

WINDMILLS AND EXTREME 2-CELLS

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ABSTRACT. In this article we prove new results about the existence of 2-cells in disc diagrams which are extreme in the sense that they are attached to the rest of the diagram along a small connected portion of their boundary cycle. In particular, we establish conditions on a 2-complex X which imply that all minimal area disc diagrams over X with reduced boundary cycles have extreme 2-cells in this sense. The existence of extreme 2-cells in disc diagrams over these complexes leads to new results on coherence using the perimeter-reduction techniques we developed in an earlier article. Recall that a group is called coherent if all of its finitely generated subgroups are finitely presented. We illustrate this approach by showing that several classes of one-relator groups, small cancellation groups and groups with staggered presentations are collections of coherent groups.

In this article we prove some new results about the existence of extreme 2-cells in disc diagrams which lead to new results on coherence. In particular, we combine the diagram results shown here with the theorems from [3] to establish the coherence of various classes of one-relator groups, small cancellation groups, and groups with relatively staggered presentations. The article is organized as follows: § 1 contains background definitions, § 2 recalls how extreme 2-cells lead to perimeter reductions and to coherent fundamental groups, § 3 introduces the concept of a windmill, § 4 uses windmills to prove that extreme 2-cells exist, and finally § 5 uses extreme 2-cells to prove that various groups are coherent. For instance, we obtain the following special case of Corollary 5.12:

Corollary 0.1. *Let $G = \langle a_1, \dots, a_r, t \mid W^N \rangle$ where W has the form*

$$t^{\epsilon_1} W_1 t^{\epsilon_2} W_2 \dots t^{\epsilon_k} W_k,$$

N is arbitrary, and for each i , ϵ_i is a nonzero integer and W_i is a reduced word in the a_i . Suppose that $\{W_1, W_2, \dots, W_k\}$ freely generate a subgroup of the free group $\langle a_1, \dots, a_r \mid - \rangle$. Then G is coherent.

1. BASIC DEFINITIONS

In this section we review some basic definitions about 2-complexes and diagrams.

Definition 1.1 (Combinatorial maps and complexes). A map $Y \rightarrow X$ between CW complexes is *combinatorial* if its restriction to each open cell of Y is a homeomorphism onto an open cell of X . A CW complex X is *combinatorial* provided

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that the attaching map of each open cell of X is combinatorial for a suitable subdivision. All complexes and maps considered in this article will be combinatorial after suitable subdivisions. In addition, we will only consider 2-complexes in which the attaching maps of 2-cells are immersions.

Definition 1.2 (Polygon). A *polygon* is a 2-dimensional disc whose cell structure has n 0-cells, n 1-cells, and one 2-cell where $n \geq 1$ is a natural number. If X is a combinatorial 2-complex then for each open 2-cell $C \hookrightarrow X$ there is a polygon R , a combinatorial map $R \rightarrow X$ and a map $C \rightarrow R$ such that the diagram

$$\begin{array}{ccc} C & \hookrightarrow & X \\ \downarrow & \nearrow & \\ R & & \end{array}$$

commutes, and the restriction $\partial R \rightarrow X$ is the attaching map of C . In this article the term *2-cell* will always mean a combinatorial map $R \rightarrow X$ where R is a polygon. The corresponding *open 2-cell* is the image of the interior of R .

A similar convention applies to 1-cells. Let e denote the graph with two 0-cells and one 1-cell connecting them. Since combinatorial maps from e to X are in one-to-one correspondence with the characteristic maps of 1-cells of X , we will often refer to a map $e \rightarrow X$ as a *1-cell* of X .

Technical difficulties with 2-complexes often arise because of the existence of redundant 2-cells and 2-cells attached by proper powers.

Definition 1.3 (Redundant 2-cells). Let X be a 2-complex. If R and S are distinct 2-cells in X with identical boundary cycles then R and S are called *redundant 2-cells*. More specifically, there must exist a combinatorial map $R \rightarrow S$ so that $\partial R \hookrightarrow R \rightarrow S \rightarrow X$ agrees with the map $\partial R \hookrightarrow R \rightarrow X$.

Definition 1.4 (Exponent of a 2-cell). Let X be a 2-complex, and let $R \rightarrow X$ be one of its 2-cells. Let n be the largest number such that the map $\partial R \rightarrow X$ can be expressed as a path W^n in X , where W is a closed path in X . This number n , which measures the periodicity of the map of $\partial R \rightarrow X$, is the *exponent* of R , and a path such as W is a *period* for ∂R . Notice that any other closed path which determines the same cycle as W will also be a period of ∂R . If the exponent n is greater than 1, then the R is said to be *attached by a proper power*.

Definition 1.5 (Disc Diagrams). A *disc diagram* D is a finite non-empty contractible 2-complex together with a specific embedding of D in \mathbb{R}^2 . A disc diagram which consists of a single 0-cell is called *trivial*. If it is homeomorphic to a disc then it is *non-singular*. It is a fundamental result in combinatorial group theory that the image of a closed (combinatorial) loop $P \rightarrow X$ is null-homotopic if and only if there is a disc diagram $D \rightarrow X$ having P as its boundary cycle [2].

Definition 1.6 (Area). Let X be a 2-complex and let $D \rightarrow X$ be a disc diagram. The *area* of D is simply the number of 2-cells it contains. Since area is a non-negative integer, for every closed loop $P \rightarrow X$ whose image is null-homotopic, there is a minimal area disc diagram $D \rightarrow X$ having P as its boundary cycle.

Definition 1.7 (Cancellable pair). Let X be a 2-complex, let $D \rightarrow X$ be a disc diagram and let R_1 and R_2 be distinct 2-cells in D . If (1) ∂R_1 and ∂R_2 are lifts of the same loop in X , (2) $\partial R_1 \cap \partial R_2$ contains a vertex v and (3) the closed path ∂R_1 can be read counterclockwise starting at v and the closed path ∂R_2 can be read

clockwise starting at v so that they have identical images in X , then R_1 and R_2 are called a *cancellable pair*. The definition of a cancellable pair is often restricted to the case where ∂R_1 and ∂R_2 contain a 1-cell in common, but this restriction is actually unnecessary.

