GEOMETRIC PRESENTATIONS FOR THE PURE BRAID GROUP

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ABSTRACT. We give several new positive finite presentations for the pure braid group that are easy to remember and simple in form. All of our presentations involve a metric on the punctured disc so that the punctures are arranged "convexly", which is why we describe them as geometric presentations. Motivated by a presentation for the full braid group that we call the "rotation presentation", we introduce presentations for the pure braid group that we call the "twist presentation" and the "swing presentation". From the point of view of mapping class groups, the swing presentation can be interpreted as stating that the pure braid group is generated by a finite number of Dehn twists and that the only relations needed are the disjointness relation and the lantern relation.

The braid group has had a standard presentation on a minimal generating set ever since it was first defined by Emil Artin in the 1920s [3]. In 1998, Birman, Ko, and Lee [5] gave a more symmetrical presentation for the braid group on a larger generating set that has become fashionable of late (see, for example, [4], [7], [8], or [11]). Our goal is to apply a similar idea to the pure braid group. The standard finite presentation for the pure braid group (also due to Artin [2]) is slightly complicated and not that easy to remember. The presentations introduced here are, we believe, simple, easy to remember and intuitively clear. The article is structured as follows. In §1 we present a variation of the Birman–Ko–Lee presentation for the full braid group that we call the rotation presentation, and in sections 2, 3, and 4 we establish increasingly simple presentations for the pure braid group that we call the modified Artin presentation, the twist presentation, and the swing presentation. For these presentations, we think of the braid group as the fundamental group of the configuration space of n points in the disk. If we reinterpret the swing presentation in terms of mapping class groups, we get a presentation where the generators are Dehn twists, and the relations are the disjointness relation and the lantern relation (see section 4). The final section explores some possible extensions.

1. Braids

This section gives an unusual presentation of the full braid group using the notion of a convexly punctured disc. In addition to proving that it is equivalent to the (closely related) Birman–Ko–Lee presentation, we introduce several notions that pave the way for our new presentations of the pure braid group.

The n-string braid group $\textsc{Braid}_n$ can be viewed as the fundamental group of the configuration space of n distinct but indistinguishable points in a disc: $\textsc{Braid}_n \cong \pi_1(\mathcal{C}(\mathbf{D}, n))$. The points are called *punctures* and the elements of $\textsc{Braid}_n$ can be thought of as homotopy classes of based loops in $\mathcal{C}(\mathbf{D}, n)$, or equivalence classes of

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motions of these n points in $\mathbf{D}$ that start and end at the same configuration. The group gets its name from Artin's original definition, which is different; the survey [6] proves these definitions are equivalent. We begin by defining some simple elements in $\textsc{Braid}_n$.

Definition 1.1 (Half-twists)**.** Fix a configuration of the n points in $\mathbf{D}$ to serve as the basepoint of $\pi_1(\mathcal{C}(\mathbf{D}, n))$ and choose an embedded arc between two punctures that only meets the set of punctures at its endpoints. Let $\mathbf{D}'$ be any subdisc of $\mathbf{D}$ that contains the arc, the two punctures it connects, and no other punctures. Since $\pi_1(\mathcal{C}(\mathbf{D}', 2)) \cong \mathbb{Z}$, we can define a *half-twist* to be a generator of $\pi_1(\mathcal{C}(\mathbf{D}', 2))$. The *positive half-twist* corresponds to the half-twist where the two points move around each other in a clockwise fashion.

The braid group is generated by various finite sets of positive half-twists. In fact, any set of $n-1$ positive half-twists along non-crossing arcs that connect the punctures in a tree-like fashion is sufficient. Typically people choose as basepoint the configuration where the punctures lie in a straight line and they use the straight arcs connecting neighboring punctures to define a generating set of positive half-twists, but this is just a convenient standardization.

A more symmetric generating set is obtained by arranging the punctures at the vertices of a convex n-gon in the disc, and using all of the positive half-twists along the line segments connecting pairs of punctures. Notice that to make such a definition, the punctured disc needs to be more than just a topological punctured disc: a metric needs to be imposed so that convexity makes sense.

Definition 1.2 (Convexly punctured discs)**.** Let $\mathbf{D}$ be a topological disc in the Euclidean plane and assume that $\mathbf{D}$ has a distinguished n-element subset that we call its *punctures*. If the disc is a convex subset of $\mathbb{R}^2$ and the boundary of the convex hull of the set of punctures is an n-gon (i.e. every puncture occurs as a vertex of the convex hull of the set of punctures) then we say that $\mathbf{D}$ is a *convexly punctured disc* and that the punctures are in *convex position*. Let P denote the convex n-gon whose vertices are the punctures. There is a natural cyclic ordering of the punctures corresponding to the clockwise orientation of the boundary cycle of P. A labeling of the punctures is said to be *standard* if it uses the set $[n] = \{1, 2, \ldots, n\}$ (or better yet $\mathbb{Z}/n\mathbb{Z}$) and the punctures are labeled in the natural cyclic order. See the left hand side of Figure 1. More generally, when the punctures are bijectively labeled by a finite set A, we refer to the convexly punctured disc $\mathbf{D}_A$.

The notion of a convexly punctured disc is inherently recursive. The convexity of the arrangement of the punctures implies the existence of a canonical convexly punctured subdisc corresponding to each nonempty subset of punctures (see the right hand side of Figure 1).

Definition 1.3 (Convexly punctured subdiscs)**.** Let $\mathbf{D}_A$ be a convexly punctured disc. If B is any subset of A, then there is a convexly punctured subdisc containing only the punctures labeled by B. In particular, define $\mathbf{D}_B$ as an ϵ-neighborhood of the convex hull of the punctures labeled by B. Since the punctures in $\mathbf{D}_A$ are in convex position, we can choose ϵ small enough so that the only punctures in the ϵ-neighborhood are those labeled by B. The resulting disc is convex and the punctures are in convex position.

Definition 1.4 (Rotating the punctures)**.** If $\mathbf{D}_A$ is a convexly punctured disc, then there is a special element of $\pi_1(\mathcal{C}(\mathbf{D}_A, |A|))$ whose definition uses the convex

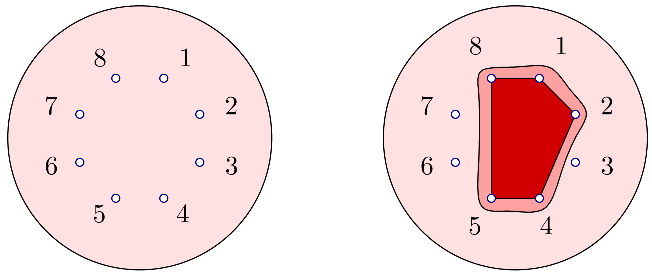

FIGURE 1. A convexly punctured disc with 8 punctures and a standard labeling and the convexly punctured subdisc $\mathbf{D}_B$ inside $\mathbf{D}_{[8]}$ when $B = \{1, 2, 4, 5, 8\}$.

position of the punctures. Let P_A denote the convex hull of the punctures and let R_A be the motion which simultaneously moves each puncture in $\mathbf{D}_A$ clockwise along one side of the polygon P_A. This is called *rotating the punctures*. See Figure 2. In the special case where $|A| = 2$, the motion we intend is the positive half-twist along the line segment P_A, and when $|A| = 1$ the rotation is the trivial motion.

