ORDER INDEPENDENCE IN ASYNCHRONOUS CELLULAR AUTOMATA

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ABSTRACT. A *sequential dynamical system*, or SDS, consists of an undirected graph Y, a vertex-indexed list of local functions $\mathfrak{F}_Y$, and a word ω over the vertex set, containing each vertex at least once, that describes the order in which these local functions are to be applied. In this article we investigate the special case where Y is a circular graph with n vertices and all of the local functions are identical. The 256 possible local functions are known as *Wolfram rules* and the resulting sequential dynamical systems are called *finite asynchronous elementary cellular automata*, or ACAs, since they resemble classical elementary cellular automata, but with the important distinction that the vertex functions are applied sequentially rather than in parallel. An ACA is said to be ω*-independent* if the set of periodic states does not depend on the choice of ω, and our main result is that for all $n > 3$ exactly 104 of the 256 Wolfram rules give rise to an ω-independent ACA. In 2005 Hansson, Mortveit and Reidys classified the 11 symmetric Wolfram rules with this property. In addition to reproving and extending this earlier result, our proofs of ω-independence also provide significant insight into the dynamics of these systems.

Our main result, as recorded in Theorem 2.2, is a complete classification of the Wolfram rules that for all $n > 3$ lead to an ω-independent finite asynchronous elementary cellular automaton, or ACA. The structure of the article is relatively straightforward. The first two sections briefly describe how an ACA can be viewed as either a special type of sequential dynamical system or as a modified version of a classical elementary cellular automaton. These two sections also contain the background definitions and notations needed to carefully state our main result. Next, we introduce several new notations for Wolfram rules in order to make certain patterns easier to discern, and we significantly reduce the number of cases we need to consider by invoking the notion of dynamical equivalence. Sections 5 and 6 contain the heart of the proof. The former covers four large classes of rules whose members are ω-independent for similar reasons, and the latter finishes off three pairs of unusual cases that exhibit more intricate behavior requiring more delicate proofs. The final section contains remarks about directions for future research.

1. SEQUENTIAL DYNAMICAL SYSTEMS

Cellular automata, or CAs, are discrete dynamical systems that have been thoroughly studied by both professional and amateur mathematicians.[1] They are defined over regular grids of cells

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[1]Stanislaw Ulam and John von Neumann were the first to study such systems, which they did while working at Los Alamos National Laboratory in the 1940s [9]. The German computer scientist Konrad Zuse proposed in 1969 that the universe is essentially one big cellular automaton [13]. In the 1970s, John Conway invented the Game of

and each cell can take on one of a finite number of states. In addition, each cell has an *update rule* that takes its own state and the states of its neighbors as input, and at each discrete time step, the rules are applied and all vertex states are simultaneously updated.

In the late 1990s, a group of scientists at Los Alamos invented a new type of discrete dynamical system that they called *sequential dynamical systems*, or SDSs. In an SDS the regular grids used to define cellular automata are replaced by arbitrary undirected graphs, and the local functions are applied sequentially rather than in parallel. Their initial motivation was to develop a mathematical foundation for the analysis, simulation and implementation of various socio-technological systems [1, 2, 3, 4].

An SDS has three components: an undirected graph Y, a list of local functions $\mathfrak{F}_Y$, and an update order ω. Start with a simple undirected graph Y with n vertices, label the vertices from 1 to n, and recall that the *neighbors* of a vertex are those vertices connected to it by an edge. If $\mathbb{F}$ is a finite field and every vertex is assigned a value from $\mathbb{F}$, then a global state of the system is described by an n-tuple $\mathbf{y}$ whose i^{th} coordinate indicates the current state of the vertex i. The set of all possible states is the vector space $\mathbb{F}^n$.

Definition 1.1 (Local functions)**.** A function $F\colon \mathbb{F}^n \to \mathbb{F}^n$ is called Y*-local at* i if (1) for each $\mathbf{y} \in \mathbb{F}^n$, $F(\mathbf{y})$ only alters the i^{th} coordinate of $\mathbf{y}$ and (2) the new value of the i^{th} coordinate only depends on the coordinates of $\mathbf{y}$ corresponding to i and its neighbors in Y. Other names for such a function include *local function* and *update rule*. We use $\mathfrak{F}_Y$ to denote a list of local functions that includes one for each vertex of Y. More precisely, $\mathfrak{F}_Y = (F_1, F_2, \ldots, F_n)$ where F_i is a function that is Y-local at i.

It is sometimes convenient to convert a local function F into another function with a severely restricted domain and range.

Definition 1.2 (Restricted local functions)**.** If i is a vertex with k neighbors in Y, then corresponding to each function F that is Y-local at i, we define a function $f\colon \mathbb{F}^{k+1} \to \mathbb{F}$ where the domain is restricted to the coordinates corresponding to i and its neighbors, and the output is the new value F would assign to the i^{th} coordinate under these conditions. It should be clear that F and f contain the same information but packaged in different ways. Each determines the other and both have their uses. Functions such as F can be readily composed, but functions such as f are easier to describe explicitly since irrelevant and redundant information has been eliminated.

The local functions that are easiest to describe are those with extra symmetries.

Definition 1.3 (Symmetric and quasi-symmetric rules)**.** Let i be a vertex in Y with k neighbors, let $F\colon \mathbb{F}^n \to \mathbb{F}^n$ be a Y-local function at i and let $f\colon \mathbb{F}^{k+1} \to \mathbb{F}$ be its restricted form. If the output of f only depends on the multiset of inputs and not their order, in other words, if the states of i and its neighbors can be arbitrarily permuted without changing the output of f, then f (and F) are called *symmetric* local functions. If they satisfy the weaker condition that at least the states of the neighbors of i can be arbitrarily permuted without changing the output, then F

Life, a two-dimensional CA, that was later popularized by Martin Gardner [5]. Beginning in 1983, Stephen Wolfram published a series of papers devoted to developing a theory of CAs and their role in science [7, 10, 11, 12]. This is also a central theme in Wolfram's 1280-page book *A New Kind of Science*, published in 2002.

and f are *quasi-symmetric*. A list of local functions $\mathfrak{F}_Y$ is symmetric or quasi-symmetric when every function in the list has this property.

The last component of an SDS is an update order.

Definition 1.4 (Update orders)**.** An *update order* ω is a finite sequence of numbers chosen from the set $\{1, \ldots, n\}$ such that every number $1 \leq i \leq n$ occurs at least once. If every number $1 \leq i \leq n$ occurs exactly once, then the update order is *simple*. Let W_Y denote the collection of all update orders and let S_Y denote the subset of simple update orders. The subscript Y indicates that we are thinking of the numbers in these sequences as vertices in the graph Y. When considering an arbitrary update order, we tend to use the notation $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ with $m = |\omega|$ (and $m \geq n$, of course), but when we restrict our attention to simple update orders, we switch to the notation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$.

Definition 1.5 (Sequential dynamical systems)**.** A *sequential dynamical system*, or SDS, is a triple $(Y, \mathfrak{F}_Y, \omega)$ consisting of an undirected graph Y, a list local functions $\mathfrak{F}_Y$, and an update order $\omega \in W_Y$. If ω is the sequence $(\omega_1, \omega_2, \ldots, \omega_m)$, then we construct the SDS *map* $[\mathfrak{F}_Y, \omega] \colon \mathbb{F}^n \to \mathbb{F}^n$ as the composition $[\mathfrak{F}_Y, \omega] := F_{\omega_m} \circ \cdots \circ F_{\omega_1}$.

