to certain finite index subgroups of the fundamental group of the space we obtain when we remove the barycenter of each 2-cell. Since the fundamental group of this space is free, and thus residually finite ([10]), these types of covers are numerous.

Theorem 14.3. *Let M be a compact 3-manifold with a combinatorial cell structure. There exists d depending on M such that the following holds: Let $B \to M \setminus M^{(0)}$ be a branched cover with at least d fold branching along each 1-cell of M. Then $\pi_1 B$ is coherent or even locally quasiconvex.*

Proof. Let M_1 denote the underlying manifold of M equipped with the cell structure dual to M. The branched covers $B \to M \setminus M^{(0)}$ branched along 1-cells of M correspond to branched covers $B' \to M_1^{(2)}$ branched along centers of 2-cells of $M_1^{(2)}$. Note that B' is a subspace of B. Furthermore, the branching degree at each 1-cell of M is the same as the branching degree at the center of the corresponding 2-cell of $M_1^{(2)}$.

The obvious strong deformation retraction of $M \setminus M^{(0)}$ onto $M_1^{(2)}$ induces a strong deformation retraction of B onto B', so that $\pi_1 B \cong \pi_1 B'$. It is therefore sufficient to prove the analogous result for branched covers $B' \to M_1^{(2)}$ where the branching occurs over the centers of 2-cells of $M_1^{(2)}$, and this is exactly what was proved in Theorem 14.1. □

We illustrate Theorem 14.3 with the following example.

Example 14.4. Let M denote the usual cell division for the 3-torus $T^3 = S^1 \times S^1 \times S^1$. We will show that any branched cover of $M \setminus M^{(0)}$ along the 1-cells of M has a coherent fundamental group provided that the degree d of branching is ≥ 3.

First observe that since M_1 is obviously isomorphic to M, it is easy to see that $M_1^{(2)}$ contains exactly three 2-cells, each of which is a square. Each piece of $M_1^{(2)}$ has length 1 and perimeter 4. If each branching degree is $\geq d$ then the presentation satisfies $C(4d)$. Since the perimeter of three consecutive pieces is $3 \cdot 4$ and the weight of each 2-cell at least $4d$, the fundamental group will be coherent when $d \geq 3$ (by Theorem 9.6) and locally quasiconvex when $d > 3$ (by Theorem 13.1).

As mentioned in the introduction, Scott and Shalen proved that all 3-manifold groups are coherent, so the coherence assertion in Theorem 14.3 is certainly not new. Nevertheless, it is interesting to be able to recover this special case from the different point of view of this paper.

The local quasiconvexity assertion is a bit trickier to obtain using prior results. However, it seems likely that the branched covers of Theorem 14.3 can be constructed from hyperbolic 3-manifolds with non-empty boundary and no cusps, by gluing along annuli. Thus the local quasiconvexity appears to follow from the following theorem of Thurston's [24] which we quote from [14, Proposition 7.1], together with the theorem due to Swarup [23] that in the case that N has no cusps, geometrically finite subgroups are quasiconvex.

Theorem 14.5 (Thurston)**.** *Let N be a geometrically finite hyperbolic manifold such that $\partial Core(N)$ is nonempty. Then every covering space N' of N with a finitely generated fundamental group is also geometrically finite.*

We believe that the branched covers with branching degree ≥ 2 of the 3-torus in Example 14.4 are atoroidal hyperbolic 3-manifolds with boundary, and hence the local quasiconvexity does follow from Theorem 14.5.

15. Related properties

There are a number of properties of groups which are closely related to coherence and local quasiconvexity. In this final section we examine briefly how our techniques can be used to produce results about three of these related topics: Howson's property, finitely generated intersections with Magnus subgroups, and the generalized word problem.

15.1. **Finitely-generated intersections.** A group is said to satisfy the *finitely generated intersection property* (or f.g.i.p.) if the intersection of any two finitely generated subgroups is also finitely generated. In 1954 Howson proved that free groups have the f.g.i.p., and as a result this property is sometimes referred to as Howson's property. As was shown in [21], every quasiconvex subgroup is finitely generated and the intersection of any two quasiconvex subgroups is again quasiconvex. Combining these two facts, one sees that every locally quasiconvex group satisfies the f.g.i.p. In particular, all of the groups we have shown to be locally quasiconvex, also have Howson's property.