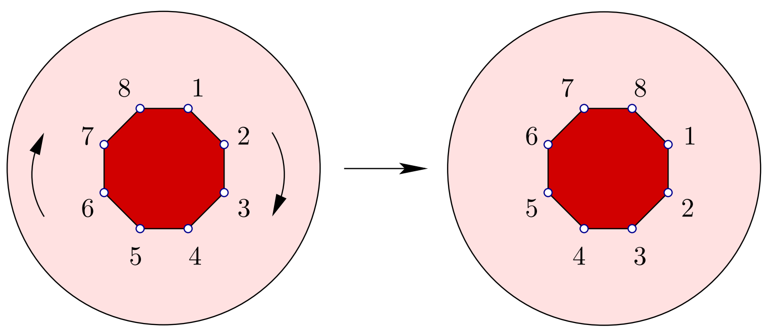

FIGURE 2. Rotating all the punctures.

Because of the recursive nature of convexly punctured discs, for each $B \subset A$ there is a well-defined element R_B inside $\pi_1(\mathcal{C}(\mathbf{D}_A, |A|))$ which rotates the punctures inside the subdisc $\mathbf{D}_B$ while leaving the remaining punctures fixed. See Figure 3 for an illustration.

When we wish to emphasize the fact that the metric on $\mathbf{D}_A$ is used to define R_B, we call this a *convex rotation*. The collection of all non-trivial convex rotations inside $\mathbf{D}_A$, denoted $\mathcal{R}_A$, is the set $\{R_B \mid B \subset A \text{ with } |B| \geq 2\}$. There are two types of relations among these rotations that are easy to establish, but in order to state these relations cleanly we need a pair of definitions.

Definition 1.5 (Non-crossing)**.** Let $\mathbf{D}_A$ be a convexly punctured disc and let B and C be disjoint subsets of A. When the convex hull of B and the convex hull of C do not intersect, B and C are said to be *non-crossing*. See Figure 4. More generally, an unordered collection $\{B_1, B_2, \ldots, B_k\}$ of pairwise disjoint subsets of A is called is non-crossing if B_i and B_j are non-crossing for each $i \neq j$.

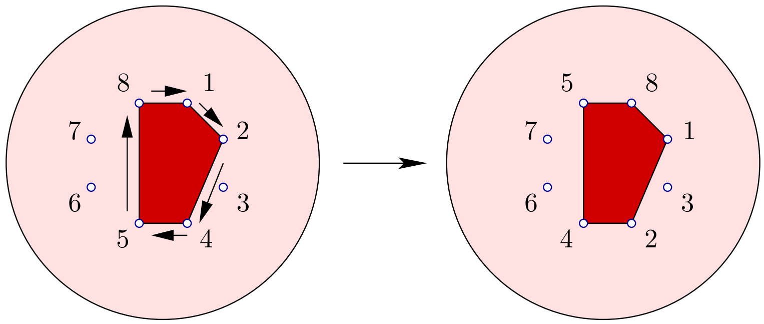

FIGURE 3. The rotation R_B inside $\mathbf{D}_{[8]}$ when $B = \{1, 2, 4, 5, 8\}$.

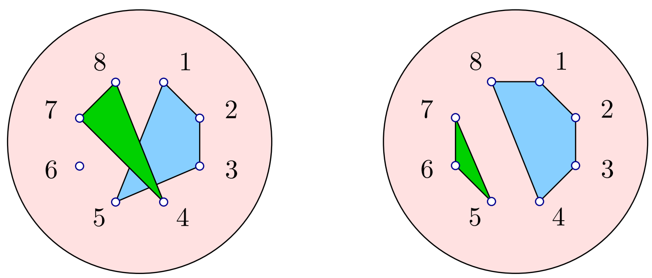

FIGURE 4. The subsets $\{1, 2, 3, 5\}$ and $\{4, 7, 8\}$ are crossing subsets of $[8]$ and the subsets $\{1, 2, 3, 4, 8\}$ and $\{5, 6, 7\}$ are non-crossing.

Definition 1.6 (Admissible partitions)**.** Let $\mathbf{D}_A$ be a convexly punctured disc. An ordered partition $(A_1, A_2, \ldots, A_k)$ of A is called an *admissible partition* (of A) if the cyclic ordering of the elements in A is consistent with the partial cyclic ordering determined by ordering of the A_i. In other words, $(A_1, A_2, \ldots, A_k)$ is an admissible partition if there is a point x on the boundary of the convex hull of the punctures so that the order in which the punctures occur starting at x and reading clockwise around the boundary consists of all of the punctures labeled by A_1, followed by all of the punctures labeled by A_2, and so on; see Figure 5. More generally, $(B_1, B_2, \ldots, B_k)$ is *admissible* inside $\mathbf{D}_A$ if it is an admissible ordering of the punctures inside the subdisc $\mathbf{D}_B$ where $B = \cup_{i=1}^{k} B_i$.

Convention 1.7. We follow the convention that uppercase letters such as A, B, and C denote sets while lowercase letters such as i, j, and k denote elements. For simplicity, unions of sets will be replaced by juxtapositions and brackets around singleton sets will be removed when there is no danger of confusion. Thus $\{i\} \cup B \cup C$ will be abbreviated as iBC. Finally, we follow the standard practice in the braid group literature [6] and compose our motions from left to right. Thus, when we write $U\ V$, the motion denoted U occurs first, followed by the motion denoted V.

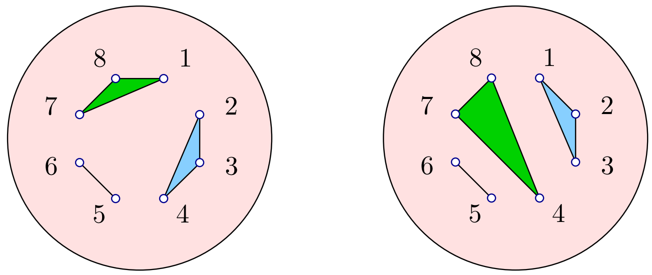

FIGURE 5. If $B = \{2,3,4\}$, $C = \{5,6\}$ and $D = \{7,8,1\}$ then (B,C,D) is an admissible partition, as is (C,D,B), but (C,B,D) is not. On the other hand, although the subsets $\{1,2,3\}$, $\{4,7,8\}$, and $\{5,6\}$ are non-crossing, no ordering of these three subsets is admissible.

Definition 1.8 (Rotation relations)**.** Let $\mathbf{D}_A$ be convexly punctured disc, let $\textsc{Braid}_A$ be $\pi_1(\mathcal{C}(\mathbf{D}_A, |A|))$, and let $\mathcal{R}_A \subset \textsc{Braid}_A$ denote the set of convex rotations inside $\mathbf{D}_A$. The elements in $\mathcal{R}_A$ satisfy the following two types of relations.

$$\begin{array}{ll} R_B R_C = R_C R_B & \text{when } B \text{ and } C \text{ are non-crossing} \\ R_{iBC} = R_{iB} R_{iC} & \text{when } (\{i\}, B, C) \text{ is admissible} \end{array}$$

The first type of relation holds because the rotations are occurring in the disjoint subdiscs $\mathbf{D}_B$ and $\mathbf{D}_C$, and the second type of relation is simply a factorization of the rotation into two smaller rotations. See Figure 6 for an illustration.

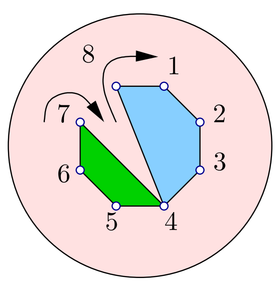

FIGURE 6. An illustration that $R_{45678123}$ is the same as $R_{4567}\ R_{48123}$.