With the usual abuse of notation, we sometimes let the SDS map $[\mathfrak{F}_Y, \omega]$ stand in for the entire SDS. The goal is to study the behavior of the map $[\mathfrak{F}_Y, \omega]$ under iteration. In this article we focus on the set of states in $\mathbb{F}^n$ that are periodic and we use $\mathsf{Per}[\mathfrak{F}_Y, \omega] \subset \mathbb{F}^n$ to denote this collection of periodic states. The set of periodic states is of interest both because it is the codomain of high iterates of the SDS map and the largest subset of states that are permuted by the map.

Definition 1.6 (ω-independence)**.** A list of Y-local functions $\mathfrak{F}_Y$ is called ω-*independent* if $\mathsf{Per}[\mathfrak{F}_Y, \omega] = \mathsf{Per}[\mathfrak{F}_Y, \omega']$ for all update orders $\omega, \omega' \in W_Y$ and π-*independent* if $\mathsf{Per}[\mathfrak{F}_Y, \pi] = \mathsf{Per}[\mathfrak{F}_Y, \pi']$ for all simple update orders $\pi, \pi' \in S_Y$.

Every ω-independent $\mathfrak{F}_Y$ is trivially π-independent. More surprisingly, Reidys has shown that these two conditions are, in fact, equivalent.

Theorem 1.7 ([8])**.** A list $\mathfrak{F}_Y$ of Y-local functions is ω-independent iff it is π-independent.

Even though ω-independence is too strong to expect generically, there are nonetheless many interesting classes of SDSs that have this property, including two classes where ω-independence is relatively easy to establish.

Proposition 1.8. If for every simple update order $\pi \in S_Y$, every state in $\mathsf{Per}[\mathfrak{F}_Y, \pi]$ is fixed by the SDS map $[\mathfrak{F}_Y, \pi]$, then $\mathfrak{F}_Y$ is π-independent and thus ω-independent.

Proof. If $\mathbf{y}$ is fixed by $[\mathfrak{F}_Y, \pi]$, then $\mathbf{y}$ is fixed by each F_i in $\mathfrak{F}_Y$ (the simplicity of π means that were F_i to change the i^{th} coordinate, there would not be an opportunity for it to change back). Being fixed by each F_i, $\mathbf{y}$ is also fixed by $[\mathfrak{F}_Y, \omega]$ for all $\omega \in W_Y$, which includes all of S_Y. Since this argument is reversible, the SDS maps with simple update orders share a common set of fixed states. If, as hypothesized, these are the only periodic states for these maps, then $\mathfrak{F}_Y$ is π-independent, and by Theorem 1.7, ω-independent. $\square$

In our second example, ω-independence is essentially immediate.

Proposition 1.9 (Bijective functions)**.** If every local function F_i in $\mathfrak{F}_Y$ is a bijection, then for every update order $\omega \in W_Y$, $\mathsf{Per}[\mathfrak{F}_Y, \omega] = \mathbb{F}^n$. As a consequence $\mathfrak{F}_Y$ is ω-independent.

Proof. Since every F_i is a bijection, so is the SDS map $[\mathfrak{F}_Y, \omega]$ and a sufficiently high iterate is the identity permutation. $\square$

This last result highlights the fact that ω-independence focuses on sets rather than cycles, since ω-independent SDSs with different update orders quite often organize their common periodic states into different cycle configurations. In fact, the restrictions of ω-independent SDS maps with different update orders to their common periodic states can be used to construct a group encoding the possible dynamics over this set [6].

Collections of ω-independent SDSs also form a natural starting point for the study of stochastic sequential dynamical systems. Stochastic finite dynamical systems are often studied through Markov chains over their state space but in general this leads to Markov chains with exponentially many states as measured by the number of cells or vertices. For ω-independent SDSs one is typically able to reduce the number of states in such a Markov chain significantly, at least when focusing on their periodic behavior.

2. Asynchronous Cellular Automata

Some of the simplest (classical) cellular automata are the one-dimensional CAs known as *elementary cellular automata*. In an elementary CA, every vertex has precisely two neighbors, the only possible vertex states are 0 or 1, and all local functions are identically defined. Since every vertex has two neighbors, the underlying graph is either a line or a circle and the restricted form of its common local function is a map $f\colon \mathbb{F}^3 \to \mathbb{F}$ where $\mathbb{F} = \mathbb{F}_2 = \{0, 1\}$ is the field with two elements. There are $2^8 = 256$ such functions, known as *Wolfram rules*, and thus 256 types of elementary cellular automata. Even in such a restrictive situation there are many interesting dynamical effects to be observed. The focus here is on the sequential dynamical systems that correspond to these classical elementary cellular automata.

Let $Y = \mathsf{Circ}_n$ denote a circular graph with n vertices labeled consecutively from 1 to n, and to avoid trivialities assume $n > 3$. (The sequential nature of the update rules in an SDS makes infinite graphs such as lines unsuitable in this context.) Since these are the only graphs considered in the remainder of the article, we replace notations such as W_Y or S_Y with W_n and S_n, etc. In Circ_n we view the vertex labels as residue classes mod n so that there is an edge connecting i to $i+1$ for every i.

Definition 2.1 (Wolfram rules)**.** Let $F_i\colon \mathbb{F}^n \to \mathbb{F}^n$ be a Circ_n-local function at i and let $f_i\colon \mathbb{F}^3 \to \mathbb{F}$ be its restricted form. Since the neighbors of i are $i-1$ and $i+1$, it is standard to list these coordinates in ascending order in $\mathbb{F}^3$. Thus, a state $\mathbf{y} \in \mathbb{F}^n$ corresponds to a triple (y_{i-1}, y_i, y_{i+1}) in the domain of f_i. Call this a *local state configuration* and keep in mind that all subscripts are viewed mod n. In order to completely specify the function F_i it is sufficient to list how the i^{th} coordinate is updated for each of the 8 possible local state configurations. More specifically, let (y_{i-1}, y_i, y_{i+1}) denote a local state configuration and let (y_{i-1}, z_i, y_{i+1}) be the local state configuration after applying F_i. The local function F_i, henceforth referred to as a *Wolfram rule*, is completely described by the following table.

(2.1)

$y_{i-1}y_iy_{i+1}$	111	110	101	100	011	010	001	000
z_i	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0

More concisely, the $2^8 = 256$ possible Wolfram rules can be indexed by an 8-digit binary number $a_7a_6a_5a_4a_3a_2a_1a_0$, or by its decimal equivalent $k = \sum_{i=0}^{7} a_i 2^i$. There is thus one *Wolfram rule* k for each integer $0 \leq k \leq 255$. For each such n, k and i let $\mathsf{Wolf}_i^{(k)}$ denote the Circ_n-local function $F_i \colon \mathbb{F}^n \to \mathbb{F}^n$ just defined, let $\mathsf{wolf}_i^{(k)}$ denote its restricted form $f_i \colon \mathbb{F}^3 \to \mathbb{F}$, and let $\mathfrak{Wolf}_n^{(k)}$ denote the list of local functions $(\mathsf{Wolf}_1^{(k)}, \mathsf{Wolf}_2^{(k)}, \ldots, \mathsf{Wolf}_n^{(k)})$. We say that Wolfram rule k is ω-independent whenever $\mathfrak{Wolf}_n^{(k)}$ is ω-independent for all $n > 3$.