Remark 1.8 (Redundant cells and proper powers). The focus of Definition 1.7 is on ∂R_1 and ∂R_2 (rather than R_1 and R_2 themselves) because of the possibility of redundant 2-cells and 2-cells attached by proper powers. If R and S are redundant 2-cells and $D \rightarrow X$ is a disc diagram containing a 2-cell R' which maps to R , then the map $D \rightarrow X$ can be modified so that R' is sent to S while keeping the rest of the map fixed. Similarly, if R is a 2-cell in X with exponent n and $D \rightarrow X$ is a disc diagram containing a 2-cell R' which is sent to R , then there are n distinct ways of sending R' to R while keeping the rest of the map fixed. Moreover, these modifications do not fundamentally change the basic properties of the disc diagram.

We will need the following lemma about minimal area diagrams. Its proof is standard and will be omitted. The basic idea is that R_1 and R_2 can be “cut out” and the resulting hole can be “sewn up”, but there are a few technicalities. See [5] or [4] for complete details.

Lemma 1.9. *Let X be a 2-complex and let $D \rightarrow X$ be a disc diagram. If D contains a cancellable pair then D does not have minimal area.*

2. PERIMETER REDUCTIONS

As mentioned in the introduction, the main goal of this article is to use structures we call “windmills” (introduced in the next section) to force disc diagrams to contain extreme 2-cells. Once this fact is known in a particular context, the machinery constructed in [3] can be used to conclude that the corresponding fundamental groups are coherent. In this short section, we briefly review the main ideas and results from [3] and very briefly explain the connection between the existence of extreme 2-cells and coherent fundamental groups.

Let Y be a subcomplex of a 2-complex X . The *perimeter* of Y in X is essentially the length of the boundary of an ϵ -neighborhood of Y in X , under the assumption that the 1-cells of X have unit length. For example, the perimeter of a single edge e is just the number of sides of 2-cells of X that are attached to e . Alternatively, the perimeter of Y in X is the total number of missing sides, where a side of a 2-cell in X is *missing* if it is attached to a 1-cell in Y but it is not a side of a 2-cell in Y . There is also a weighted version where the sides of the 2-cells of X are given non-negative weights (subject to minor restrictions). The *weighted perimeter* of Y in X is then the sum of the weights of the missing sides.

The main idea of [3] is to use perimeter calculations to force the termination of the following algorithm. Let X be 2-complex with a finitely generated fundamental group and let Y be a compact subcomplex of X such that the induced map $\pi_1 Y \rightarrow \pi_1 X$ is onto. Note that such a Y always exists since we can use the union of closed loops representing a finite generating set. At this point the map from $\pi_1 Y$ to $\pi_1 X$ may or may not be π_1 -injective. If it is, then $\pi_1 Y = \pi_1 X$ and the compactness of Y implies that $\pi_1 X$ is finitely presented. If this map is not π_1 -injective then it is natural to focus attention on a closed loop $P \rightarrow Y \subset X$ that is essential in Y and null-homotopic in X . Being null-homotopic in X there is a disc diagram $D \rightarrow X$ with P as its boundary cycle and being essential in Y there is at least one 2-cell of

D that is not in Y . If we enlarge Y by adding the 2-cells from D , then this new complex has a fundamental group that still maps onto $\pi_1 X$, it is still compact and it is closer to being π_1 -injective. In general, this process of enlargement might need to happen infinitely many times.

If, however, all the disc diagrams over X always have 2-cells where most of their boundary cycle is contained in the closed loop P , then it is at least conceivable that we can guarantee the existence of a 2-cell in D whose addition to Y results in a larger subcomplex with a smaller (weighted) perimeter. Under such conditions, the iterative procedure described above must stop since at each stage the non-negative integral perimeter of the resulting subcomplexes is steadily decreasing, and when it stops, $\pi_1 Y' = \pi_1 X$ and the compactness of Y' implies that $\pi_1 X$ is finitely presented as above.

Many variations on this proof-scheme are described in [3] along with precise definitions and statements of the results. In this article we focus on producing 2-complexes for which every disc diagram has an extreme 2-cell. The conclusion that the corresponding fundamental groups are coherent will follow from the fact that in the contexts described weights can be found so that the hypotheses of Theorem 7.6 of [3] are satisfied.

3. WINDMILLS

In this section we introduce a particular type of (weak) subcomplex of a 2-complex that we call a windmill. These structures will be used to force the existence of extreme 2-cells in disc diagrams.

Definition 3.1 (Subcomplexes). Let X and Y be 2-complexes and let $Y \hookrightarrow X$ be a topological embedding. If X and Y can be subdivided so that $Y \hookrightarrow X$ is combinatorial, then we will call Y a *subcomplex* of X even though its image is not a subcomplex in the original cell structure of X . We will use the term *true subcomplex* if $Y \subset X$ is a subcomplex in the traditional sense - without subdivisions. Finally, given a subcomplex Y in X , the closure of $X \setminus Y$ will be another subcomplex that we will call its *complement*.

The fact that the image of Y need not be a subcomplex of X in the traditional sense could have been avoided if we had assumed at the start that X and Y were already suitably subdivided. We will not, however, carry out such subdivisions since the cell structures of X and Y carry information of interest in applications. In fact we will mostly be interested in the other extreme: subcomplexes where the image of Y^1 is, in some sense, transverse to X^1 .

Definition 3.2 (Windmills). Let X be a 2-complex, let $Y \hookrightarrow X$ be a subcomplex and let $Z \hookrightarrow X$ be its complement subcomplex, and let $\Gamma = Y \cap Z$ be the subgraph of X which separates them. If $\phi : R \rightarrow X$ is a 2-cell of X , then we will say R is a windmill with respect to Z if, roughly speaking, $\phi^{-1}(Z)$ looks like a windmill. An example is shown in Figure 1. The dark portion of this 2-cell belongs to $\phi^{-1}(Z)$ and there are eight 1-cells in its interior which separate the light and dark areas. Other examples are shown in Figures 2 and 3.