In Theorem 1.10 we show that the convex rotations subject only to the rotation relations give a finite presentation for the braid group. To facilitate the proof we first review the Birman–Ko–Lee presentation of $\textsc{Braid}_A$.

Definition 1.9 (The Birman–Ko–Lee presentation)**.** Let $\mathbf{D}_A$ be a convexly punctured disc and let $\textsc{Braid}_A$ be $\pi_1(\mathcal{C}(\mathbf{D}_A, |A|))$. The presentation of $\textsc{Braid}_A$ introduced by Birman, Ko, and Lee can be readily restated using the language we have introduced above. The generators they use are the rotations R_{ij} and the presentation they give is the following:

$$\textsc{Braid}_A \cong \left\langle \{R_{ij}\} \;\middle|\; \begin{array}{ll} R_{ij}\ R_{kl} = R_{kl}\ R_{ij} & \text{when } ij \text{ and } kl \text{ are non-crossing} \\ R_{ijk} = R_{ij}\ R_{ik} & \text{when } (\{i\},\{j\},\{k\}) \text{ is admissible} \end{array} \right\rangle$$

Strictly speaking, they never introduce the elements R_{ijk} and instead equate its three possible factorizations, $R_{ij}\ R_{ik} = R_{ik}\ R_{jk} = R_{jk}\ R_{ij}$, but the net effect is the same.

It is now easy to show that the convex rotations subject only to the rotation relations given another presentation of the braid group.

Theorem 1.10 (Rotation presentation)**.** *If $\mathbf{D}_A$ is a convexly punctured disc,* $\textsc{Braid}_A$ *is $\pi_1(\mathcal{C}(\mathbf{D}_A, |A|))$, and $\mathcal{R}_A$ is the set of convex rotations inside $\mathbf{D}_A$, then*

$$\textsc{Braid}_A \cong \left\langle \mathcal{R}_A \;\middle|\; \begin{array}{ll} R_B\ R_C = R_C\ R_B & \textit{when } B \textit{ and } C \textit{ are non-crossing} \\ R_{iBC} = R_{iB}\ R_{iC} & \textit{when } (\{i\}, B, C) \textit{ is admissible} \end{array} \right\rangle$$

Proof. Let G_A be the group defined by this presentation, let B_A be the group defined by the Birman–Ko–Lee presentation and let $\textsc{Braid}_A$ be $\pi_1(\mathcal{C}(\mathbf{D}_A, |A|))$. Since all of the relations of G_A hold in $\pi_1(\mathcal{C}(\mathbf{D}_A, |A|))$, there is a group homomorphism $f : G_A \to \textsc{Braid}_A$. Similarly, all of the relations of the Birman–Ko–Lee presentation are included as relations in G_A, so there is a group homomorphism $g : B_A \to G_A$. Because the composition $f \circ g$ is nothing other than the standard isomorphism between B_A and $\textsc{Braid}_A$, we know that g must be one-to-one. Finally, the factorization rules show that the rotations of the form R_{ij} are enough to generate G_A. Thus g is onto. Since g and $f \circ g$ are both isomorphisms, so is f, and all three groups are isomorphic. □

Theorem 1.10 can be summarized as follows: if the basepoint of $\pi_1(\mathcal{C}(\mathbf{D}, n))$ is a convexly punctured disc, then this group is generated by the convex rotations and all of its relations are consequences of the semi-obvious rotation relations: disjoint rotations commute and larger rotations can be factored into smaller ones.

2. Pure braids

In this section we give a new finite positive presentation for the pure braid group that is similar in many ways to the rotation presentation for the full braid group (Theorem 1.10). If $\mathbf{D}_A$ is a convexly punctured disc, then the *pure braid group* $\textsc{PBraid}_A$ is the subgroup of $\textsc{Braid}_A$ where the punctures must return to their original positions. In other words, $\textsc{PBraid}_A$ is the fundamental group of the configuration space of $|A|$ distinct and distinguishable points in $\mathbf{D}$. Algebraically, $\textsc{PBraid}_A$ is the kernel of the natural projection $\textsc{Braid}_A \to \textsc{Sym}_A$ that forgets everything about the motion except the induced permutation of the punctures (where $\textsc{Sym}_A$, of course, denotes the group of permutations of $|A|$ elements). Because the group $\textsc{PBraid}_A$ is a finite-index subgroup of $\textsc{Braid}_A$, every element of $\textsc{Braid}_A$ has a power that lies in $\textsc{PBraid}_A$. This idea produces our first class of pure braid elements.

Definition 2.1 (Swinging the punctures)**.** Let $\mathbf{D}_A$ be a convexly punctured disc and let R_B be one of its convex rotations. The smallest power of R_B that lies in $\textsc{PBraid}_A$ is the motion $S_B = (R_B)^{|B|}$. We refer to this as *swinging the punctures* labeled B, swinging being an apt term for a vigorous rotation. To emphasize the fact that the metric on $\mathbf{D}_A$ is used to define S_B, we call it a *convex swing.*

The collection of all convex swings inside $\mathbf{D}_A$ is the finite set $\mathcal{S}_A = \{S_B \mid B \subset A \text{ with } |B| \geq 1\}$. The convex swings around single punctures, though trivial, are included in this set to facilitate extensions in later sections. In this section, the focus is on the convex swings of the form S_{ij}, which are known to generate the pure braid group. In fact, these were the elements Artin used in his original presentation.

Definition 2.2 (Artin's presentation of the pure braid group)**.** Let $\mathbf{D}_A$ be a convexly punctured disc with a standard labeling and let $\text{PB}\textsc{raid}_A$ be its pure braid group. Artin's presentation for $\text{PB}\textsc{raid}_A$ is generated by the elements of the form S_{ij} subject to the following five types of relations:

$$S_{rs}^{-1} S_{ij} S_{rs} = \begin{cases} S_{ij} & \text{if } r < s < i < j \\ S_{ij} & \text{if } i < r < s < j \\ S_{rj} S_{ij} S_{rj}^{-1} & \text{if } r < i = s < j \\ (S_{ij} S_{sj}) S_{ij} (S_{ij} S_{sj})^{-1} & \text{if } r = i < s < j \\ (S_{rj} S_{sj} S_{rj}^{-1} S_{sj}^{-1}) S_{ij} (S_{rj} S_{sj} S_{rj}^{-1} S_{sj}^{-1})^{-1} & \text{if } r < i < s < j \end{cases}$$

This presentation enabled Artin to establish a normal form for the pure braids and to use this normal form to prove that $\text{PB}\textsc{raid}_A$ is poly-free (i.e. iteratively constructed using extensions by free groups), but it seems clear from the final paragraph of the article that he felt it has its limitations:

> *Although it has been proved that every braid can be deformed into a similar normal form the writer is convinced that any attempt to carry this out on a living person would only lead to violent protests and discrimination against mathematics. He would therefore discourage such an experiment.*
>
> Emil Artin [2]

We reformulate Artin's presentation is in terms of crossing, non-crossing, and admissible partitions and eliminate the need for a standard labeling.