For each update order ω there is an SDS $(\mathsf{Circ}_n, \mathfrak{Wolf}_n^{(k)}, \omega)$ that can be thought of as an elementary CA, but with the update functions applied asynchronously (and possibly more than once). For this reason, such systems are called *asynchronous cellular automata* or ACAs. We now state our main result.

Theorem 2.2. There are exactly 104 Wolfram rules that are ω-independent. More precisely, $\mathfrak{Wolf}_n^{(k)}$ is ω-independent for all $n > 3$ iff $k \in \{$0, 1, 4, 5, 8, 9, 12, 13, 28, 29, 32, 40, 51, 54, 57, 60, 64, 65, 68, 69, 70, 71, 72, 73, 76, 77, 78, 79, 92, 93, 94, 95, 96, 99, 102, 105, 108, 109, 110, 111, 124, 125, 126, 127, 128, 129, 132, 133, 136, 137, 140, 141, 147, 150, 152, 153, 156, 157, 160, 164, 168, 172, 184, 188, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 204, 205, 206, 207, 216, 218, 220, 221, 222, 223, 224, 226, 228, 230, 232, 234, 235, 236, 237, 238, 239, 248, 249, 250, 251, 252, 253, 254, 255$\}$.

The main result of [6] states that precisely 11 of the 16 *symmetric* Wolfram rules are ω-independent over Circ_n for all $n > 3$. Theorem 2.2 significantly extends this result, reproving it in the process. In addition to identifying a large class of ω-independent ACAs, the proof also provides further insight into the dynamics of these systems at both periodic and transient states and thus serves as a foundation for the future study of their stochastic properties. We conclude this section with two remarks about the role played by computer investigations of these systems.

Remark 2.3 (Unlisted numbers)**.** The “only if” portion of this theorem was established experimentally. For each $4 \leq n \leq 9$, for each $0 \leq k \leq 255$, and for each simple update order $\pi \in S_n$, a computer program written by the first and third authors calculated the set $\mathsf{Per}[\mathfrak{Wolf}_n^{(k)}, \pi]$. For each of the 152 values of k not listed above, there were distinct simple update orders that led to distinct sets of periodic states, leaving the remaining 104 rules as the only ones with the potential to be ω-independent for all $n > 3$. Moreover, since a counterexample for one value of n leads to similar counterexamples for all multiples of n, these 104 rules are also the only ones that are eventually ω-independent for all sufficiently large values of n. Because these brute-force calculations are explicit yet tedious they have been omitted, but the interested reader should feel free to contact the third author for a copy of the software that performed the calculations.

Remark 2.4 (Computational guidance)**.** These early computer-aided investigations also had a major impact on the “if” portion of the proof. Once the computer results highlighted the 104 rules that were ω-independent for small values of n, we identified patterns and clusters among

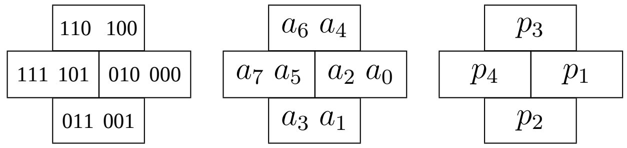

FIGURE 1. Grid notation for Wolfram rules

the 104 rules, which led to conjectured lemmas, and eventually to proofs that our conjectures were correct. The computer calculations thus provided crucial data that both prompted ideas and tempered our search for intermediate results.

3. WOLFRAM RULE NOTATIONS

Patterns among the 104 numbers listed in Theorem 2.2 are difficult to discern because the conversion from binary to decimal obscures many structural details. In this section we introduce other ways to describe the Wolfram rules that makes their similarities and differences more immediately apparent.

Definition 3.1 (Grid notation)**.** For each binary number $k = a_7a_6a_5a_4a_3a_2a_1a_0$ we arrange its digits in a grid. The 8 local state configurations can be viewed as the vertices of a 3-cube and we arrange them according to the conventional projection of a 3-cube into the plane. See the left-hand side of Figure 1. Next, we can place the binary digits of k at these positions as shown in the center of Figure 1. The boxes have been added because the local state configurations come in pairs. When a local function is applied, the states of the neighbors of i are left unchanged, so that the resulting local state configuration is located in the same box. We call this the *grid notation* for k. The grid notation for Wolfram rule 29 = 00011101 is shown on the left-hand side of Figure 2.

Because grid notation is sometimes cumbersome to work with we also define a very concise 4 symbol tag for each Wolfram rule that respects the box structure of the grid.

Definition 3.2 (Tags)**.** When we look at the grid notation for a Wolfram rule, in each box we see a pair of numbers, 11, 00, 10, or 01, and we encode these configurations by the symbols `1`, `0`, `-`, and `x`, respectively. In other words '`1`' = $\boxed{1\quad 1}$, '`0`' = $\boxed{0\quad 0}$, '`-`' = $\boxed{1\quad 0}$, and '`x`' = $\boxed{0\quad 1}$. The symbols are meant to indicate that when the states of the neighbors place us in this box, the local function updates the i^{th} coordinate by converting it to a 1, converting it to a 0, leaving it unchanged, or always changing it. We label the symbols for the four boxes p_1, p_2, p_3 and p_4 as shown on the right-hand side of Figure 1 and we define the *tag* of k to be the string $p_4p_3p_2p_1$. The numbering and the order of the p_i's has been chosen to match the binary representation as closely as possible, with the hope of easing conversions between binary and tag representations. The process of converting Wolfram rule 29 to its tag `0x-1` is illustrated in Figure 2.

Definition 3.3 (Symmetric and asymmetric)**.** The middle row of the grid contains the positions where the states of the neighbors are equal and the top and bottom rows contain the positions where the states of the neighbors are different. We call the middle row the *symmetric* portion of

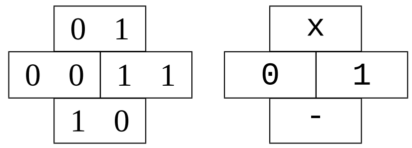

FIGURE 2. Converting Wolfram rule 29 = 00011101 into its tag `0x-1`.

the grid and the top and bottom rows the *asymmetric* portion. In the tag representation, the beginning and end of a tag describes how the rule responds to a symmetric neighborhood configuration and the middle of a tag describes how it responds to an asymmetric neighborhood configuration. With this in mind we call p_4p_1 the *symmetric part* of the tag $k = p_4p_3p_2p_1$ and we call p_3p_2 its *asymmetric part*.

Table 1 shows the 104 ω-independent Wolfram rules listed in Theorem 2.2 arranged according to the symmetric and asymmetric parts of their tags. The rows list all 16 possibilities for the symmetric part of the tag while the columns list only 10 of the 16 possibilities for the asymmetric part since only these 10 occur among the 104 rules. In addition, each row and column label has a decimal equivalent, listed next to the row and column headings, that add up to k. In this format the benefits of the tag representation should be clear. Far from being distributed haphazardly, the ω-independent rules appear clustered together in large blocks. Table 1 reveals a lot of structure, but some patterns remain slightly hidden due to the order in which the rows and columns are listed. For example, there is a 4-by-4 block of bijective rules obtained by restricting attention to the four rows that show up in the last column and the four columns that show up in the last row.

