INTEGRATING POLYNOMIALS IN SECANT AND TANGENT

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This note provides a relatively painless way to integrate arbitrary polynomials in secant and tangent without ever invoking integration by parts or anything beyond elementary polynomial and trigonometric identities. The techniques involved also introduce students to some of the ideas behind the construction of Laurent polynomials, although the manner in which they do so is rather indirect. We begin with a theorem which covers almost all of the possibilities.

Theorem 1. *For every polynomial $P(s,t)$ in two variables, there are polynomials F and G in one variable and a constant c such that*

$$\int P(\sec x, \tan x) \sec x \; dx = F(u) - G(v) + c\ln(u) + C$$

where $u = \sec x + \tan x$ and $v = \sec x - \tan x$.

Proof. Once we define $u = \sec x + \tan x$ and $v = \sec x - \tan x$, it is easy to check that $\sec x = \frac{u+v}{2}$, $\tan x = \frac{u-v}{2}$ and that $uv = \sec^2 x - \tan^2 x = 1$. If we replace $\sec x$ and $\tan x$ with their equivalents in terms of u and v, then the polynomial P in $\sec x$ and $\tan x$ becomes a polynomial in u and v instead. Moreover, since $uv = 1$, any monomial containing both variables quickly reduces to one which contains only a single variable. In other words, the resulting polynomial in u and v can always be written in the form $f(u)u + g(v)v + c$ where f and g are polynomials of a single variable and c is a constant.

Next consider the differentials du and dv. Since $du = \sec x \; \tan x + \sec^2 x \; dx$ and $dv = \sec x \; \tan x - \sec^2 x \; dx$, we find that $\sec x \; dx = \frac{1}{u} du = -\frac{1}{v} dv$. We are now ready to calculate the original integral.

$$\begin{aligned} \int P(\sec x, \tan x) \sec x \; dx &= \int (f(u)u + g(v)v + c) \sec x \; dx \\ &= \int f(u) \; du - \int g(v) \; dv + c \int \frac{1}{u} \; du \\ &= F(u) - G(v) + c\ln|u| + C \end{aligned}$$

where $F(u)$ and $G(v)$ represent the antiderivatives of the polynomials $f(u)$ and $g(v)$, respectively. $\square$

Example 2. Consider the integral $\int 16 \sec^5 x \; dx$. We find that

$$\begin{aligned} P(s,t) &= 16s^4 \\ P\left(\frac{u+v}{2}, \frac{u-v}{2}\right) &= (u+v)^4 \\ &= (u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + v^4) \\ &= (u^4 + 4u^2 + 6 + 4v^2 + v^4) \end{aligned}$$

Date: April 24, 1998.

Thus $f(u) = u^4 + 4u^2$, $g(v) = 4v^2 + v^4$, and $c = 6$ and we immediately conclude that

$$\int 16 \sec^5 x \; dx = \left(\frac{u^5}{5} + \frac{4u^3}{3}\right) - \left(\frac{4v^3}{3} + \frac{v^5}{5}\right) + 6 \ln |u| + C$$

where $u = \sec x + \tan x$ and $v = \sec x - \tan x$ as above. By way of contrast, the standard approach would involve performing integration by parts twice in order to reduce the exponent of the integrand.

Theorem 1 is nearly comprehensive in the sense that the only monomials in secant and tangent which are not covered are the constant term and those of the form $\tan^n x$, and even these latter terms are almost within the reach of the thereom. Consider, for example, the monomial $\tan^7 x$. By repeatedly applying the identity $\tan^2 x = \sec^2 x - 1$ we see that

$$\begin{aligned} \tan^7 x &= \tan^5 x \; \sec^2 x - \tan^5 x \\ &= \tan^5 x \; \sec^2 x - \tan^3 x \; \sec^2 x + \tan^3 x \\ &= \tan^5 x \; \sec^2 x - \tan^3 x \; \sec^2 x + \tan x \; \sec^2 x - \tan x \end{aligned}$$

In the final expression, the first three terms can be integrated using Theorem 1, so that only $\tan x$ remains. More generally, given any polynomial in secant and tangent the pure powers of tangent can be modified in this way so that the result is the sum of a constant term, a constant multiple of $\tan x$, and a polynomial to which Theorem 1 can be applied. This completes the proof of the following result.

Theorem 3. *For every polynomial $P(s,t)$ in two variables, there are polynomials F and G in one variable and constants a, b, and c such that*

$$\int P(\sec x, \tan x) \; dx = F(u) - G(v) + a \ln |u| - b \ln |\cos x| + cx + C$$

where $u = \sec x + \tan x$ and $v = \sec x - \tan x$.

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