Theorem 2.3 (Artin's presentation, modified)**.** *Let $\mathbf{D}_A$ be a convexly punctured disc and let $\text{PB}\textsc{raid}_A$ be its pure braid group. The group $\text{PB}\textsc{raid}_A$ is generated by the convex swings S_{ij} and every relation is can be derived from the following three types of relations (assume all indices are distinct):*

1. $[S_{ij}, S_{rs}] = 1$ *when* $\{i, j\}$ *and* $\{r, s\}$ *are non-crossing,*
2. $[S_{ij}, S_{js} S_{rs} S_{js}^{-1}] = 1$ *when* $\{r, s\}$ *and* $\{i, j\}$ *cross in cyclic order* r, i, s, j*,*
3. $S_{sj} S_{rs} S_{rj} = S_{rs} S_{rj} S_{sj} = S_{rj} S_{sj} S_{rs}$ *when* (r, s, j) *is admissible.*

Each of these relations can be viewed as an assertion that two elements commute. The configurations needed for each type of relation are shown in Figure 7. In relation (2), the element $S_{js} S_{rs} S_{js}^{-1}$ corresponds to a (non-convex) swing along the dotted arc.

Proof. Since the generators are the same and it is striaghtforward to check that the given relations hold in $\text{PB}\textsc{raid}_n$, it suffices to show that Artin's original relations can be derived from relations (1), (2), and (3). Relation (1) implies the first two of Artin's relations. In Artin's third relation $i = s$. After replacing i with s and rearranging, the third relation is equivalent to $S_{sj} S_{rs} S_{rj} = S_{rs} S_{rj} S_{sj}$ with $r < s < j$, which is the first equality of relation (3). In Artin's fourth relation $i = r$. After replacing i with r and rearranging, the fourth relation is equivalent to:

$$(A4') \qquad S_{rj} S_{rs} S_{rj} S_{sj} = S_{rs} S_{rj} S_{sj} S_{rj}$$

with $r < s < j$. Relation $(A4')$ can be derived from the second equality in relation (3) by starting with this relation, right multiplying both sides by S_{rj}, and then

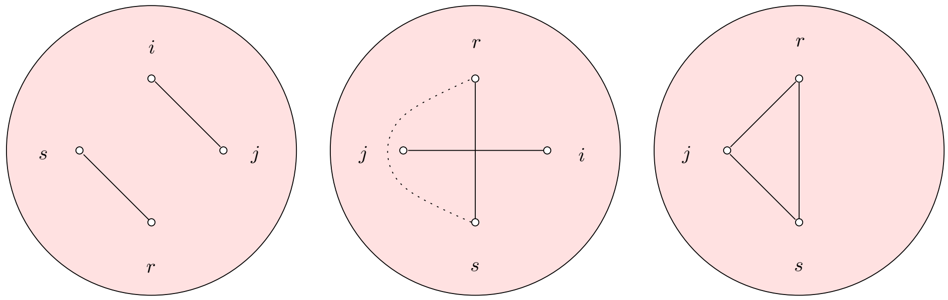

FIGURE 7. The configurations of punctures for Theorem 2.3. The pictures from left to right correspond to relations (1), (2), and (3).

applying the relation to the left hand side as indicated:

$$\begin{aligned} S_{rj}S_{sj}S_{rs} &= S_{rs}S_{rj}S_{sj} \\ S_{rj}(S_{sj}S_{rs}S_{rj}) &= S_{rs}S_{rj}S_{sj}S_{rj} \\ S_{rj}S_{rs}S_{rj}S_{sj} &= S_{rs}S_{rj}S_{sj}S_{rj} \end{aligned}$$

Finally, Artin's fifth relation can be derived from relations (2) and (3) as follows:

$$\begin{aligned} S_{ij}S_{sj}S_{rs}S_{sj}^{-1} &= S_{sj}S_{rs}S_{sj}^{-1}S_{ij} \\ S_{ij}(S_{sj}S_{rs}S_{rj})S_{rj}^{-1}S_{sj}^{-1} &= (S_{sj}S_{rs}S_{rj})S_{rj}^{-1}S_{sj}^{-1}S_{ij} \\ S_{ij}S_{rs}S_{rj}S_{sj}S_{rj}^{-1}S_{sj}^{-1} &= S_{rs}S_{rj}S_{sj}S_{rj}^{-1}S_{sj}^{-1}S_{ij} \end{aligned}$$

The first line is relation (2) slightly rearranged and the second line freely reduces to the first. We used relation (3) as indicated to go from the second line to the third, which is Artin's fifth relation rearranged. □

3. TWISTS

Our second presentation of the pure braid group is defined in in terms of "convex twists" that we like to think of as do-si-dos. A *do-si-do* is a movement in American square dancing where two dancers approach each other and circle back to back (whence the French term *dos-à-dos*), and then return to their original positions. The direction they face is unchanged throughout. There is a similar motion in the pure braid group.

Definition 3.1 (Twisting the punctures)**.** Let $\mathbf{D}_A$ be a convexly punctured disc and let B and C be non-crossing subsets that partition A. Treating the convexly punctured subdiscs $\mathbf{D}_B$ and $\mathbf{D}_C$ as though they were rigid punctures, we can define a positive full twist between them. In keeping with 'R' for rotation and 'S' for swing, we use 'T' for twist and we denote this motion $T_{B,C}$. The motion $T_{B,C}$ is identical to $T_{C,B}$, so the subscripts should be considered unordered. As in the square dancing move, the subdiscs should be moving by pure translations throughout. See Figure 8 for an illustration. More generally, we define $T_{B,C}$ whenever B and C are merely non-crossing by having the twist take place inside the convex subdisc $\mathbf{D}_{BC}$. We call these elements *convex twists* and we let $\mathcal{T}_A$ denote the collection of all convex twists inside $\mathbf{D}_A$ (i.e. $\mathcal{T}_A = \{T_{B,C} \mid B, C \subset A$ with B and C non-crossing$\}$).

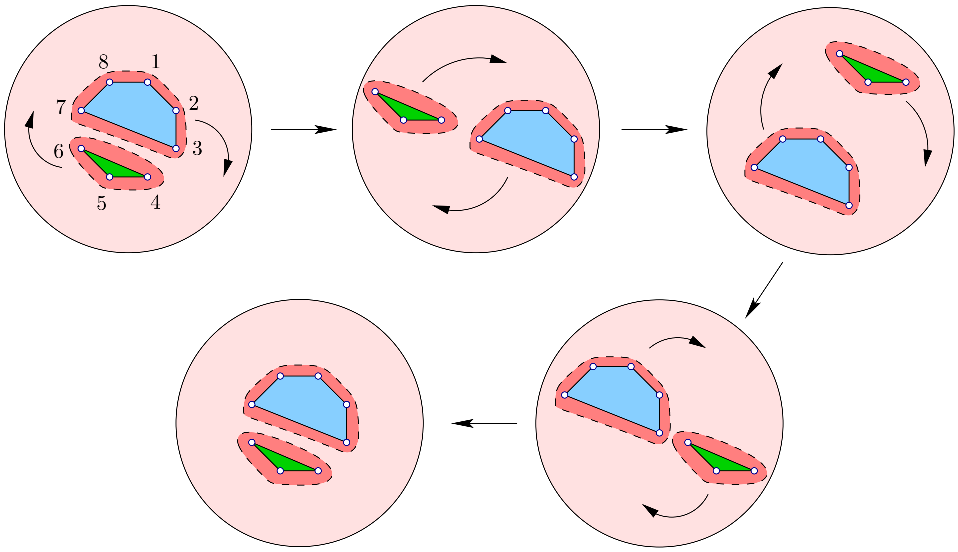

FIGURE 8. The convex twist $T_{B,C}$ when $B = \{4, 5, 6\}$ and $C = \{7, 8, 1, 2, 3\}$.