In this subsection we show how the algorithm of Section 6 can be used to explicitly construct the finitely generated intersection of two finitely generated subgroups using the perimeter techniques we have already introduced. The method is a 2-dimensional generalization of that described by Stallings for graphs [22]. The construction of the finitely generated intersection will proceed in two steps. The first step will be to reduce this property to a property of the fiber product of the spaces corresponding to these subgroups. The second step will be to show how this property can be achieved using perimeter reductions.

Definition 15.1 (Fiber products)**.** Let X be a complex, let $A \to X$ and $B \to X$ be maps, and let $D \subset X \times X$ be the *diagonal* of $X \times X$ so $D = \{(x,x) \mid x \in X\}$. The *fiber-product* $A \otimes B \to X$ is defined to be the subspace of $A \times B$ which is the preimage of D in $A \times B \to X \times X$. Identifying D with X, there is a natural map $A \otimes B \to X$, and in fact, the following diagram commutes:

$$\begin{array}{ccc} A \otimes B & \to & B \\ \downarrow & \searrow & \downarrow \\ A & \to & X \end{array}$$

Notice that if both of the maps $A \to X$ and $B \to X$ have compact domain, then so does $A \otimes B \to X$, and if both maps are immersions, then so is their fiber product. Notice also that if ϕ is the map $A \to X$ and B is a subcomplex of X, then $A \otimes B$ is $\phi^{-1}(B)$. Note that the map $A \otimes B \to X$ induces a cell structure on $A \otimes B$ such that the map $A \otimes B \to X$ is combinatorial. Furthermore, if A, B, and X are based spaces and the maps from A and B preserve basepoints, then $X \times X$ and $A \otimes B$ have natural basepoints. The *based component* C of $A \otimes B$ is the component of $A \otimes B$ containing this basepoint. In particular we have $\pi_1(A \otimes B) = \pi_1 C$.

Lemma 15.2. *Let $A \to X$ and $B \to X$ be based maps and let a, b, and x be their basepoints. If for every element of $\pi_1 A \cap \pi_1 B$ there is a closed path $P \to X$ based at x which lifts to closed paths $P \to A$ and $P \to B$ based at a and b, then the image of $\pi_1(A \otimes B)$ in $\pi_1 X$ is the intersection of the images of $\pi_1 A$ and $\pi_1 B$. In particular, when A and B are compact, the fiber product $A \otimes B$ will also be compact, and $\pi_1(A \otimes B)$ is finitely generated.*

Proof. Since the based component of $A \otimes B$ factors through A and B, the image of its fundamental group must be contained in the intersection of the images of their fundamental groups. On the other hand, by hypothesis, each element in this intersection has a representative which lifts both to A and to B, and thus to $A \otimes B$ as well. In particular, $\pi_1(A \otimes B)$ also maps onto this intersection. $\square$

While we have already proven in Theorem 13.1 that the following groups are locally quasiconvex and hence have the finitely generated intersection property, the following theorem gives an explicit and relatively efficient method of computing this intersection.

Theorem 15.3 (f.g.i.p using i-shells)**.** *Let X be a weighted $C(4)$-$T(4)$-complex $[C(6)$-$T(3)]$. If every i-shell with $i \leq 2$ $[i \leq 3]$ is perimeter-reducing then $\pi_1 X$ has the finitely generated intersection property and the intersection of two finitely generated subgroups of $\pi_1 X$ can be constructed explicitly.*

Proof. First note that by Theorem 9.4, X satisifies the 2-cell reduction hypothesis. Let H and K be finitely generated subgroups of $\pi_1 X$ and let $A_1 \to X$ and $B_1 \to X$ be compact complexes chosen so that the images of their fundamental groups are H and K. We will now show by construction that $H \cap K$ is finitely generated. Let $A_2 \to X$ and $B_2 \to X$ be obtained from A_1 and B_1 by running the perimeter reduction algorithm described in Section 6. When the algorithm stops, no further folds or 2-cell perimeter reductions can be performed on $A_2 \to X$ or $B_2 \to X$. Thus, by the remark at the end of Definition 5.5, $\pi_1 A_2$ and $\pi_1 B_2$ can now be viewed as subgroups of $\pi_1 X$. Next, let $A_3 \to X$ and $B_3 \to X$ be complexes obtained from A_2 and B_2 by attaching to every vertex v, a copy of each 2-cell in X whose boundary cycle contains the image of v in X. These copies of 2-cells are attached only at the vertex v, and the maps into X are extended in the obvious way. Finally, let $A_4 \to X$ and $B_4 \to X$ be obtained by rerunning the perimeter reduction algorithm on A_3 and B_3. Note that each of these steps adds only a finite number of 2-cells, and consequently, since A_1 and B_1 are compact, so are A_4 and B_4. It should also be clear that $\pi_1 A_2 = \pi_1 A_3$ and $\pi_1 B_2 = \pi_1 B_3$, and since running the perimeter reduction algorithm does not change the image of the fundamental group in X, $\pi_1 A_4 = \pi_1 A_2 = H$ and $\pi_1 B_4 = \pi_1 B_2 = K$. It only remains to show that $A_4 \to X$ and $B_4 \to X$ satisfy the conditions of Lemma 15.2.