The precise definition we will use goes as follows: R is a *windmill with respect to* Z if $\phi^{-1}(Z \setminus \Gamma)$ is connected and $\phi^{-1}(\Gamma)$ is homeomorphic to a collection of isolated points in ∂R plus $n \geq 2$ disjoint closed 1-cells whose endpoints lie in ∂R and whose

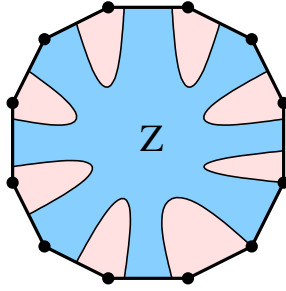
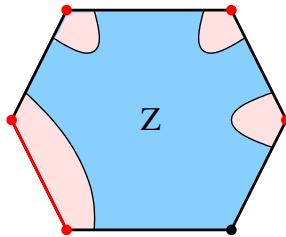


FIGURE 1. A windmill configuration in a 2-cell.

interiors lie entirely in the interior of R . If each 2-cell of X is a windmill with respect to Z , then Z is a *windmill in X* .

Note that if Z is a windmill in X then $\Gamma \cap X^1$ is a finite set of points. We will now give two concrete methods of creating windmills which we will need for our applications in Section 5.

Definition 3.3 (∂A). Let X be a connected 2-complex and let A be a true subcomplex of X^1 . Let Y be the closure of a regular neighborhood of A , let Z denote the complementary subcomplex, and let ∂A denote the intersection Γ of Y and Z . Then Z is a windmill if $\phi^{-1}(A)$ is disconnected for each 2-cell $\phi : R \rightarrow X$. For example, if X consists of a single hexagonal 2-cell and A consists of five of its 0-cells and one of its 1-cells, then the windmill created by ∂A is shown in Figure 2.

FIGURE 2. A windmill created by ∂A .

Our second construction is similar.

Definition 3.4 ($\bar{\partial} A$). Let X be a connected 2-complex, let A and B be true subcomplexes of X^1 such that $(A \cup B) = X^1$ and $(A \cap B) \subset X^0$. We will now define $\bar{\partial} A$ and simultaneously define Y and Z so that they extend A and B respectively. Define the vertices of $\bar{\partial} A$ to be the 0-cells in $A \cap B$. Suppose $R \rightarrow X$ is a 2-cell. If $\partial R \rightarrow X$ only contains 1-cells from A then R will also be a 2-cell of Y and if ∂R only contains 1-cells from B then R will belong to Z . Finally, if it contains 1-cells from A and B , then the boundary cycle $\partial R \rightarrow X$ can be uniquely partitioned into non-trivial paths which alternate between paths in A and paths in B . For each non-trivial path in A , we add an edge to $\bar{\partial} A$ which starts and ends at the endpoints of this path and runs parallel to it through the interior of R . The regions of R thus created which border 1-cells from A will belong to Y and the unique remaining

region will belong to Z . This procedure will create a windmill Z if $\phi^{-1}(A)$ has more than one non-trivial component for each 2-cell $\phi : R \rightarrow X$. If X consists of a single hexagonal 2-cell, A contains three of its 1-cells (the two leftmost 1-cells and the 1-cell in the upper right) and B contains the other three, then the windmill created by ∂A is shown in Figure 3.

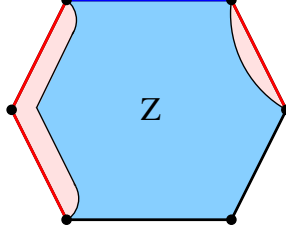


FIGURE 3. A windmill created by ∂A .

Remark 3.5 (∂A versus $\bar{\partial} A$). Despite their similar definitions, in general, neither $\partial A \hookrightarrow X$ nor $\bar{\partial} A \hookrightarrow X$ is homotopic to a subgraph of the other. To pass from $\partial A \hookrightarrow X$ to $\bar{\partial} A \hookrightarrow X$ requires shrinking some “trivial” loops and identifying distinct vertices. Moreover, these definitions will lead to independent applications.

The windmills of primary interest will be those where a particular inclusion map is π_1 -injective. When X has no redundant 2-cells and no 2-cells attached by proper powers (and Γ is the subgraph which separates a windmill from its complement), we will require that the inclusion $\Gamma \hookrightarrow X$ be π_1 -injective. In the general case, we will only need to focus on a particular portion of Γ that we call its essence.

Definition 3.6 (Essence of a subgraph). Let X be a 2-complex and let $\Gamma \hookrightarrow X$ be the subgraph which separates a windmill in X from its complement. If Γ partitions redundant 2-cells R and S in similar ways, then the portion of Γ in R and the portion in S perform similar functions in disc diagrams over X and we will not need both. Similarly, if R is a 2-cell with exponent $n > 1$ and the windmill-like structure in R respects this n -fold symmetry, then we will only need “ $\frac{1}{n}$ -th” of $\Gamma \cap R$. Both types of redundancies may occur in ∂A and in $\bar{\partial} A$. These two observations define an equivalence relation on the 1-cells of Γ . Let $\text{ESSENCE}(\Gamma) \hookrightarrow X$ be a graph in X which results from picking one 1-cell from each equivalence class. In the end the exact choice of 1-cells is irrelevant since, if $\Gamma' \hookrightarrow X$ and $\Gamma'' \hookrightarrow X$ are any two possibilities for $\text{ESSENCE}(\Gamma) \hookrightarrow X$ then Γ' and Γ'' are homeomorphic and the maps are homotopic. To see the homotopy, note that distinct choices of representative 1-cells can be pushed to the same path in X^1 while keeping their endpoints fixed.

Definition 3.7 (Splitting windmills). Let X be a 2-complex and let Γ be the subgraph which separates a windmill Z in X from its complement. If the embedding $\text{ESSENCE}(\Gamma) \hookrightarrow X$ is π_1 -injective on each connected component then Z is a *splitting windmill*.

4. EXTREME 2-CELLS

In this section we prove that minimal area disc diagrams over 2-complexes with splitting windmills have 2-cells which are extreme in the sense that they are attached to the rest of D along a very small portion of their boundary cycle (Theorem 4.14). The key property of splitting windmills that enables the proof is that they partition minimal area disc diagrams in a very restrictive manner. Given any map to X (such as a disc diagram) we can pull back the partitioning of X determined by the windmill and its complement to define a partitioning of the domain. Recall that a graph with no cycles is a *forest*, a connected forest is a *tree*, and a vertex of valence 1 is a *leaf*.