Notice that the element $T_{B,C}$ of $\pi_1(\mathcal{C}(\mathbf{D}, n))$ can be realized as a loop in $\mathcal{C}(\mathbf{D}, n)$ where the points of C stay fixed throughout and the points of B move around those of C. This perspective makes the third twist relation (Definition 3.3) easier to understand. The next step is to make some elementary observations about the relations satisfied by the convex twists. An extra definition makes the relations easier to state.

Definition 3.2 (Nested pairs)**.** Let $\mathbf{D}_A$ be a convexly punctured disc and let (B, C) and (D, E) be admissible subsets of A, not necessarily disjoint. We say that (B, C) and (D, E) are *nested* if one of the following four conditions hold: $BC \subset D$, $BC \subset E$, $DE \subset B$ or $DE \subset C$. For example, if $\mathbf{D}_A$ is a convexly punctured disc with a standard labeling by $A = [8]$ and $B = \{7, 8, 1, 2, 3\}$, $C = \{4, 5, 6\}$, $D = \{7, 1\}$ and $E = \{2, 3\}$, then (B, C) and (D, E) are nested because $D \cup E$ is a subset of B.

Definition 3.3 (Twist relations)**.** Let $\mathbf{D}_A$ be convexly punctured disc, let $\mathrm{PB}\textsc{raid}_A$ be its pure braid group, and let $\mathcal{T}_A \subset \mathrm{PB}\textsc{raid}_A$ be the finite set of convex twists inside $\mathbf{D}_A$. The elements in $\mathcal{T}_A$ satisfy the following three types of relations that we call the *convex twist relations*:

$$\begin{aligned} T_{B,C}\ T_{D,E} &= T_{D,E}\ T_{B,C} && \text{when } BC \text{ and } DE \text{ are non-crossing} \\ T_{B,C}\ T_{D,E} &= T_{D,E}\ T_{B,C} && \text{when } (B,C) \text{ and } (D,E) \text{ are nested} \\ T_{B,CD} &= T_{B,C}\ T_{B,D} && \text{when } (B,C,D) \text{ is admissible} \end{aligned}$$

The first relation holds because the twists are occurring in the disjoint subdiscs $\mathbf{D}_{BC}$ and $\mathbf{D}_{DE}$, and the third relation is simply a factorization of the twist into two smaller twists. See Figure 9 for an illustration. Thus, the only relation that need to be explained is the second one. In this case, it is useful to prove a stronger result first.

Lemma 3.4 (Twists and braids)**.** *If $\mathbf{D}_A$ is a convexly punctured disc and $T_{B,C}$ is a convex twist in $\mathbf{D}_A$, then $T_{B,C}\ U = U\ T_{B,C}$ for all U in* $\textsc{Braid}_B$.

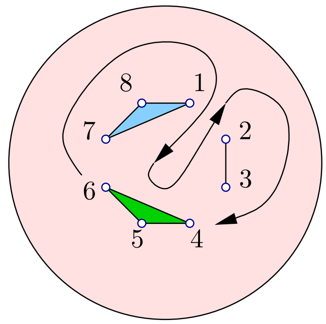

FIGURE 9. An illustration that $T_{B,CD}$ is the same as $T_{B,C}\ T_{B,D}$ when $B = \{4,5,6\}$, $C = \{7,8,1\}$ and $D = \{2,3\}$.

Proof. The 3-dimensional model of a braid (in Artin's original definition) is obtained from its representation as an element of $\pi_1(\mathcal{C}(\mathbf{D}_A, |A|))$ by tracing the paths of the punctures in the disc over time. This gives a solid cylinder $\mathbf{D}_A \times [0,1]$ with $|A|$ strands inside it. If we keep track of the convex subdisc $\mathbf{D}_B$ during the convex twist $T_{B,C}$, the result looks something like Figure 10. The key observation is the solid tube which tracks $\mathbf{D}_B$ over time is internally untwisted. Thus the action of any element of $\pi_1(\mathcal{C}(\mathbf{D}_B, |B|))$ on $\mathbf{D}_B$ that takes place after $T_{B,C}$ can be pushed back through this tube so that takes place before $T_{B,C}$. □

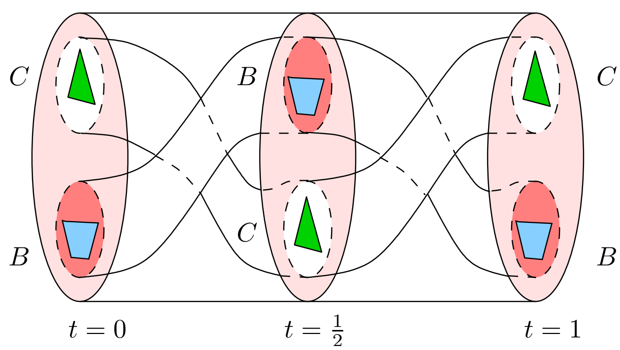

FIGURE 10. The 3-dimensional trace of a convex twist over time.

The reason why $T_{B,C}$ and $T_{D,E}$ commute when (B,C) and (D,E) are nested should now be clear. If, for example, $DE \subset B$, then $T_{D,E}$ is an element of $\mathrm{B{\scriptsize RAID}}_B$ and by Lemma 3.4 they commute. The other three cases are similar. We show in Theorem 3.7 that the convex twists subject only to the convex twist relations give a (finite positive) presentation of the pure braid group. To facilitate the proof we first establish that the modified Artin relations (Theorem 2.3) can be derived from the convex twist relations, starting with the following lemma.

Lemma 3.5. *Let $\mathbf{D}_A$ be a convexly punctured disc. If (B,C,D) is admissible, then*

$$T_{C,B}\ T_{B,D}\ T_{D,C} = T_{B,D}\ T_{D,C}\ T_{C,B} = T_{D,C}\ T_{C,B}\ T_{B,D}$$

holds in $\mathrm{PB{\scriptsize RAID}}_A$ and these relations are consequences of the convex twist relations.

Proof. To prove that the first two expressions are equal it suffices to show that $T_{C,B}$ commutes with $T_{B,D}\ T_{D,C}$. But $T_{B,D}\ T_{D,C} = T_{D,B}\ T_{D,C} = T_{D,BC}$ by the factoring relation (since (D, B, C) is also admissible) and this commutes with $T_{C,B}$ because nested twists commute. The second equality is proved similarly. □

Lemma 3.6. *Let $\mathbf{D}_A$ be a convexly punctured disc. The modified Artin relations (Theorem 2.3) are derivable from the convex twist relations (Definition 3.3).*

Proof. Since the convex swing S_{ij} is another name for the convex twist $T_{i,j}$, each modified Artin relation can be easily rewritten in terms of the convex twists. The first relation of Theorem 2.3 is covered by the assertion that non-crossing convex twists commute and the third relation is a special case of Lemma 3.5, so only the second relation remains to be derived. For later use, note that the second equality of relation (3) in Theorem 2.3 can be written as:

$$T_{s,j}\ T_{r,s}\ T_{s,j}^{-1} = T_{r,j}^{-1}\ T_{r,s}\ T_{r,j} \tag{3'}$$

The fact that this relation holds in $\mathrm{PB{\small RAID}}_A$ is clear from Figure 7; both sides describe the nonconvex twist along the dotted arc. To derive the second modified Artin relation, we start with a pair of commuting nested convex twists (second convex twist relation), decompose them into smaller convex twists (third convex twist relation), rearrange the equality, and finally apply relation (3′) to the right hand side as follows:

$$\begin{aligned}
T_{ris,j}\ T_{r,s} &= T_{r,s}\ T_{ris,j}\\
T_{r,j}\ T_{i,j}\ T_{s,j}\ T_{r,s} &= T_{r,s}\ T_{r,j}\ T_{i,j}\ T_{s,j}\\
T_{i,j}\ T_{s,j}\ T_{r,s}\ T_{s,j}^{-1} &= (T_{r,j}^{-1}\ T_{r,s}\ T_{r,j})\ T_{i,j}\\
T_{i,j}\ (T_{s,j}\ T_{r,s}\ T_{s,j}^{-1}) &= (T_{s,j}\ T_{r,s}\ T_{s,j}^{-1})\ T_{i,j}
\end{aligned}$$