For each element of $H \cap K$ we can choose closed paths $P \to A_2$ and $Q \to B_2$ whose images in X represent this element. Since the concatenation $P \to X$ followed by the inverse of $Q \to X$ is null-homotopic in X, there is a minimal area disc diagram $D \to X$ whose boundary cycle is PQ^{-1}. Let $P \to A_2$ and $Q \to A_2$ be chosen (among combinatorial paths homotopic to them) so that the corresponding disc diagram D is of minimal area.

Observe that neither $P \to D$ nor $Q \to D$ contains a boundary arc which is the complement of at most 3 pieces in D (2 pieces in the $C(4)$-$T(4)$ case). To see this, suppose that such a subpath existed in P. By hypothesis, there would then be a perimeter reduction on $P \to X$ which would yield a corresponding reduction of $P \to A_2$ (since no perimeter reductions exists for $A_2 \to X$). But this would yield a new path P and a new disc diagram D with fewer 2-cells. We therefore conclude by Corollary 9.5, that D is K-thin. It is now easy to see that $P \to X$ actually lifts to B_4 as well as A_4. Thus $A_4 \to X$ and $B_4 \to X$ satisfy Lemma 15.2 and the proof is complete. □

The fact that Theorem 15.3 uses the 2-cell reduction hypothesis rather than the path reduction hypothesis is crucial. The examples below satisfy the path reduction hypothesis but fail to have the finitely generated intersection property.

Example 15.4. Let X be the standard 2-complex of the presentation $F_2 \times \mathbb{Z} = \langle a, b, t \mid [a,t], [b,t] \rangle$. If we assign a weight of 0 to every side incident at the edge labeled t and a weight of 1 to every other side of the complex, the result is a $C(4)$-$T(4)$ complex which satisfies the path reduction hypothesis. However, the subgroup $\langle a, b \rangle \cap \langle at, bt \rangle$ is not finitely generated.

Similarly, if we let X be the standard 2-complex of the presentation $\langle a, b, t \mid t^{-1}a^2tb^3 \rangle$, and we assign a weight of 1 to all of the sides incident at the edge labeled t and a weight of 0 to all of the other sides, the result is a $C(6)$-$T(3)$-complex which satisfies the path reduction hypothesis. Since this group has a free factor which is commensurable with $F_2 \times Z$, it also fails to have Howson's property.

Remark 15.5 (f.g.i.p using fans)**.** We note that the construction above works to explicitly compute the intersection between two subgroups of $\pi_1 X$ if X satisfies the more general criterion of Theorem 12.5. The main difference is that one attaches fans in $\mathcal{T}$ instead of i-shells, and one adds a diameter J "neighborhood" in passing from A_2 and B_2 to A_3 and B_3.

15.2. **Finitely-generated intersections with Magnus subgroups.** In this subsection we show how the intersections between specific subgroups can sometimes be shown to be finitely generated under a weaker set of assumptions. For example, even if we only assume the path reduction hypothesis, we can sometimes prove that certain subgroups $H \subset G$ have the property that $H \cap K$ is finitely generated for every finitely generated subgroup $K \subset G$.