Theorem 4.1 (Forest). *Let X be a 2-complex and let $\Gamma \hookrightarrow X$ be the subgraph which separates a splitting windmill from its complement. If $P \rightarrow X$ is a non-trivial null-homotopic immersed combinatorial path and $\psi : D \rightarrow X$ is a minimal area disc diagram having P as its boundary cycle, then $\Gamma' = \psi^{-1}(\Gamma)$ is a forest and every leaf in Γ' lies in ∂D .*

Proof. Let Y and Z denote complementary subcomplexes in X , one of which is a splitting windmill. Which letter represents the windmill will be irrelevant since the proof is symmetric with respect to Y and Z . The second assertion is immediate since every 1-cell in Γ' traverses an open 2-cell of D in which one side belongs to $\phi^{-1}(Y)$ and the other to $\phi^{-1}(Z)$, whereas if D contained a leaf in its interior, both sides of its unique 1-cell would necessarily belong to the same preimage.

Suppose that Γ' contains a cycle. By choosing an innermost cycle we can find a cycle Q in $\Gamma' \subset D$ so that the portion of D to the left of Q belongs entirely to $\psi^{-1}(Y)$ or entirely to $\psi^{-1}(Z)$ as Q is traversed counterclockwise. Without loss of generality assume it belongs to $\psi^{-1}(Z)$. If $\psi(Q)$ is not an immersed loop in Γ , then the 2-cells containing the portion of Q immediately before and after a point which fails to be an immersion will form a cancellable pair in D . Note that we need the fact that the portion of D to the left of Q lies in $\psi^{-1}(Z)$ to conclude that these 2-cells have opposite orientations. Since by Lemma 1.9 this contradicts our assumption that D has minimal area, $\psi(Q)$ must be immersed. Moreover, since Γ is a graph, $\psi(Q)$ is an essential in Γ .

Next, let $\phi : \Gamma \rightarrow \text{ESSENCE}(\Gamma)$ be the natural projection which sends each 1-cell in Γ to the 1-cell in $\text{ESSENCE}(\Gamma)$ which represents its equivalence class. We claim that $\phi(\psi(Q))$ is immersed – hence essential – in $\text{ESSENCE}(\Gamma)$. If not, then as above, the 2-cells containing the portion of Q immediately before and after the point which fails to be an immersion will form a cancellable pair in D . The difference is that this time the two 2-cells are not sent to X in identical ways; they might be sent to redundant 2-cells or in different ways to a single 2-cell attached by a proper power. Finally, $\phi(\psi(Q))$ is essential in X since the inclusion $\text{ESSENCE}(\Gamma) \hookrightarrow X$ is π_1 -injective by assumption. On the other hand, D is simply-connected, so Q is null-homotopic in D and its image should be null-homotopic in X . This contradiction shows that Γ' is a forest. \square

Remark 4.2 (Structure of Γ'). The conclusion of Theorem 4.1 does not preclude the existence of trivial components in the interior of D since the arguments given need an edge to get started. Such isolated interior points can arise if Γ passes through a 0-cell of X . Another complication is that the components of Γ' can be quite complicated trees. Figure 4 illustrates how such branching can occur.

Although we will not need this simplification, we note that neither complication will occur when $\Gamma \cap X^0$ is empty.

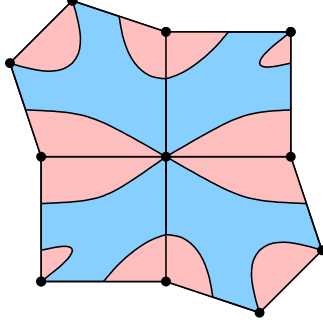


FIGURE 4. An example of branching in Γ'

Despite the fact that Γ' might branch in D , there is enough structure to ensure that D is constructed by gluing together components in a tree-like fashion. To make this precise we introduce the idea of a connection graph.

Definition 4.3 (Connection graph). If D is a disc diagram and $\Gamma \hookrightarrow D$ is a graph in D , then we define its *connection graph* $\text{CONN}(\Gamma, D)$ as follows. The vertices of $\text{CONN}(\Gamma, D)$ are the path components of Γ and the path components of $D \setminus \Gamma$, and we have an edge from u to v when u represents a component Γ_0 of Γ , v represents a component D_0 of $D \setminus \Gamma$, and $\Gamma_0 \cap \partial D_0 \neq \emptyset$.

Remark 4.4 (Paths). Since the components involved are path connected and the edges represent adjacency in D , for any combinatorial path $P \rightarrow \text{CONN}(\Gamma, D)$, we can create a path $Q \rightarrow D$ which traces through the corresponding components in the exact same order. Moreover, if P is simple, we can choose Q to be simple. Conversely, generic paths $Q \rightarrow D$ determine combinatorial paths $P \rightarrow \text{CONN}(\Gamma, D)$ which simply trace the components traversed. Our standing assumption that maps can be suitably subdivided to be combinatorial, rules out pathological paths which wiggle across a single edge in Γ infinitely often in a decaying manner.

Lemma 4.5 (Tree-like). *Let D be a disc diagram and let $\Gamma \hookrightarrow D$ be a forest in D . If the leaves of Γ lie in ∂D , then its connection graph, $T = \text{CONN}(\Gamma, D)$, is a tree. If in addition no isolated vertices of Γ are contained in the interior of D , then the components of $D \setminus \Gamma$ are simply connected.*

Proof. Suppose $P \rightarrow T$ is a nontrivial closed simple cycle. We will reach a contradiction by showing that P has a *backtrack* meaning that it traverses an edge followed by its inverse. Let $Q \rightarrow D$ be the closed simple cycle from Remark 4.4. Then Q bounds a disc diagram $D' \subset D$. Since P is nontrivial, Q intersects Γ . An innermost component of $D' - \Gamma$ determines a backtrack of P .