Proposition 3.4 (Bijective rules)**.** Wolfram rules 51, 54, 57, 60, 99, 102, 105, 108, 147, 150, 153, 156, 195, 198, 201 and 204 are ω-independent.

Proof. The 16 rules listed have tags where each p_i is either `-` or `x`. These (and only these) Wolfram rules correspond to bijective local functions and by Proposition 1.9 the ACAs these rules define are ω-independent. □

4. DYNAMICAL EQUIVALENCE

In this section we use the notion of dynamical equivalence to reduce the proof of Theorem 2.2 to a more manageable size. Two sequential dynamical systems $(Y, \mathfrak{F}_Y, \omega)$ and $(Y, \mathfrak{F}'_Y, \omega')$ defined over the same graph Y are said to be *dynamically equivalent* if there is a bijection $H\colon \mathbb{F}^n \to \mathbb{F}^n$ between their states such that $H \circ [\mathfrak{F}_Y, \omega] = [\mathfrak{F}'_Y, \omega'] \circ H$. The key fact about dynamically equivalent SDSs, which is also easy to show, is that H establishes a bijection between their periodic states. In particular, $H(\mathsf{Per}[\mathfrak{F}_Y, \omega]) = \mathsf{Per}[\mathfrak{F}'_Y, \omega']$. Thus, if $\mathfrak{F}'_Y$ is an ω-independent SDS and for each $\omega \in W_Y$ there exists an $\omega' \in W_Y$ such that $(Y, \mathfrak{F}_Y, \omega)$ and $(Y, \mathfrak{F}'_Y, \omega')$ are dynamically equivalent using *the same function* H, then $\mathfrak{F}_Y$ is also ω-independent.

Although there are 256 Wolfram rules, many give rise to dynamically equivalent ACAs. In particular, there are three relatively elementary ways to alter an ACA to produce another one that appears different on the surface, but which is easily seen to be dynamically equivalent to the

	p_3	-	-	0	0	-	1	1	-	x	x
	p_2	-	0	-	0	1	-	1	x	-	x
p_4p_1		72	64	8	0	74	88	90	66	24	18
--	132	204	196	140	132	206	220	222	198	156	150
0-	4	76	68	12	4	78	92	94	70	28	
-0	128	200	192	136	128	202	216	218	194	152	
1-	164	236	228	172	164	238	252	254	230	188	
-1	133	205	197	141	133	207	221	223	199	157	
10	160	232	224	168	160	234	248	250	226	184	
01	5	77	69	13	5	79	93	95	71	29	
00	0	72	64	8	0						
x0	32		96	40	32						
0x	1	73	65	9	1						
-x	129	201	193	137	129				195	153	147
x-	36	108				110	124	126	102	60	54
x1	37	109				111	125	127			
1x	161					235	249	251			
11	165	237				239	253	255			
xx	33	105							99	57	51

TABLE 1. The 104 ω-independent Wolfram rules arranged by the symmetric and asymmetric parts of their tags.

original. These are obtained by (1) renumbering the vertices in the opposite direction, (2) systematically switching all 1s to 0s and 0s to 1s, or (3) doing both at once. We call these alterations *reflection*, *inversion* and *reflection-inversion* of the ACA, respectively. The term reflection highlights the fact that this alteration makes it appear as though we picked up the circular graph and flipped it over. We begin by describing the effect renumbering has on individual local functions.

Definition 4.1 (Renumbering)**.** The renumbering of the vertices we have in mind is achieved by the map $r\colon \mathsf{Circ}_n \to \mathsf{Circ}_n$ that sends vertex i to vertex $n+1-i$. For later use we extend this to a map $r\colon W_n \to W_n$ on update orders by applying r to each entry in the sequence. More specifically, if $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$, then $r(\omega) = (r(\omega_1), r(\omega_2), \ldots, r(\omega_m))$. Finally, on the level of states we define a map $R\colon \mathbb{F}^n \to \mathbb{F}^n$ that sends sends $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ to $(y_n, \ldots, y_2, y_1)$, and we note that R is an involution.

Definition 4.2 (Reflected rules)**.** If the vertices of Circ_n are renumbered, rule $\mathsf{Wolf}_i^{(k)}$ is applied, and then the renumbering is reversed, the net effect is the same as if a different Wolfram rule were applied to the vertex $r(i)$. Let ℓ be the number that represents this other Wolfram rule. The differences between k and ℓ are best seen in grid notation. The renumbering not only changes the vertex at which the rule seems to be applied, but it also reverses the order in which the 3 coordinates are listed in the restricted local form. Only the asymmetric local state configurations, i.e. the top and bottom rows of the grid, are altered by this change so that the grid for ℓ looks like a reflection of the grid for k across a horizontal line. We call ℓ the *reflection* of k and we define a map $\mathsf{refl}\colon \{0,\ldots,255\} \to \{0,\ldots,255\}$ with $\mathsf{refl}(k) = \ell$. On the level of tags, the only change is to switch order of p_2 and p_3, so, for example ℓ=`01-x` is the reflection of k=`0-1x`.

In short, when $\ell = \mathsf{refl}(k)$, $R \circ \mathsf{Wolf}_i^{(k)} \circ R = \mathsf{Wolf}_{r(i)}^{(\ell)}$ and, since R is an involution, this can be rewritten as $R \circ \mathsf{Wolf}_i^{(k)} = \mathsf{Wolf}_{r(i)}^{(\ell)} \circ R$.

Proposition 4.3. If $\ell = \mathsf{refl}(k)$, then $\mathfrak{Wolf}_n^{(k)}$ is ω-independent iff $\mathfrak{Wolf}_n^{(\ell)}$ is ω-independent.

Proof. The value of ℓ was defined so that $R\circ\mathsf{Wolf}_i^{(k)} = \mathsf{Wolf}_{r(i)}^{(\ell)}\circ R$. As a result, for any $\omega \in W_n$, the ACA $(\mathsf{Circ}_n, \mathfrak{Wolf}_n^{(k)}, \omega)$ is dynamically equivalent to the ACA $(\mathsf{Circ}_n, \mathfrak{Wolf}_n^{(\ell)}, r(\omega))$ since

$$\begin{aligned} R \circ [\mathfrak{Wolf}_n^{(k)}, \omega] &= R \circ \mathsf{Wolf}_{\omega_m}^{(k)} \circ \cdots \circ \mathsf{Wolf}_{\omega_2}^{(k)} \circ \mathsf{Wolf}_{\omega_1}^{(k)} \\ &= \mathsf{Wolf}_{r(\omega_m)}^{(\ell)} \circ \cdots \circ \mathsf{Wolf}_{r(\omega_2)}^{(\ell)} \circ \mathsf{Wolf}_{r(\omega_1)}^{(\ell)} \circ R \\ &= [\mathfrak{Wolf}_n^{(\ell)}, r(\omega)] \circ R. \end{aligned}$$

The argument at the beginning of the section now shows that the ω-independence of $\mathfrak{Wolf}_n^{(\ell)}$ implies that of $\mathfrak{Wolf}_n^{(k)}$, but since $\ell = \mathsf{refl}(k)$ implies $k = \mathsf{refl}(\ell)$, the converse also holds. $\square$

Similar results hold for inversions as we now show.