The last line is the second modified Artin relation. □

Theorem 3.7 (Twist presentation)**.** *If $\mathbf{D}_A$ is a convexly punctured disc, then its pure braid group is generated by convex twists and all of its relations are consequences of the convex twist relations. In particular, $\mathrm{PB{\small RAID}}_A$ is isomorphic to the group defined by the following finite presentation:*

$$\left\langle \mathcal{T}_A \;\middle|\; \begin{array}{cl} T_{B,C}\ T_{D,E} = T_{D,E}\ T_{B,C} & \textit{when } BC \textit{ and } DE \textit{ are non-crossing}\\ T_{B,C}\ T_{D,E} = T_{D,E}\ T_{B,C} & \textit{when } (B,C) \textit{ and } (D,E) \textit{ are nested}\\ T_{B,CD} = T_{B,C}\ T_{B,D} & \textit{when } (B,C,D) \textit{ is admissible} \end{array} \right\rangle$$

Proof. Let G_A be the group defined by this listed presentation, let PB_A be the group defined by Artin's presentation and let $\mathrm{PB{\small RAID}}_A$ be the pure braid group of $\mathbf{D}_A$. Since all of the relations of G_A hold in the pure braid group $\mathrm{PB{\small RAID}}_A$, there is a group homomorphism $f : G_A \to \mathrm{PB{\small RAID}}_A$. Similarly, by Lemma 3.6, all of the relations of Artin's presentation are induced by relations in G_A, so there is a group homomorphism $g : PB_A \to G_A$. Because the composition $f \circ g$ is nothing other than the standard isomorphism between PB_A and $\mathrm{PB{\small RAID}}_A$, we know that g must be one-to-one. Finally, the factorization rules show that the twists of the form $T_{i,j}$ are enough to generate G_A. Thus g is also onto. Since g and $f \circ g$ are both isomorphisms so is f, and all three groups are isomorphic. □

Said differently, and perhaps more memorably, the pure braid group is generated by do-si-dos and the only relations needed are: nested do-si-dos commute, non-crossing do-si-dos commute, and do-si-dos decompose.

4. Swings

In this section we introduce our final finite presentation for the pure braid group, this time inspired by mapping class groups and particulary simple in form. The *relative mapping class group* of the pair (Σ, Δ), where Σ is a surface and $\Delta \subset \Sigma$, is denoted $\textsc{Mod}(\Sigma, \Delta)$, and is defined by

$$\textsc{Mod}(\Sigma, \Delta) = \pi_0(\text{Homeo}^+(\Sigma, \Delta))$$

where $\text{Homeo}^+(\Sigma, \Delta)$ denotes the set of orientation preserving homeomorphisms of Σ that fix the set Δ pointwise. The set $\textsc{Mod}(\Sigma, \Delta)$ forms a group under function composition. In order to be consistent with the earlier sections, we continue to use the algebraic rather than functional convention for these compositions (i.e., left to right, not right to left).

Mapping class groups are relevant because the pure braid group can be described in this language:

$$\text{P}\textsc{Braid}_A \cong \textsc{Mod}(\mathbf{D}_A, A \cup \partial \mathbf{D}_A)$$

where $\mathbf{D}_A$. For a proof of this isomorphism, see [6].

We begin by defining the simple elements of the mapping class group that are traditionally used as generators.

Definition 4.1 (Dehn twists)**.** If α is a simple closed curve in $\mathbf{D}_A$ disjoint from $A \cup \partial \mathbf{D}_A$ then there is a homeomorphism of $\mathbf{D}_A$ that looks like Figure 11 on a small regular neighborhood of α and the identity outside of this neighborhood. The homeomorphism described is well-defined up to an isotopy of α, so long as the intermediate curves remain disjoint from $A \cup \partial \mathbf{D}_A$. If a denotes this isotopy class of curves, the element of $\text{P}\textsc{Braid}_A$ it defines is called the *Dehn twist* about a and denoted S_a. The reason for this notation is discussed below.

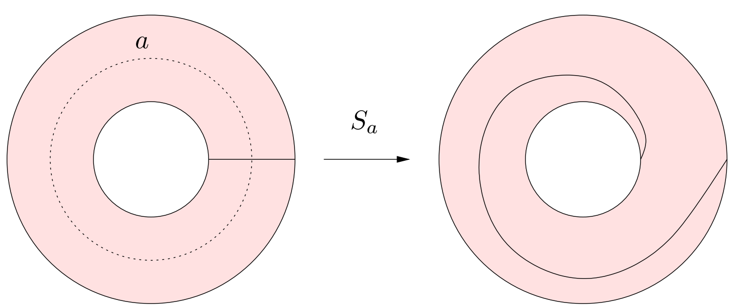

FIGURE 11. The non-identity portion of a Dehn twist.

Definition 4.2 (Convex Dehn twists)**.** A simple closed curve α in a convexly punctured disc $\mathbf{D}_A$ is *convex* if its interior (i.e., the component of $\mathbf{D} - \alpha$ disjoint from $\partial \mathbf{D}$) is convex. We call the Dehn twists about isotopy classes of convex simple closed curves *convex Dehn twists*. Let $\mathcal{D}_A$ denote the set of all convex Dehn twists in $\text{P}\textsc{Braid}_A$. This set is finite since each isotopy class is uniquely determined by the subset B of punctures contained in the interior of any representative convex curve. The trivial class that surrounds no punctures is excluded from this set.

Convex Dehn twists correspond, in fact, to motions we have already encountered.

Lemma 4.3 (Convex swings and convex Dehn twists)**.** *Let $\mathbf{D}_A$ be a convexly punctured disc. Every convex swing S_B, when viewed as an element of the mapping class group, is equal to the convex Dehn twist S_b where b is the isotopy class of the boundary cycle of the convex subdisc $\mathbf{D}_B$. Conversely, every convex Dehn twist S_b corresponds to the convex swing S_B where B labels the set of punctures contained in the interior of b.*

It is because of this close connection between convex swings and convex Dehn twists that we have chosen to use S_b to denote a Dehn twist even when the isotopy class b does not contain a convex representative. In the sequel, we use the Dehn twist notation (S_b) when no assumptions are made about convexity, but we typically switch to swing notation (S_B) when all the isotopy classes contain convex curves. Dehn twists satisfy three well-known types of relations.