Theorem 15.6. *Let X be a compact based 2-complex with non-negative integer weights assigned to the sides of its 2-cells, and let M be a based subgraph of $X^{(1)}$ with $\mathbf{P}(M) = 0$. If the weight of each 2-cell in X is positive and X satisfies the path reduction hypothesis, then for any finitely generated subgroup H in $\pi_1 X$, the intersection $\pi_1 M \cap H$ is also finitely generated.*

Proof. Let $\widehat{X}$ be the based covering space of X corresponding to the subgroup H. Let $K \subset X$ be a based π_1-surjective subcomplex. According to Theorem 3.7, there exists a compact connected subcomplex $Y \subset \widehat{X}$ such that $K \subset Y$, and such that $Y \to X$ is of minimal perimeter among all such complexes and hence such that $\pi_1 Y \to \pi_1 \widehat{X}$ is an isomorphism. Let ϕ denote the map from Y to X, and let C denote the connected component of $\phi^{-1}(M)$ which contains the basepoint of Y. Note that $\phi^{-1}(M)$ is the fiber product $Y \otimes M$ of the maps $Y \to X$ and $M \hookrightarrow X$. We will show that $H = \pi_1(Y \otimes M) = \pi_1(C)$ where C is the based component of the fiber product.

It is obvious that $\pi_1(Y \otimes M) \subset (\pi_1 Y \cap \pi_1 M)$ and so it is sufficient to show that $\pi_1 Y \cap \pi_1 M \subset \pi_1(Y \otimes M)$. Suppose that $P \to M \subset X$ is a closed immersed path which represents an element of $\pi_1\widehat{X}$, then P lifts to a closed path $\widehat{P}$ in $\widehat{X}$. Let $Y' = Y \cup \widehat{P}$ and note that Y' is compact and connected. Since $\widehat{P}$ is a closed path, Y' is formed from Y by adding finitely many arcs whose endpoints lie in Y. In particular, $Y' - Y$ consists of a set of 0-cells and perimeter zero 1-cells, and so $\mathbf{P}(Y') = \mathbf{P}(Y)$. If $Y' \neq Y$ then $\pi_1 Y$ is a proper free factor of $\pi_1 Y'$ and hence the map $Y' \to \widehat{X}$ is not π_1-injective. Part A of Theorem 3.7 provides a complex Y_t with $Y \subset Y' \subset Y_t$ and $\mathbf{P}(Y_t) < \mathbf{P}(Y') = \mathbf{P}(Y)$. This contradicts the minimality of $\mathbf{P}(Y)$.

In conclusion $Y' = Y$ and $P \to X$ lifts to Y. By Lemma 15.2 it now follows that $\pi_1 M \cap H = \pi_1(C)$, and by the compactness of C this intersection is finitely generated. $\square$

We note that when X satisfies the strict hypothesis, generators for an intersection can be computed. To see that the hypothesis that $\mathbf{P}(M) = 0$ cannot be dropped, let M denote the $\langle a, b\rangle$ subcomplex of Example 15.4. The following application is an example from small cancellation theory:

Example 15.7. In the group $G = \langle a, b, c, d \mid abcddacbbadc\rangle$, the subgroup generated by a and b has a finitely generated intersection with every other finitely generated subgroup of G. To apply Theorem 15.6 we let X be the standard 2-complex of the presentation, and let M be the graph consisting of the unique 0-cell of X together with the edges corresponding to the letters a and b. Note that X is a $C(12) - T(4)$-complex and that all its pieces are of length 1. Also, if we assign a weight of 0 to all of the sides present at the edges labeled a and b, and a weight of 1 to all of the others, then the unique 2-cell has weight 6, and the perimeter of each piece is at most 3. Thus X satisfies the path reduction hypothesis.

Corollary 15.8. *A Magnus subgroup of $G = \langle A | W^n\rangle$ has finitely generated intersection with any f.g. subgroup of $\langle A|W^n\rangle$ provided that $n \geq |W|$.*

Proof. Recall that a Magnus subgroup of G is any subgroup which is generated by a proper subset $B \subset A$. Let X be the standard 2-complex of G and let M be the graph formed using the unique 0-cell and the 1-cells corresponding to the generators in B. We assign a weight of 0 to every side incident at an edge in M and a weight of 1 to all of the other sides. To show that X, with this weighting, satisfies the path reduction hypothesis, by Theorem 8.2 we only need to show that the weight of the packet of the 2-cell is at most the perimeter of a subword of (a cyclic conjugate of) W. In order to apply Theorem 8.2, the weight of R must be strictly positive. This corresponds to the existence of a letter in W which is not in M. When this is not the case, the subgroup generated by B is a free factor of G and the result follows immediately from the theory of free products. Thus, we may assume that the weight of W is indeed positive, and that Theorem 8.2 applies.