Definition 4.4 (Inverting). Let $\mathbf{1}$ and $\mathbf{0}$ denote the special states $(1,1,\ldots,1)$ and $(0,0,\ldots,0)$ in $\mathbb{F}^n$. Since the function $i(a) = 1 - a$ changes 1 to 0 and 0 to 1, the map $I\colon \mathbb{F}^n \to \mathbb{F}^n$ sending $\mathbf{y}$ to $\mathbf{1} - \mathbf{y}$, has this effect on each coordinate of $\mathbf{y}$. The map I is an involution like R, and from their definitions it is easy to check that they commute with each other.

Definition 4.5 (Inverted rules). If the states of Circ_n are inverted, rule $\mathsf{Wolf}_i^{(k)}$ is applied, and then the inversion is reversed, the net effect is the same as if a different Wolfram rule were applied at vertex i. Let ℓ be the number that represents this other Wolfram rule. The differences between k and ℓ are again best seen in grid notation. The pre-inversion of states effects the local state configurations as though the grid had been rotated $180°$. The second inversion merely changes every entry so that 1s becomes 0s and 0s become 1s. Thus the grid for ℓ can be obtained from the grid for k by rotating the grid and altering every entry. We call ℓ the *inversion* of k and define a map $\mathsf{inv}\colon \{0,\ldots,255\} \to \{0,\ldots,255\}$ with $\mathsf{inv}(k) = \ell$. On the level of tags, there are two changes that take place. Boxes p_1 and p_4 switch places as do boxes p_2 and p_3, but in process the boxes are turned over and the numbers changed. If we look at what this does to the entries in a box, 11 becomes 00, 00 becomes 11, while 10 and 01 are left unchanged. To formalize this, define a conjugation map $c\colon \{\mathtt{1}, \mathtt{0}, -, \mathtt{x}\} \to \{\mathtt{1}, \mathtt{0}, -, \mathtt{x}\}$ with $c(\mathtt{1}) = \mathtt{0}$, $c(\mathtt{0}) = \mathtt{1}$, $c(-) = -$, and $c(\mathtt{x}) = \mathtt{x}$. When k has tag $p_4p_3p_2p_1$, ℓ has tag $c(p_1)c(p_2)c(p_3)c(p_4)$, so, for example, ℓ = `x0-1` is the inversion of k = `0-1x`.

In short, when $\ell = \mathsf{inv}(k)$, $I \circ \mathsf{Wolf}_i^{(k)} \circ I = \mathsf{Wolf}_i^{(\ell)}$ and, since I is an involution, this can be rewritten as $I \circ \mathsf{Wolf}_i^{(k)} = \mathsf{Wolf}_i^{(\ell)} \circ I$.

Proposition 4.6. If $\ell = \mathsf{inv}(k)$, then $\mathfrak{Wolf}_n^{(k)}$ is ω-independent iff $\mathfrak{Wolf}_n^{(\ell)}$ is ω-independent.

	p_3	`0`	`0`
	p_2	`0`	`-`
p_4p_1		0	8
`--`	132	132	140
`0-`	4	4	12
`-0`	128	128	136
`00`	0	0	8
`-1`	133	133	141
`01`	5	5	13
`-x`	129	129	137
`0x`	1	1	9
`1-`	164	164	172
`10`	160	160	168
`x0`	32	32	40

	p_3	`-`	`x`	`x`
	p_2	`-`	`-`	`x`
p_4p_1		72	24	18
`--`	132	204	156	150
`x-`	36	108	60	54
`xx`	33	105	57	51
`-0`	128	200	152	
`10`	160	232	184	
`0-`	4	76	28	
`01`	5	77	29	
`00`	0	72		
`0x`	1	73		

FIGURE 3. The 41 ω-independent Wolfram rules up to equivalence, separated into two tables by their behavior in asymmetric contexts.

Proof. The value of ℓ was defined so that $I \circ \mathsf{Wolf}_i^{(k)} = \mathsf{Wolf}_i^{(\ell)} \circ I$. As in the proof of Proposition 4.3 this implies that for any $\omega \in W_n$, the ACA $(\mathsf{Circ}_n, \mathfrak{Wolf}_n^{(k)}, \omega)$ is dynamically equivalent to the ACA $(\mathsf{Circ}_n, \mathfrak{Wolf}_n^{(\ell)}, \omega)$. The argument at the beginning of the section and the fact that $\ell = \mathsf{inv}(k)$ implies $k = \mathsf{inv}(\ell)$, complete the proof as before. $\square$

As an immediate corollary of Propositions 4.3 and 4.6, when $\ell = \mathsf{refl}(\mathsf{inv}(k)) = \mathsf{inv}(\mathsf{refl}(k))$, $\mathfrak{Wolf}_n^{(k)}$ is ω-independent iff $\mathfrak{Wolf}_n^{(\ell)}$ is ω-independent. If we partition the 256 Wolfram rules into equivalence classes of rules related by reflection, inversion or both, then there are 88 distinct equivalence classes and the 104 rules listed in Theorem 2.2 are the union of 41 of them.

Figure 3 displays representatives of these 41 classes in pared down versions of Table 1. We used reflection and inversion to eliminate 5 of the 10 columns. Every rule with a `1` in the asymmetric portion of its tag is the inversion of a rule with a `0` instead. In particular, the entries in the 3 columns headed `-1`, `1-` and `11` are inversions of the entries in the columns headed `0-`, `-0` and `00`, respectively. Next, since reflections switch p_2 and p_3 we can also eliminate the columns headed `-0`, `-x` as redundant. This leaves the 5 columns headed `00`, `0-`, `--`, `x-` and `xx`. Since the last 3 do not contain `0`s or `1`s, further inversions, or inversion-reflections can be used to identify redundant rows in these columns.

As mentioned above, the 41 rules listed in Figure 3 are representatives of the 41 distinct equivalence classes of rules whose ω-independence needs to be established in order to prove Theorem 2.2. The rows in each table have been arranged to correspond as closely as possible with the structure of the proof. For example, the first three rows of the table on the right-hand side are the 9 equivalence classes shown to be ω-independent by Proposition 1.9.

$$\begin{array}{c} \boxed{*\;\;0} \\ \boxed{*\;\;0}\boxed{*\;\;0} \\ \boxed{*\;\;0} \end{array} \qquad \begin{array}{c} \boxed{*\;\;0} \\ \boxed{1\;\;1}\boxed{*\;\;0} \\ \boxed{*\;\;0} \end{array} \qquad \begin{array}{c} \boxed{*\;\;0} \\ \boxed{*\;\;0}\boxed{1\;\;1} \\ \boxed{*\;\;0} \end{array} \qquad \begin{array}{c} \boxed{*\;\;0} \\ \boxed{*\;\;0}\boxed{0\;\;1} \\ \boxed{*\;\;0} \end{array}$$

FIGURE 4. Four major classes of ω-independent Wolfram rules.