Definition 4.4 (Dehn twist relations)**.** Let $\mathbf{D}_A$ be punctured disc and let $\text{PB}\textsc{raid}_A$ be its pure braid group. The Dehn twists in $\text{PB}\textsc{raid}_A$ satisfy the following:

$$\begin{array}{rcll} S_b & = & 1 & \text{when } b \text{ surrounds a single puncture} \\ S_bS_c & = & S_cS_b & \text{when } b \text{ and } c \text{ have disjoint representatives} \\ S_aS_bS_cS_d & = & S_xS_yS_z & \text{when the representative curves look like Figure 12} \end{array}$$

The first relation is trivial. The second relation, called the *disjointness relation*, is obvious since the annuli in which the twisting takes place can be chosen to be disjoint. The third relation, called the *lantern relation*, was known to Dehn [9]. The lantern relation is the only relation where the order matters and only on one side. If we were using functional notation, the lantern relation would be $S_dS_cS_bS_a = S_zS_yS_x$. The left hand side could be rewritten as $S_aS_bS_cS_d$ since these Dehn twists pairwise commute, but the order of S_x, S_y, and S_z is important.

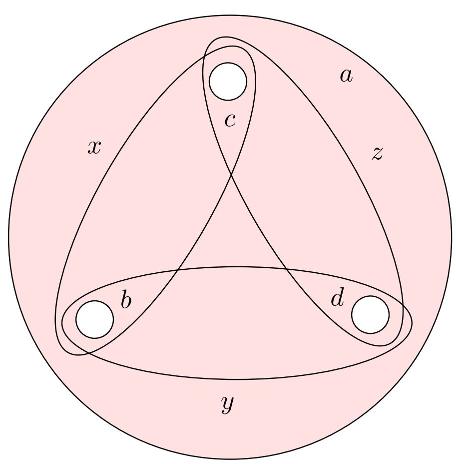

FIGURE 12. The curves in the lantern relation.

The convex twist version of the lantern relation is established in the course of proving Lemma 4.9. The first step is to convert the Dehn twist relations to convex form. Let $\mathbf{D}_A$ be a convexly punctured disc. We call two sets $B, C \subset A$ *compatible* if $B \subset C$, $C \subset B$, or B and C are non-crossing. This definition is useful since these are exactly the conditions under which the boundary curves $\partial\mathbf{D}_B$ and $\partial\mathbf{D}_C$ can be chosen to be disjoint.

Definition 4.5 (Swing relations)**.** If $\mathbf{D}_A$ is a convexly punctured disc and we restrict our attention to the Dehn twist relations that hold among the convex Dehn twists, then we can rewrite them as relations among the convex swings. We call these the *convex swing relations*.

$$\begin{array}{rcll} S_B & = & 1 & \text{when } |B| = 1 \\ S_B S_C & = & S_C S_B & \text{when } B \text{ and } C \text{ are compatible} \\ S_{BCD} S_B S_C S_D & = & S_{CB} S_{BD} S_{DC} & \text{when } (B, C, D) \text{ is admissible} \end{array}$$

Notice that if we choose isotopy classes of convex curves a, b, c, d, x, y and z so that they surround the puncture sets BCD, B, C, D, BC, BD, and CD, respectively, then these seven curves can be arranged as in Figure 12.

In Lemma 4.9 we show that the convex twist relations can be derived from the convex swing relations. To show this we first need to establish the connection between convex swings and convex twists.

Lemma 4.6 (Twists as Swings)**.** *Let $\mathbf{D}_A$ be a convexly punctured disc. Every convex twist $T_{B,C}$ in $\mathbf{D}_A$ can be rewritten as a product of three commuting convex swings (or their inverses). In particular, $T_{B,C} = S_B^{-1} S_C^{-1} S_{BC}$ written in any order and $S_{BC} = S_B S_C T_{B,C}$ written in any order.*

Proof. One approach is to first establish the equality $S_{BC} = S_B S_C T_{B,C}$. The natural motion for S_{BC} can be viewed as enacting the motions S_B, S_C, and $T_{B,C}$ simultaneously with the convex swings S_B and S_C taking place inside the convex subdiscs $\mathbf{D}_B$ and $\mathbf{D}_C$ as they perform their do-si-do. As in the proof of Lemma 3.4, the motions S_B and S_C can be pushed forward to back through these untwisted tubes so that the resulting elements take place in any order. Alternatively, let b, c, and z be the isotopy classes of the convex boundary cycles of $\mathbf{D}_B$, $\mathbf{D}_C$ and $\mathbf{D}_{BC}$, respectively. A visual proof that $T_{B,C}$ is equal to the product $S_b^{-1} S_c^{-1} S_z$ of three commuting Dehn twists is shown in Figure 13. □

An easy consequence of Lemma 4.6 is that the swing S_A is central in $\mathrm{PB{\small RAID}}_A$.

Theorem 4.7 (S_A is central)**.** *If $\mathbf{D}_A$ is a convexly punctured disc then for all $i, j \in A$, there is an element U such that $S_{ij} U = U S_{ij} = S_A$. As a consequence, the element S_A is central in* $\mathrm{PB{\small RAID}}_A$.

Proof. When $\{i, j\} = A$ we can set U equal to the identity element and there is nothing to prove, so assume $|A| > 2$. If i and j are consecutive in the standard cyclic order (say i followed by j) then there is a nonempty set B where (i, j, B) is an admissible partition of A. By Lemma 4.6, $S_A = S_{ij}(S_B T_{ij,B}) = (S_B T_{ij,B}) S_{ij}$, so the assertion is true with $U = S_B T_{ij,B}$. Finally, when i and j are not consecutive, there are nonempty sets B and C such that (i, B, j, C) is an admissible partition of A. Applying Lemma 4.6 twice we find that $S_A = S_{ijC} S_B T_{B,ijC} =$

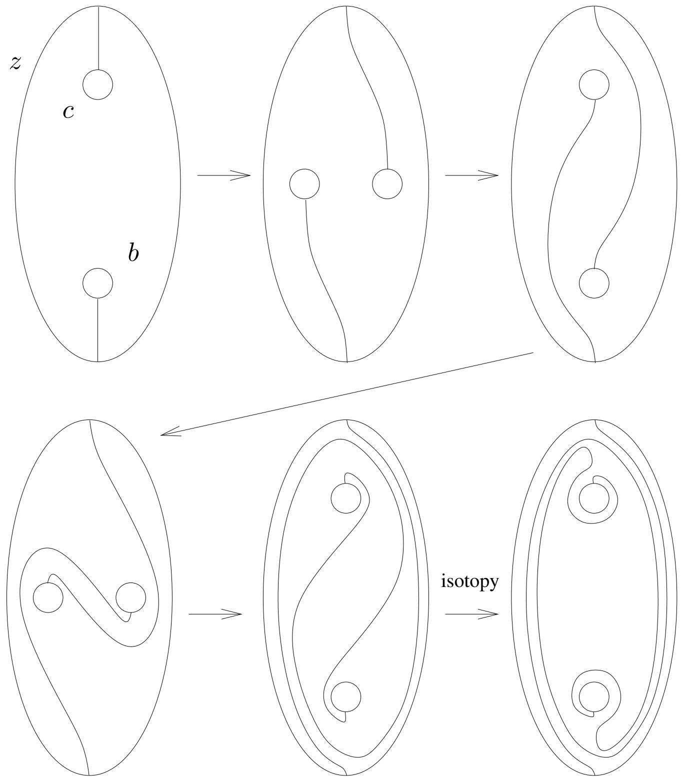

FIGURE 13. A visual proof of Lemma 4.6.