Suppose some component D_0 of $D \setminus \Gamma$ is not simply-connected. Let $Q \subset D_0$ be an essential simple closed curve. Let D' be the region bounded by Q . Then D' cannot be a disc since Q is essential. Thus D' contains some component of Γ , which is necessarily a trivial component since any nontrivial component intersects ∂D by Theorem 4.1. \square

In order to take full advantage of Lemma 4.5 we introduce the notion of a modified preimage.

Definition 4.6 (Modified preimages). Let X be 2-complex and let Y and Z be complementary subcomplexes separated by $\Gamma = Y \cap Z$. If $\psi : D \rightarrow X$ is a disc diagram over X , then we partition D into sets Y' , Z' and Γ' as follows. Let Γ' be $\psi^{-1}(\Gamma)$ with any isolated points in the interior of D removed and let $Z' = \psi^{-1}(Z) \setminus \Gamma'$ and $Y' = \psi^{-1}(Y) \setminus \Gamma'$. We will call Γ' , Y' and Z' the *modified preimages* of Γ , Y and Z , respectively. Notice that Y' and Z' are open in D and Γ' is closed.

The sets Z' and Y' are almost the same as $\psi^{-1}(Z \setminus \Gamma)$ and $\psi^{-1}(Y \setminus \Gamma)$ except that the isolated points of $\psi^{-1}(\Gamma)$ in the interior of D have been added to the regions which contain them. Adding these points will ensure that the components of Z' and Y' will be simply-connected whenever Γ' is a forest with all its leaves in ∂D . In particular, the following corollary is an immediate consequence of Theorem 4.1, Lemma 4.5, and Definition 4.6.

Corollary 4.7 (Simply-connected). *Let X be a 2-complex and let $\Gamma \hookrightarrow X$ be the subgraph which separates a splitting windmill Z from its complement Y . If $P \rightarrow X$ is a non-trivial null-homotopic immersed combinatorial path and $\psi : D \rightarrow X$ is a minimal area disc diagram having P as its boundary cycle, then each component of each modified preimage, Γ' , Y' and Z' is simply-connected.*

A non-singular subdiagram of a disc diagram D which is attached to the rest of D at a single point is a *dangling subdiagram*. As a quick illustration of Lemma 4.5 we give a short proof of the well-known result that certain disc diagrams must contain dangling subdiagrams.

Lemma 4.8 (Dangling subdiagrams exist). *If X is a 2-complex, $P \rightarrow X$ is a non-trivial null-homotopic immersed combinatorial loop, and $D \rightarrow X$ is a disc diagram having P as its boundary cycle, then either D itself is non-singular or D contains at least two dangling subdiagrams.*

Proof. Let Γ be the collection of 0-cells of D whose removal disconnects D (i.e. cut vertices) and note that a disc diagram without cut vertices is either trivial, a single 1-cell, or non-singular. By Lemma 4.5, $T = \text{CONN}(\Gamma, D)$ is a tree, and by construction each vertex of Γ corresponds to a vertex of T with valence at least 2. Thus the leaves of T correspond to components of $D \setminus \Gamma$ attached to the rest of D at a single point. Since trivial subdiagrams cannot be separated off by cut vertices and 1-cells attached at a single point are prohibited since P is immersed, the leaves of T correspond to dangling subdiagrams. Similarly, if T is trivial, then D is non-singular since the restrictions on P ensure that D is not a single 0-cell or single 1-cell. The result now follows from the observation that finite trees are either trivial or have at least two leaves. \square

Our second application is only slightly more complicated. In order to state the result we will need the notion of an outermost component.

Definition 4.9 (Outermost components). Let Z be a subcomplex of a 2-complex X , let $\psi : D \rightarrow X$ be a disc diagram, and let Z' be the modified preimage of Z . A component Z_0 of Z' is *outermost* if $Z' \setminus Z_0$ is contained in a single connected component of $D \setminus Z_0$.

Lemma 4.10 (Outermost components exist). *Let Z be a splitting windmill in a 2-complex X , let $P \rightarrow X$ be a non-trivial null-homotopic immersed combinatorial path and let $\psi : D \rightarrow X$ be a minimal area disc diagram having P as its boundary cycle. If D is non-singular and Z' is the modified preimage of Z in D , then either Z' is connected or Z' has at least two outermost components. Moreover, for each outermost component Z_0 of D there exists a simple path $Q \rightarrow D$ in ∂Z_0 so that $D \setminus Q$ is disconnected and Z_0 lies in a different connected component of $D \setminus Q$ from the rest of Z' .*

Proof. By Theorem 4.1, Γ' is a forest with its leaves in ∂D , and so by Lemma 4.5, the connection graph $T = \text{CONN}(D, \Gamma')$ is a tree. Consider the smallest subtree T' of T which contains all of the vertices corresponding to components of Z' . This subtree is either trivial, in which case Z' is connected, or it has at least two leaves. By minimality of T' each leaf corresponds to a component of Z' , and using Remark 4.4, we see that a component of Z' is an outermost component if and only if it corresponds to a leaf in T' .

The final assertion can be shown as follows. Let Z_0 be an outermost component which corresponds to a leaf v in T' and let Γ_0 be the component of Γ' which corresponds to the unique vertex u in T' connected to v . The intersection $\partial Z_0 \cap \Gamma_0$ will be a path with the required properties. In particular, the intersection $\partial Z_0 \cap \Gamma_0$ is a simple path Q (rather than more complicated 1-complex) since Γ_0 is a tree with its leaves in ∂D and ∂Z_0 is a circle, so $\partial Z_0 \cap \Gamma_0$ consists of at most one arc since T is a tree. The separation properties for Q follow immediately from the position of u in T' and Remark 4.4. \square

Our third application of Lemma 4.5 will show that certain disc diagrams contain 2-cells which are extreme in the following sense.

Definition 4.11 (Extreme 2-cells). Let X be a 2-complex, let Y and Z be complementary subcomplexes, and let $\Gamma = Y \cap Z$ be the subgraph which separates them. A 2-cell R in a disc diagram $\psi : D \rightarrow X$ is *extreme with respect to Z* if ∂R is the concatenation of two paths S and Q where Q is a subpath of ∂D and $S \cap \psi^{-1}(Z)$ has at most one non-trivial component (isolated points in the intersection are ignored). Figure 5 contains a sketch of a disc diagram which contains four copies of the 2-cell from Figure 2. The one in the lower lefthand corner is not extreme; the other three are extreme.