5. MAJOR CLASSES

In this section we prove that four large sets of Wolfram rules are ω-independent. All of the proofs are similar and, when combined with Proposition 1.9, leave only 6 equivalence classes of Wolfram rules that need to be discussed separately. The main tool we use is the notion of a potential function.

Definition 5.1 (Potential functions)**.** Let $F\colon X \to X$ be a map whose dynamics we wish to understand. A *potential function for* F is any map $\rho\colon X \to \mathbb{R}$ such that $\rho(F(x)) \leq \rho(x)$ for all $x \in X$. A potential function narrows our search for periodic points since any element x with $\rho(F(x)) < \rho(x)$ cannot be periodic: further applications of F can never return x to its original potential, hence the name. The only elements in X that are possibly periodic under F are those whose potential under ρ never drops at all. If we call the inverse image of a number in $\mathbb{R}$ a *level set of* ρ, then to find all periodic points of F, we only need to examine its behavior on each of these level sets. Finally, it should be clear that when non-decreasing functions are used in the definition instead of non-increasing ones, the effect is the same.

Definition 5.2 (SDS potential functions)**.** A potential function for an SDS such as $(Y, \mathfrak{F}_Y, \omega)$ is a map $\rho\colon \mathbb{F}^n \to \mathbb{R}$ that is a potential function, in the sense defined above, for the SDS map $[\mathfrak{F}_Y, \omega]$. The easiest way to create such a function is to find one that is a potential function for every local function F_i in $\mathfrak{F}_Y$. Of course, ρ should be either a non-decreasing potential function for each F_i or a non-increasing potential function for each F_i, rather than a mixture of the two, for the inequalities to work out. When ρ has this stronger property we call it a *potential function for* $\mathfrak{F}_Y$ since such a ρ is a potential function for $(Y, \mathfrak{F}_Y, \omega)$ for every choice of update order ω.

Proposition 5.3. Rules 0, 4, 8, 12, 72, 76, 128, 132, 136, 140 and 200 are ω-independent.

Proof. If k is one of the numbers listed above, then its grid notation matches the leftmost form shown in Figure 4. (Each $*$ is to be interpreted as either a 0 or a 1 so that 16 rules share this form, the 11 listed in the statement and 5 that are equivalent to the listed rules or to previously known cases.) The 4 specified values mean that local functions never remove 0s. Thus, the map ρ sending $\mathbf{y} \in \mathbb{F}^n$ to the number of 0s it contains is a non-decreasing potential function for $\mathfrak{Wolf}_n^{(k)}$. Moreover, the local functions $\mathsf{Wolf}_i^{(k)}$ cannot change $\mathbf{y}$ without raising $\rho(\mathbf{y})$, so all periodic states are fixed states (for any update order), and by Proposition 1.8 $\mathfrak{Wolf}_n^{(k)}$ is ω-independent. $\square$

For the next potential function, additional definitions are needed.

Definition 5.4 (Blocks)**.** A state $\mathbf{y} \in \mathbb{F}^n$ is thought of as a cyclic binary n-bit string with the indices taken mod n, and a *substring* of $\mathbf{y}$ corresponds to a set of consecutive indices. We refer to maximal substrings of all 0s as *0-blocks* and maximal substrings of all 1s as *1-blocks*. If a block contains only a single number it is *isolated* and if it contains more than one number it is *non-isolated*. The state $\mathbf{y} = 010110$, for example contains one isolated 0-block and one non-isolated 0-block of length 2 that wraps across the end of the word.

We study how these blocks evolve as the local functions are applied. The decomposition of a Wolfram rule into its symmetric and asymmetric parts is particularly well adapted to the study of these evolutions. The asymmetric rules either make no change or shrink a non-isolated 1-block or 0-block from the left or the right, depending on which of the 4 asymmetric rules we are considering. Similarly, the 4 symmetric rules either do nothing, they remove an isolated block or they create an isolated block in the interior of a long block.

Proposition 5.5. Rules 160, 164, 168, 172 and 232 are ω-independent.

Proof. If k is one of the numbers listed above, then its grid notation matches the second form shown in Figure 4. The specified values mean that (1) the only 0s ever removed are the isolated 0s and (2) isolated 0s are never added. In particular, non-isolated blocks of 0s persist indefinitely, they might grow but they never shrink or split, and the isolated 0s, once removed, never return. Thus, the map ρ that sends $\mathbf{y}$ to the number of non-isolated 0s in $\mathbf{y}$ minus the number of isolated 0s in $\mathbf{y}$ is a non-decreasing potential function for $\mathfrak{Wolf}_n^{(k)}$. As before, the local functions $\mathsf{Wolf}_i^{(k)}$ cannot change $\mathbf{y}$ without raising $\rho(\mathbf{y})$, so all periodic states are fixed states (for any update order), and by Proposition 1.8 $\mathfrak{Wolf}_n^{(k)}$ is ω-independent. $\square$

Proposition 5.6. Rules 5, 13, 77, 133 and 141 are ω-independent.

Proof. If k is one of the numbers listed above, then its grid notation matches the third form shown in Figure 4. This time the specified values mean that (1) the only 0s that are removed create isolated 1s, and (2) isolated 1s are never removed and they never stop being isolated. Thus the map ρ that sends $\mathbf{y}$ to the number of 0s in $\mathbf{y}$ plus *twice* the number of isolated 1s in $\mathbf{y}$ is a non-decreasing potential function for $\mathfrak{Wolf}_n^{(k)}$. Once again, the local functions $\mathsf{Wolf}_i^{(k)}$ cannot change $\mathbf{y}$ without raising $\rho(\mathbf{y})$, so all periodic states are fixed states (for any update order), and by Proposition 1.8 $\mathfrak{Wolf}_n^{(k)}$ is ω-independent. $\square$

The argument for the fourth collection is slightly more complicated.

Proposition 5.7. Rules 1, 9, 73, 129 and 137 are ω-independent.

Proof. If k is one of the numbers listed above, then its grid notation matches the rightmost form shown in Figure 4. This time the specified values mean that (1) the only 0s that are removed create isolated 1s, but (2) isolated 1s can also be removed. The map ρ that sends $\mathbf{y}$ to the number of 0s in $\mathbf{y}$ plus the number of isolated 1s in $\mathbf{y}$ is a non-decreasing potential function for $\mathsf{Wolf}_n^{(k)}$, but the difficulty is that there are local changes with $\rho(\mathsf{Wolf}_i^{(k)}(\mathbf{y}) = \rho(\mathbf{y})$. This is true for the local change $000 \to 010$ and for the local change $010 \to 000$. All other local changes raise the potential, but the existence of these two equalities indicates that there might be (and there are) states that are periodic under the action of some SDS map $[\mathfrak{Wolf}_n^{(k)}, \omega]$ without being fixed. Rather than appeal to a general theorem, we calculate its periodic states explicitly in this case.