$(S_{ij}S_CT_{ij,C})S_BT_{B,ijC}$. Since all five of these elements pairwise commute, the assertion is true with $U = S_BS_CT_{ij,C}T_{B,ijC}$. The final assertion is nearly immediate. If we select the element U that is paired with the element S_{ij}, then parenthesizing $S_{ij}US_{ij}$ in two different ways shows that S_A commutes with S_{ij} for all $i, j \in A$. Because these elements generate PBRAID_A, S_A is central. □

Theorem 4.7 is, of course, obvious from the point of view of mapping class groups (since disjoint twists commute), configuration spaces (since nested swings commute), and in Artin's original pictorial definition (just by staring at the pictures). The point here is that the formal presentation we are introducing makes it easy to establish facts such as this without appealing to topological intuition.

Lemma 4.8 (Swings as Twists)**.** *Let* $\mathbf{D}_A$ *be a convexly punctured disc. Every convex swing* S_B *in* $\mathbf{D}_A$ *can be rewritten as a product of convex twists.*

Proof. Using the identity $S_{BC} = S_BS_CT_{B,C}$ from Lemma 4.6 repeatedly we can decompose any convex swing involving more than 2 punctures into a convex twist and two convex swings involving strictly fewer punctures. After finitely many such steps the original convex swing has been rewritten as a product of convex twists and convex swings involving single punctures. Since the latter are trivial, they drop out of the product, proving that the original convex swing is a product of convex twists. □

Lemma 4.9. *Let $\mathbf{D}_A$ be a convexly punctured disc. The convex twist relations (Theorem 3.7) are derivable from the convex swing relations (Definition 4.4).*

Proof. Let $T_{B,C}$ and $T_{D,E}$ be nested or non-crossing convex twists. When interpreted as elements of the mapping class group using Lemma 4.6, both elements become a product of three convex swings or their inverses. The nested / non-crossing hypothesis implies that all six convex curves can be chosen to be simultaneously pairwise disjoint. The fact that these convex twists commute now follows immediately from the commuting convex swing relations.

Finally, the third twist relation is closely related to the lantern relation. Let (B, C, D) be an admissible partition of $\mathbf{D}_A$. Using Lemma 4.6, the third twist relation $T_{B,CD} = T_{B,C}T_{B,D}$ is equivalent to $S_{BCD}S_B^{-1}S_{CD}^{-1} = (S_{BC}S_B^{-1}S_C^{-1})(S_{BD}S_B^{-1}S_D^{-1})$. This is the relation we wish to establish. Since the convex swings S_B, S_C, and S_D commute with all of the other convex swings in the relation, they can be collected on the left hand side, cancelling out an S_B^{-1} in the process. Finally, multiplying both sides on the right by S_{CD} produces the convex swing version of the lantern relation. Since these steps are reversible, the third twist relation can be derived from the lantern relation and the disjointness relation. □

Theorem 4.10 (Swing presentation)**.** *If $\mathbf{D}_A$ is a convexly punctured disc, then its relative mapping class group (i.e., its pure braid group) is generated by its convex Dehn twists and all of its relations are consequences of the obvious triviality and disjointness relations among the generators, together with a finite number of lantern relations derived from admissible partitions. More concretely,* $\text{PB}\textsc{raid}_A$ *is isomorphic to the group defined by the following finite convex swing presentation:*

$$\left\langle \mathcal{S}_A \;\middle|\; \begin{array}{cl} S_B = 1 & \textit{when } |B| = 1 \\ S_B S_C = S_C S_B & \textit{when } B \textit{ and } C \textit{ are compatible} \\ S_{BCD}S_B S_C S_D = S_{CB}S_{BD}S_{DC} & \textit{when } (B,C,D) \textit{ is admissible} \end{array} \right\rangle$$

Alternatively (and equivalently), $\text{PB}\textsc{raid}_A$ *is isomorphic to the group defined by the following finite convex Dehn twist presention:*

$$\left\langle \mathcal{S}_A \;\middle|\; \begin{array}{cl} S_b = 1 & \textit{when } b \textit{ surrounds a single puncture} \\ S_b S_c = S_c S_b & \textit{when } b \textit{ and } c \textit{ have disjoint representatives} \\ S_a S_b S_c S_d = S_z S_y S_x & \textit{when these isotopy classes look like Figure 12} \end{array} \right\rangle$$

Proof. Let H_A be the group defined by the convex swing presentation and let G_A be the group defined by the convex twist presentation given in Theorem 3.7. There is a group homomorphism $f : H_A \to \text{PB}\textsc{raid}_A$ since each of the relations in the presentation are known to hold in the pure braid group, and there is a group homorphism $g : G_A \to H_A$ extending the natural map that rewrites convex twists as a product of convex swings since every convex twist relation in the presentation of G_A can be derived from the convex swing relations in the presentation of H_A (Lemma 4.9). Because the composition $f \circ g$ is the previously established isomorphism between G_A and $\text{PB}\textsc{raid}_A$ (Theorem 3.7), the map g is injective. The map g is also onto since by Lemma 4.8 the set $g(G_A)$ includes the generating set of H_A. Because g and $f \circ g$ are isomorphisms, so is f. Finally, the conversion from the convex swing presentation to the convex Dehn twist presentation is merely a change of notation. □

5. Final Comments

In this final section we mention some possible extensions. The (pure) braid groups are paradigmatic examples of two distinct classes of groups: mapping class groups and Artin groups.

5.1. **Other mapping class groups.**

In the world of mapping class groups there is a natural modification where the punctures are replaced by boundary components.

Theorem 5.1. *Let $\mathbf{D}_A$ be a convexly punctured disc and let $\mathbf{D}'_A$ be $\mathbf{D}_A$ with small open neighborhoods of the punctures removed. The relative mapping class group of $\mathbf{D}'_A$ is generated by its convex Dehn twists and all of its relations are consequences of the obvious disjointness relations among the generators, together with a finite number of lantern relations derived from admissible partitions. More concretely,* $\textsc{Mod}(\mathbf{D}'_A, \partial\mathbf{D}'_A)$ *is isomorphic to the group defined by the following finite presentation:*

$$\left\langle \mathcal{S}_A \;\middle|\; \begin{array}{cc} S_B S_C = S_C S_B & \text{when } B \text{ and } C \text{ are compatible} \\ S_B S_C S_D S_{BCD} = S_{CB} S_{BD} S_{DC} & \text{when } (B,C,D) \text{ is admissible} \end{array} \right\rangle$$

The group defined above is an abelian extension the pure braid group since there is a natural map from it to $\text{PB}\textsc{raid}_A$ and the kernel is generated by the elements S_i which are central. If $\mathbf{D}''_A$ is $\mathbf{D}_A$ with small neighborhoods of only some of the punctures removed then its relative mapping class group has a similar positiive finite presentation with all of the convex disjointness and lantern relations and only those triviality relations that correspond to the remaining punctures.

The presentation for the relative mapping class group of a sphere with discs removed naturally leads to nice presentations for the pure stabilizers of any set of closed curves in the mapping class group of a closed surface whose complements have genus zero and there is at least a chance that a relatively simple presentation for the full mapping class group of a closed surface could result.

5.2. **Other Artin groups.**

In [1] Daniel Allcock gave orbifold descriptions for most of the other irreducible Artin groups of finite-type (i.e. those that correspond to the irreducible finite Coxeter groups). Because of the existence of an orbifold description, there should be presentations for the pure Artin groups of finite-type that are very similar in nature to the ones given here. In particular, for each pure Artin group G of finite-type the squares of the standard dual generators should be a generating set, these generators should be identifiable with some set of basic (convex) moves in the orbifold picture, and there should be enough relations among the geometrically obvious convex commutations, factorizations, and lantern relations to define a presentation for G.

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