We will prove three versions of the following result under successively weaker hypotheses.

Lemma 4.12 (Extreme 2-cells exist: first version). *Let X be a 2-complex, let Z be a splitting windmill in X with complement Y , let $P \rightarrow X$ be a non-trivial null-homotopic immersed combinatorial path, and let $\psi : D \rightarrow X$ be a minimal area disc diagram having P as its boundary cycle. If D is non-singular, and the modified preimage of Z in D is connected, then either D consists of a single 2-cell or D contains at least two 2-cells which are extreme with respect to Z .*

Proof. Let Y' and Z' denote the modified preimages of Y and Z , and let Δ be the graph $D^1 \cap Z'$. Observe that Δ is a forest, for otherwise there would be a simple closed curve Q in Δ and, since D is simply-connected, Q would bound a non-singular subdiagram of D containing at least one 2-cell. Consequently, Q would lie in Z' but contain points of Y' in its interior, contradicting the fact that

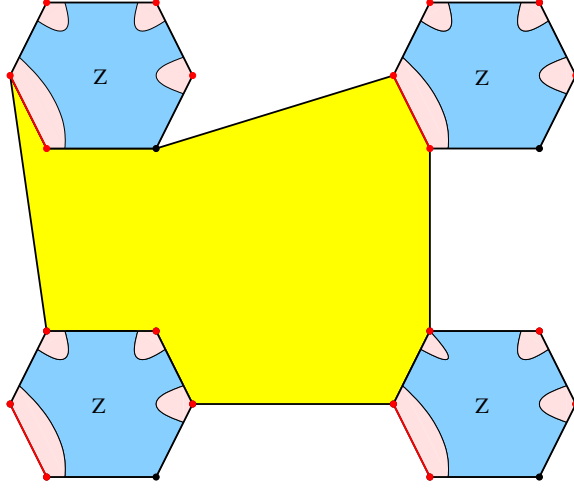


FIGURE 5. Three extreme 2-cells in a disc diagram.

Z' is simply-connected (Corollary 4.7). Moreover, the leaves of Δ must lie in $\partial Z'$ because the 2-cells of X are attached along immersed paths (Definition 1.1). Thus, by Lemma 4.5, the connection graph $T = \text{CONN}(Z', \Delta)$ is a tree.

A similar argument shows that for each 2-cell R in D , the distinct portions of $\partial R \cap Z'$ (recall that there are at least two by the definition of a windmill) belong to distinct components of Δ . If not, a simple path in Δ connecting distinct portions, combined with a simple path connecting them through $R \cap Z'$ (which exists because $R \cap Z'$ is connected) forms a simple closed path in Z' which surrounds points in $Y' \cup \Gamma'$ (in particular there are points of this type in ∂R separating the distinct intervals of $\partial R \cap Z'$ we have connected). This contradicts that Z' is simply-connected, proving the claim. Consequently, all leaves of T are components of Δ .

Finally, let T' be the smallest subtree in T which contains all of the vertices corresponding to components of $Z' \setminus \Delta$. Since the components of $Z' \setminus \Delta$ also correspond to the 2-cells in D , T' is a single vertex if and only if D consists of a single 2-cell. Moreover, when D has more than one 2-cell it is easy to see that a 2-cell of D is extreme with respect to Z if and only if it corresponds to a leaf of T' . \square

Using Lemma 4.10 we can remove the assumption that Z' is connected.

Lemma 4.13 (Extreme 2-cells exist: second version). *Let X be a 2-complex, let Z be a splitting windmill in X with complement Y , let $P \rightarrow X$ be a non-trivial null-homotopic immersed combinatorial path, and let $\psi : D \rightarrow X$ be a minimal area disc diagram having P as its boundary cycle. If D is non-singular, then either D consists of a single 2-cell or D contains at least two 2-cells which are extreme with respect to Z .*

Proof. Let Z' be the modified preimage of Z in D . By Lemma 4.12, we may assume Z' is disconnected and by Lemma 4.10, D must contain at least two outermost components. If each outermost component Z_0 contributes at least one extreme 2-cell, we will be done.

Let D_0 be the union of the 2-cells of D which intersect Z_0 non-trivially. Notice that $D_0 \cap Z' = Z_0$ since the intersection of Z' with each 2-cell is connected. We claim that D_0 is a nonsingular disc diagram which is attached to the rest of D along a path Q' contained in $Y' \cup \Gamma'$. To see that D_0 is simply-connected, suppose not. Then there is a simple closed path in $\partial D_0 \cap (Y' \cup \Gamma')$ which bounds a subdiagram of D (it cannot contain points in Z' since Z' is open in D). This subdiagram contains at least one 2-cell and hence a point in Z' . And finally the boundary of this component of Z' must be an essential cycle in Γ' contradicting Corollary 4.7. Thus D_0 is a disc diagram. Since it is a union of 2-cells and Z_0 is open in D , it is also non-singular. Finally, by Lemma 4.10, Z_0 can be separated from the rest of Z' by a path Q in Γ' . Let Q' be the portion of ∂D_0 which has the same endpoints as Q and which avoids Z_0 . The path Q' exists since Q separates and $Z' \cap D_0$ is connected.

By Lemma 4.12 D_0 is either a single 2-cell or it contains at least two 2-cells which are extreme with respect to Z . Since a single 2-cell attached to the rest of D along a path Q' in $Y' \cup \Gamma'$ is always extreme with respect to Z , we may assume D_0 has at least two extreme 2-cells. Finally, when such a D_0 is attached to the rest of the diagram along a path Q' in $Y' \cup \Gamma'$, at most one of these 2-cells loses its status as an extreme 2-cell, and the proof is complete. \square

Finally, using Lemma 4.8 we can remove the assumption that D is non-singular.