Fix an update order $\omega \in W_n$ and, for convenience, let $F\colon \mathbb{F}^n \to \mathbb{F}^n$ denote the SDS map $[\mathfrak{Wolf}_n^{(k)}, \omega]\colon \mathbb{F}^n \to \mathbb{F}^n$. If $a_3 = 0$ and $\mathbf{y}$ contains a substring of the form 011, then $\rho(F(\mathbf{y})) > \rho(\mathbf{y})$ and $\mathbf{y}$ is not periodic under F. This is because either (1) the substring remains unaltered until its central coordinate is updated, at which point it changes to 0 and ρ is raised, or (2) it is altered ahead of time by switching the 1 on the right to a 0 (also raising ρ), or by switching the 0 on the left to a 1 (impossible since $a_1 = a_5 = 0$). Analogous arguments show that if $a_6 = 0$ and $\mathbf{y}$ contains the substring 110, or if $a_7 = 0$ and $\mathbf{y}$ contains the substring 111, then $\mathbf{y}$ is not periodic under F. Let P be the subset of $\mathbb{F}^n$ where these situations do not occur. More specifically, if $a_3 = 0$ remove the states with 011 substrings, if $a_6 = 0$ remove the states with 110 substrings, and if $a_7 = 0$ remove the states with 111 substrings. If all three are equal to 1, then $P = \mathbb{F}^n$.

We claim that $P = \mathsf{Per}[\mathfrak{Wolf}_n^{(k)}, \omega]$, independent of the choice of ω. We have already shown $P \subset \mathsf{Per}[\mathfrak{Wolf}_n^{(k)}, \omega]$. Note that P is invariant under F (in the sense that $F(P) \subset P$) since the allowed local changes are not able to create the forbidden substrings when they do not already exist. Moreover, F restricted to P agrees with rule 201 = `---x`, the rule of this form with $a_3 = a_6 = a_7 = 1$, since whenever a_3, a_6 or a_7 is 0, P has been suitably restricted to make this fact irrelevant. Finally, for every ω rule 201 is bijective, thus F is injective on P, F permutes the states in P and a sufficiently high power of F is the identity, showing every state in P is periodic independent of our choice of ω. □

6. Exceptional Cases

At this point there are only 6 remaining rules whose ω-independence needs to be established and they come in pairs: 28 and 29, 32 and 40, and 152 and 184. These final 6 rules exhibit more intricate dynamics and the proofs are, of necessity, more delicate. We treat them in order of difficulty.

Proposition 6.1. Rules 32 and 40 are ω-independent.

Proof. Let k be 32 or 40, let $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in S_n$ be a simple update order, and let $F\colon \mathbb{F}^n \to \mathbb{F}^n$ denote the SDS map $[\mathfrak{Wolf}_n^{(k)}, \pi]\colon \mathbb{F}^n \to \mathbb{F}^n$. The listed rules share the leftmost form shown in Figure 5 and it is easy to see that $\mathbf{0}$ is the only fixed state ($\mathbf{1}$ is not fixed and $a_2 = a_6 = 0$ means the rightmost 1 in any 1-block converts to 0 when updated). We also claim $\mathbf{0}$ is the only periodic state of F. Once this is established, the ω-independence of $\mathfrak{Wolf}_n^{(k)}$ follows immediately from Proposition 1.8.

The values $a_0 = a_1 = a_4 = 0$ mean non-isolated 0-blocks persist indefinitely, they do not shrink or split. Moreover, $a_2 = a_6 = 0$ means that each non-isolated 0-block adds at least one 0 on its left-hand side with each application of F. In particular, any state $\mathbf{y} \neq \mathbf{0}$ with a non-isolated 0-block eventually becomes the fixed point $\mathbf{0}$. Thus no such $\mathbf{y}$ is periodic.

The rest of the argument is by contradiction. Suppose that $\mathbf{y}$ is a periodic point of F other than $\mathbf{0}$ and consider the i^{th} coordinates in $\mathbf{y}$, $F(\mathbf{y})$ and $F(F(\mathbf{y}))$. We claim that at least one of these coordinates is 0 and at least one of these is 1. This is because at least 4 out of the 5 local state configurations that do not involve non-isolated 0s change the coordinate (and when $k = 32$ all 5 of them make a change). The only way that y_i does not change value in $F(\mathbf{y})$ is if immediately prior to the application of $\mathsf{Wolf}_i^{(k)}$, the local state configuration is 011. Between this application

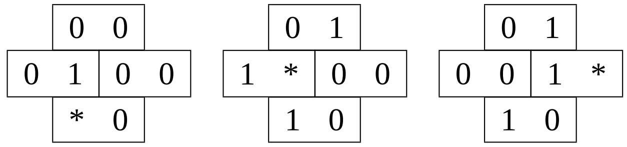

FIGURE 5. Three final pairs of ω-independent Wolfram rules.

of $\mathsf{Wolf}_i^{(k)}$ and the next, the 0 to the left is updated. It either is no longer isolated at this point (contradicting the periodicity of $\mathbf{y}$) or it now becomes a 1. In the latter case, the application of $\mathsf{Wolf}_i^{(k)}$ during the second iteration of F changes the i^{th} coordinate from 1 to 0. Note that we used the simplicity of the update order to ensure that each coordinate is updated only once during each pass through F. Finally, suppose that $i = \pi_1$ and choose $\mathbf{y}$, $F(\mathbf{y})$ or $F(F(\mathbf{y}))$ so the $(i+1)^{\text{st}}$ coordinate is a 0. As soon as $\mathsf{Wolf}_{\pi_1}^{(k)}$ is applied, there is a non-isolated 0-block, contradicting the claim that $\mathbf{y} \neq \mathbf{0}$ is a periodic point. □

Since it was easy to show that every state is periodic under the bijective Wolfram rule 156 (with tag `-x--`), we did not examine the evolution of its blocks. We do so now since its behavior is relevant to our study of the 4 remaining rules.

Example 6.2 (Wolfram rule 156)**.** Because the symmetric part of rule 156 is `--` no isolated blocks are ever created or destroyed and thus the number of blocks is invariant under iteration. Moreover, the four values $a_1 = a_5 = 0$ and $a_2 = a_3 = 1$ mean that substrings of the form 01 are fixed indefinitely, leaving the right end of every 0-block and the left end of every 1-block permanently unchanged. The other type of boundary can and does move since p_3 = x, and it is its behavior that we want to examine. Let $\pi \in S_n$ be a simple update order and let $F \colon \mathbb{F}^n \to \mathbb{F}^n$ denote the SDS map $[\mathfrak{Wolf}_n^{(156)}, \pi] \colon \mathbb{F}^n \to \mathbb{F}^n$. So long as $\mathbf{y}$ is not $\mathbf{0}$ or $\mathbf{1}$, there is a 1-block followed by a 0-block and a corresponding substring of the form $01 \cdots 10 \cdots 01$. (If $\mathbf{y}$ only contains one 0-block and one 1-block, then the first two digits are the same as the last two digits, but that is irrelevant here.) As remarked above, the beginning of the 1-block and the end of the 0-block are fixed, but the boundary between them can vary.

Suppose both blocks are non-isolated and consider the central substring 10 at positions i and $i+1$. These are the only positions in the entire substring that can vary and the first one to be updated *will* change value. Assume the 0 is updated first. The 1-block grows, the 0-block shrinks and the boundary shifts one step to the right. As we cycle through the local functions, the simplicity of π guarantees that the $(i+2)^{\text{nd}}$ coordinate is updated before the $(i+1)^{\text{st}}$ coordinate is updated a second time. Thus the boundary shifts one more step to the right. This argument continues to be applicable until the 0-block shrinks to an isolated 0. At this point, the 0 is still updated before the 1 to its left is updated again, but this time the 0 remains unchanged. When the 1 to its left is updated it changes back to a 0, the 1-block shrinks, the 0-block grows and the boundary shifts to the left. The same argument with left and right reversed shows that now the 0-block continues to grow until the 1-block shrinks to an isolated 1, at which point the shifting stops and the boundary starts shifting back in the other direction.