Let k be the number of times that elements outside of M occur in W. The weight of the unique 2-cell is then nk, and the weight of its packet is

n^2k. On the other hand, since the perimeter of a single edge in X is at most nk, the perimeter of a subword of W is at most $nk|W|$. Thus, the path reduction hypothesis is satisfied whenever $n \geq |W|$. □

15.3. **Generalized word problem.** Our final related property concerns a generalization of the word problem for a group.

Definition 15.9 (Generalized word problem)**.** A subgroup of G generated by elements $V_1, \ldots, V_r$ is said to have *solvable membership problem* provided it is decidable whether an element $U \in G$ lies in $\langle V_1, \ldots, V_r \rangle$. If the membership problem is solvable for every finitely generated subgroup of G, then G is said to have a *solvable generalized word problem.* The name alludes to the fact that it includes the question of membership in the trivial subgroup (the word problem for G) as a particular case.

We will need the following lemma.

Lemma 15.10. *Let X be a 2-complex with non-negative integer weights assigned to the sides of its 2-cells. If X satisfies the (weak) path reduction hypothesis and $Y \to X$ is a map which does not admit a (weak) perimeter reduction or a fold, then every path $P \to Y$ whose image is a closed and null-homotopic path in X will be a closed path in Y as well.*

Proof. Let $P_i \to X$ $(i = 1, \ldots t)$ be a sequence of folds and perimeter reductions which starts with the path $P \to Y \to X$ and ends with the trivial path. Since $Y \to X$ does not admit any folds or (weak) perimeter reductions, all of the alterations to the path $P \to X$ can be mimicked in the path $P \to Y$. In particular, since the final path P_t is the trivial path and since the endpoints of P_i are the same throughout this process, the original path $P \to Y$ must have been closed. □

Theorem 15.11. *Let X be a 2-complex with non-negative integer weights assigned to the sides of its 2-cells. If X satisfies the path reduction hypothesis, then $\pi_1 X$ has a solvable generalized word problem.*

Proof. Let $V_1, \ldots, V_r, U$ be a set of closed paths in X with a common basepoint. To decide whether the element of $\pi_1 X$ represented by U is in the subgroup H generated by the elements corresponding to the V_i we proceed as follows. Let Y_1 be the wedge of r closed paths and a single open path attached to the others at only one of its endpoints. We define the map $Y_1 \to X$ so that it agrees with the maps $V_i \to X$ and $U \to X$. Let p denote the basepoint of Y_1, and let q be the other endpoint of the open path in Y_1 (see Figure 8). Let Y_t be the final complex produced by running the perimeter reduction algorithm on the map $Y_1 \to X$. Recall that the process also constructs a π_1-surjective map $Y_1 \to Y_t$ such that the composition $Y_1 \to Y_t \to X$ is the original map $Y_1 \to X$. We claim that vertices p and q are identified under the map $Y_1 \to Y_t$ if and only if the element corresponding to the path $U \to X$ lies in the subgroup H. Both directions of this implication need to be verified.

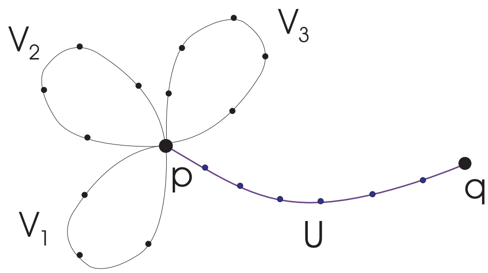

FIGURE 8. The complex Y_1 described in Theorem 15.11

First, observe that if p and q are identified in Y_t, then $U \to Y_1$ is sent to a closed path $U \to Y_t$. Since by the remark at the end of Definition 5.5, $\pi_1 Y_t$ can be considered a subgroup of $\pi_1 X$ whose image is the same as the image of $\pi_1 Y_1$ (which is H), the element represented by U must lie in the subgroup H.

Conversely, suppose that the element represented by U is indeed in the subgroup H. Then there exists a closed path $V \to Y_1$ whose image in X is homotopic to U. In particular, the path VU^{-1} is a path in Y_1 whose image in X is null-homotopic. By Lemma 15.10, the image of this path in Y_t must be closed. $\square$

We conclude this section with the following corollary.

Corollary 15.12. *If G is a group which satisfies the hypotheses of Theorem 8.2 or Theorem 9.6, then each of its finitely generated subgroups has a decidable membership problem.*

Proof. The proofs of these theorems actually show that these groups satisfy the path reduction hypothesis in addition to the perimeter reduction hypothesis. Thus, Theorem 15.11 can be applied. $\square$

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