Theorem 4.14 (Extreme 2-cells exist). *If X is a 2-complex, Z is a splitting windmill in X with complement Y , $P \rightarrow X$ is a non-trivial null-homotopic immersed combinatorial path, and $\psi : D \rightarrow X$ is a minimal area disc diagram having P as its boundary cycle, then either D consists of a single 2-cell or D contains at least two 2-cells which are extreme with respect to Z .*

Proof. We may assume that D is singular by Lemma 4.13, so D must contain at least two dangling subdiagrams by Lemma 4.8. If each dangling subdiagram D' contributes at least one extreme 2-cell, we will be done. By Lemma 4.13 D' is either a single 2-cell or it contains at least two 2-cells which are extreme with respect to Z . Since a single 2-cell attached to the rest of D at a point is always extreme with respect to Z , we may assume D' has at least two extreme 2-cells. Finally, when such a D' is attached to the rest of the diagram at a point, at most one of these 2-cells loses its status as an extreme 2-cell, and the proof is complete. \square

When the hypotheses of Theorem 4.14 hold, we will say that *disc diagrams over X have extreme 2-cells*.

5. APPLICATIONS TO COHERENCE

In this final section we combine the constructions ∂A and $\bar{\partial} A$ with Theorem 4.14 to show that various groups are coherent. Throughout this section let X be a 2-complex, let A be a portion of its 1-skeleton, let Γ be either ∂A or $\bar{\partial} A$, and let Y and Z be as defined in Definitions 3.3 or 3.4, respectively. In order to apply Theorem 4.14, we need to know that Z is a splitting windmill. As we noted in the definitions of ∂A and $\bar{\partial} A$, there are easy conditions on A which ensure that Z is a windmill, so the main issue becomes whether $\text{ESSENCE}(\Gamma) \rightarrow X$ is π_1 -injective. Moreover, since the inclusion map $\text{ESSENCE}(\Gamma) \rightarrow X$ can be homotoped to a map $\text{ESSENCE}(\Gamma) \rightarrow A \subseteq X$ by pushing the regular neighborhood of A back into A in

the obvious way, it is sufficient to establish that this new map is π_1 -injective and that the inclusion $A \rightarrow X$ is π_1 -injective. Here are three common situations where $A \rightarrow X$ is known to be π_1 -injective.

Theorem 5.1 (Freiheitsatz). *Let X be the standard 2-complex of a presentation whose single relator is reduced and cyclically reduced. If A is a non-empty subgraph of X^1 that omits at least one 1-cell contained in the boundary cycle of the relator, then the inclusion $A \hookrightarrow X$ is π_1 -injective.*

The Freiheitsatz for one-relator groups, which was first proven by Magnus, can be generalized in various ways. One of these generalizations involves the notion of staggered 2-complex (see [2] or [1]).

Definition 5.2 (Staggered). Let X be a 2-complex with a subgraph $A \subset X^1$ such that each 2-cell of X contains a 1-cell not in A on its boundary. Suppose that there is a linear ordering on the 1-cells of X which are not in A , and a linear ordering on the 2-cells of X . For each 2-cell α , we let $\max(\alpha)$ and $\min(\alpha)$ denote the highest and lowest 1-cells not in A which occur in $\partial\alpha$. We then say that the pair X, A is *staggered* provided that if α and β are 2-cells with $\alpha < \beta$ then $\max(\alpha) < \max(\beta)$ and $\min(\alpha) < \min(\beta)$.

The following generalization of the Freiheitssatz is proven in [2] (see also [1]).

Theorem 5.3. *If X, A is staggered (for some linear orderings) then the inclusion map $A \hookrightarrow X$ is π_1 -injective on all components.*

Our third example is an immediate corollary of the fundamental theorem of small cancellation theory. See [2] or [4] for small-cancellation definitions and further details.

Theorem 5.4. *Let X be a $C(6) [C(4) - T(4)]$ small-cancellation complex, and let A be a subgraph of X^1 . If there does not exist a path S in A and a path Q in X such that Q is the concatenation of at most 3 pieces [2 pieces] in X and QS is the attaching map of a 2-cell of X , then $A \hookrightarrow X$ is π_1 -injective.*

Thus in each of these three contexts we merely need to check that $\text{ESSENCE}(\Gamma) \rightarrow A$ is π_1 -injective in order for Theorem 4.14 to apply.

5.1. Combinatorial descriptions. To understand the situation, we now provide a combinatorial description of $\partial A \rightarrow A$ and $\tilde{\partial} A \rightarrow A$.

Definition 5.5. Let X be the standard 2-complex of the presentation

$$\langle a_1, \dots, a_p, b_1, \dots, b_q \mid W_1, \dots, W_r \rangle$$

Let A and B be the subgraphs of X^1 corresponding to the a_i and b_i edges. For each i , the word W_i can be written uniquely in the form

$$W_{i0} b_{i1}^{\epsilon_{i1}} W_{i1} b_{i2}^{\epsilon_{i2}} W_{i2} \dots b_{is_i}^{\epsilon_{is_i}} W_{is_i}$$

where each ϵ_{ij} is ± 1 , each b_{ij} is a generator in B , and each word W_{ij} is a (possibly empty) word in the generators of A . By replacing W_i with one of its cyclic conjugates we can assume that W_{i0} is empty. We now form a graph ∂A from the set of W_i words as follows: For each i we form a $2s_i$ -sided polygon whose edges are directed and labeled by the elements $b_{ij}^{\epsilon_{ij}}$ and W_{ij} in exactly the same order as in W_i . For each k we identify edges which are labeled by b_k according to their

orientations. Finally for each k , we remove the interior of the edge labeled b_k . The resulting graph ∂A has $2q$ vertices and $\sum_{i=1}^r s_i$ edges.

By assumption, each word W_{ij} is a word in the free group generated by A , and there is an induced label-preserving map from ∂A to A . Note that the edges which are labeled by the trivial element are mapped to vertices. The graph ∂A is *injective* if this map is π_1 -injective on each component. An important special case where ∂A is injective is when the words W_{ij} form a basis for a subgroup of the free group generated by A .