Proposition 6.3. Rules 152 and 184 are ω-independent.

Proof. Let k be 152 or 184, let $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in S_n$ be a simple update order, and let $F\colon \mathbb{F}^n \to \mathbb{F}^n$ denote the SDS map $[\mathfrak{Wolf}_n^{(k)}, \pi]\colon \mathbb{F}^n \to \mathbb{F}^n$. The listed rules share the second form shown in Figure 5 and it is easy to see that $\mathbf{0}$ and $\mathbf{1}$ are the only fixed states (since $a_2 = a_6 = 0$ means the rightmost 1 in any 1-block converts to 0 when updated). We also claim $\mathbf{0}$ and $\mathbf{1}$ are the only periodic states of F. Once this is established, the ω-independence of $\mathfrak{Wolf}_n^{(k)}$ follows immediately from Proposition 1.8.

Since isolated blocks are never created, the map ρ that sends $\mathbf{y}$ to the number of blocks it contains is a non-increasing potential function for $\mathfrak{Wolf}_n^{(k)}$. Moreover, since the only differences between rule 156 and rules 152 and 184 are that rule 152 removes isolated 1-blocks and rule 184 removes both isolated 1-blocks and isolated 0-blocks, the map F agrees with $[\mathfrak{Wolf}_n^{(156)}, \pi]$ so long as it is not called upon to update an isolated 1-block (or an isolated 0-block when k = 184). The long-term behavior of rule 156, however, as described in Example 6.2, shows that under iteration every $\mathbf{y}$ not equal to $\mathbf{0}$ or $\mathbf{1}$ eventually updates such an isolated block, removing it and decreasing ρ, thus showing that such a $\mathbf{y}$ is not periodic. $\square$

Finally, the argument for Wolfram rules 28 and 29 is a combination of the difficulties found in the proofs of Propositions 5.7 and 6.3.

Proposition 6.4. Rules 28 and 29 are ω-independent.

Proof. Let k be 28 or 29, let $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in S_n$ be a simple update order, and let $F\colon \mathbb{F}^n \to \mathbb{F}^n$ denote the SDS map $[\mathfrak{Wolf}_n^{(k)}, \pi]\colon \mathbb{F}^n \to \mathbb{F}^n$. The listed rules share the rightmost form shown in Figure 5 and the values $a_5 = 0$ and $a_2 = 1$ mean that isolated blocks are never removed. Thus the map ρ that sends $\mathbf{y}$ to the number of blocks it contains is a non-decreasing potential function for $\mathfrak{Wolf}_n^{(k)}$. The four values $a_1 = a_5 = 0$ and $a_2 = a_3 = 1$ mean that substrings of the form 01 persist indefinitely, as in Wolfram rule 156. In fact, so long as ρ is unchanged, the behavior of F under iteration is indistinguishable from iterations of the map $[\mathfrak{Wolf}_n^{(156)}, \pi]$. Consider a substring of the form $01 \cdots 10 \cdots 01$ and suppose that the length of the 1-block on the left plus the length of the 0-block on the right is at least 4. We claim that any $\mathbf{y}$ containing such a substring is not periodic under F. If it were, the evolution of this substring would oscillate as described in Example 6.2 and at the point where the 0-block shrinks to an isolated 0, the 1-block on the left contains the substring 111. Moreover, between the point when that penultimate 0 becomes a 1 and the point when it is to switch back, the substring 111 is updated, increasing ρ. When k is 29, a similar increase in ρ can occur when the 1-block shrinks to an isolated 1 and the 0-block contains the substring 000. In neither case can a state containing a 1-block followed by a 0-block with combined length at least 4 be periodic under F.

Next, note that when k = 29 both of the special states $\mathbf{0}$ and $\mathbf{1}$ are not fixed, but that for k = 28 $\mathbf{1}$ is not fixed, while $\mathbf{0}$ is fixed. Let P be the set of states containing both 0s and 1s that do not contain a 1-block followed by a 0-block with combined length at least 4, and, when k = 28, include the special state $\mathbf{0}$ as well. Because we understand the way that such states $\mathbf{y} \in P$ evolve under Wolfram rule 156 (Example 6.2), we know that at no point in the future does a descendent of $\mathbf{y}$ ever contain a substring of the form 111 or 000. Thus P has been restricted enough to make the values of a_7 and a_0 irrelevant, and F sends P into itself. Moreover, since F agrees with $[\mathfrak{Wolf}_n^{(156)}, \pi]$ on P, and this map is injective, F is injective on P, F permutes the states in P and

Number of flips	0	1	2	3	4	5	6	7	8
Number of ω-independent rules	1	8	26	34	26	4	4	0	1
Number of rules	1	8	28	56	70	56	28	8	1
Percentage	100%	100%	93%	61%	37%	7%	14%	0%	100%

TABLE 2. The number of flips and the probability of ω-independence.

a sufficiently high power of F is the identity, showing every state in P is periodic, independent of our choice of π. Now that we know that $\mathfrak{Wolf}_n^{(k)}$ is π-independent, ω-independent follows from Theorem 1.7. $\square$

7. CONCLUDING REMARKS

Now that the proof of Theorem 2.2 is complete, we pause to make a few comments about it and the 104 ω-independent Wolfram rules it identifies. For each of the 8 local state configurations, Wolfram rule k either leaves the central coordinate unchanged or it "flips" its value. The number of local state configurations that are flipped in this way is strongly correlated with the probability that a given rule is ω-independent. See Table 2. The numbers in the third row are the binomial coefficients $\binom{8}{i}$, since they clearly count the number of Wolfram rules with exactly i flips. The key facts illustrated by Table 2 are that virtually all of the rules with at most 2 flips are ω-independent, the percentage drops off rapidly between 2 and 6 flips, and ω-independence is very rare among rules with 6 or more flips. In fact, all 5 such rules are ω-independent because they are bijective. It would interesting to know whether this observation can be quantitatively (or even qualitatively) extended to a rigorous assertion about more general SDSs.

Next, there are two aspects of Theorem 2.2 that we found slightly surprising. First, we did not initially expect the set of rules that were ω-independent for small values of n to match exactly the set of rules that were ω-independent for all values of $n > 3$. The second surprise was that the during the course of the proof we found that the Wolfram rules truly are local rules, in the sense that their set of periodic points tended to have essentially local characterizations.

Finally, although the focus of this article was solely the classification of the 104 ω-independent Wolfram rules, and not the dynamics of these rules per se, many interesting dynamical properties arose in the course of the proof. We are currently studying the dynamics and periodic sets for all 256 Wolfram rules in greater detail, as well as examining how the sets of periodic states under an ω-independent Wolfram rule get permuted as the update order is altered. The latter situation involves an object called the *dynamics group* of an ω-independent SDSs. We plan on publishing these further results in a future article that builds on the results described here.

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