Definition 5.6 (Generator Graphs). Let W be an arbitrary word and let t be one of the generators it contains. If we single out all of the instances of t in W then we can write W uniquely in the form $W_0 t^{\epsilon_1} W_1 t^{\epsilon_2} W_2 \dots t^{\epsilon_r} W_r$ where each ϵ_i is an integer and each word W_i is a non-empty word which does not contain the letter t . If we replace W with one of its cyclic conjugates we can assume that W_0 is empty. We now form a graph $\partial t(W)$, called the *generator graph* of W for the generator t , as follows: We begin with the $|W|$ -sided polygon whose edges are directed and labeled by the generators so that the label of the entire boundary is the word W . Next we identify all of the t -edges according to their orientations, and finally we remove the interior of the unique edge labeled t in the quotient. The resulting graph will be $\partial t(W)$. Notice that it contains either one or two connected components.

More generally, let B and C be disjoint sets of letters and let W be a word of the form $W = B_1 C_1 B_2 C_2 \dots B_k C_k$ where B_i and C_i are non-empty reduced words using generators from B and C respectively. The generator graph $\partial B(W)$ is formed as follows: Take the $|W|$ -sided polygon as before, and identify all of the instances of the generator $b \in B$ according to their orientations, and repeat this for each generator in B that occurs in W . Finally, remove the interior of the edges labeled by elements of B . The resulting graph is $\partial B(W)$. This more general graph may contain quite a few components.

Since each C_i is a word in the free group generated by C , there is an induced label-preserving map from $\partial B(W)$ to the bouquet of circles labeled by the $c_i \in C$. The graph $\partial B(W)$ is *injective* if this map is π_1 -injective on each component. An important special case where $\partial B(W)$ is injective is when the words C_i form a basis for a subgroup of the free group generated by C .

5.2. Applications of ∂A . Here is an application to coherence of one-relator groups.

Theorem 5.7. *Consider a one-relator group of the form*

$$G = \langle a_1, a_2, \dots, b \mid (b^{\epsilon_1} W_1 b^{\epsilon_2} W_2 \dots b^{\epsilon_r} W_r)^n \rangle$$

where for each i , $\epsilon_i = \pm 1$, n is arbitrary, and W_i is a word in the a_i . Suppose that ∂A is injective. Let P denote a reduced word representing the trivial element, then P contains a subword Q such that QS is equal to a cyclic conjugate of $W^{\pm n}$ and $b^{\pm 1}$ occurs at most once in S . As a consequence, G is coherent.

Proof. By the Freiheitssatz (Theorem 5.1), the a_i elements form a basis for a free group. Let R denote the unique 2-cell of X . Let each side of R at b have weight 1, and let each side of R not at b have weight 0. Then X satisfies the \leq condition for the perimeter reduction hypothesis of [3, Thm 7.6], and is therefore coherent. \square

We can now state a generalization of Theorem 5.7 to staggered 2-complexes.

Theorem 5.8. *Let X be the standard 2-complex of the presentation*

$$\langle a_1, \dots, a_p, t_1, \dots, t_q \mid W_1, \dots, W_r \rangle$$

and let A denote the subgraph of X^1 corresponding to the a_i edges. If X, A is relatively staggered for some linear orderings and the inclusion $\partial A \rightarrow X$ is injective, then extreme 2-cells exist in disc diagrams over X .

Theorem 5.9. *Let X be a $C(6)[C(4) - T(4)]$ small-cancellation complex, and suppose that A is a subgraph of X^1 such that there does not exist a path $S \rightarrow A$ such that QS is the attaching map of a 2-cell of X , where Q is the concatenation of at most 3 pieces [2 pieces] in X . Then $A \rightarrow X$ is π_1 -injective. Consequently, if $\partial A \rightarrow A$ is injective, then extreme 2-cells exist in disc diagrams over X .*

In both cases, the restricted nature of the extreme 2-cells, combined with Theorem 7.6 of [3], leads to new tests for coherence.

5.3. ∂A applications. The following theorem is merely the conclusion of Theorem 4.14 translated into a more group theoretic language.

Theorem 5.10. *Let $W = A_1B_1A_2B_2 \dots A_kB_k$ be a word where the A_i are non-empty words using generators in A and the B_i are non-empty words using generators from B (disjoint from A), and let G be the one-relator group $G = \langle A \cup B \mid W^n \rangle$. If $\text{ESSENCE}(\partial A) \rightarrow A$ is π_1 -injective and P is a cyclically reduced word representing the trivial element in G , then there are words Q and S such that Q is a subword of P , QS is a cyclic conjugate of $W^{\pm n}$ and S is a subword of $B_{i-1}A_iB_i$ for some i where the subscripts are considered $\pmod k$.*

When k is at least 2 then this theorem gives a refinement of the B.B. Newman spelling theorem in the sense that it further restricts the size of the possible complements S . As with the spelling theorem, this leads immediately to a corresponding weight test. We refer the reader to [3] for the definition of $\text{Perimeter}(A_i)$ and $\text{Weight}(W^n)$.

Corollary 5.11. *Let $G = \langle A \cup B \mid W^n \rangle$ be a one relator group with torsion where A and B are disjoint sets of generators and W has the form $A_1B_1A_2B_2 \dots A_kB_k$ for some non-empty words A_i and B_i using generators from A and B respectively. If $\text{ESSENCE}(\partial A) \rightarrow A$ is π_1 -injective and $\text{Perimeter}(A_i) \leq \text{Weight}(W^n)$ for all i , then G is coherent.*

Proof. Let X be the standard 2-complex of the presentation. Assign a weight of 1 to each side labeled by an element of A and a weight of 0 to each side labeled by an element of B . By Theorem 5.10, the Perimeter Reduction Hypothesis of [3, Thm 7.6] is satisfied and so $G \cong \pi_1 X$ is coherent. \square

The most important Corollary, and the easiest to apply, is the following.

Corollary 5.12. *Let $G = \langle A, t \mid W^N \rangle$ where W has the form $t^{\epsilon_1}W_1t^{\epsilon_2}W_2 \dots t^{\epsilon_k}W_k$ and for each i , ϵ_i is an integer and W_i is a reduced word over A . If $\text{ESSENCE}(\partial A) \rightarrow A$ is injective, then G is coherent.*

Proof. Since $\text{Perimeter}(t) = \text{Weight}(W)$, Corollary 5.11 applies. \square

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