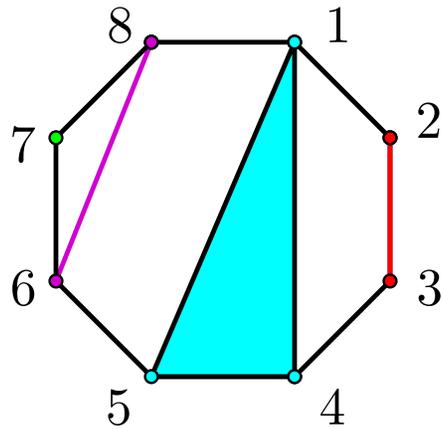


# Finding parking when not commuting



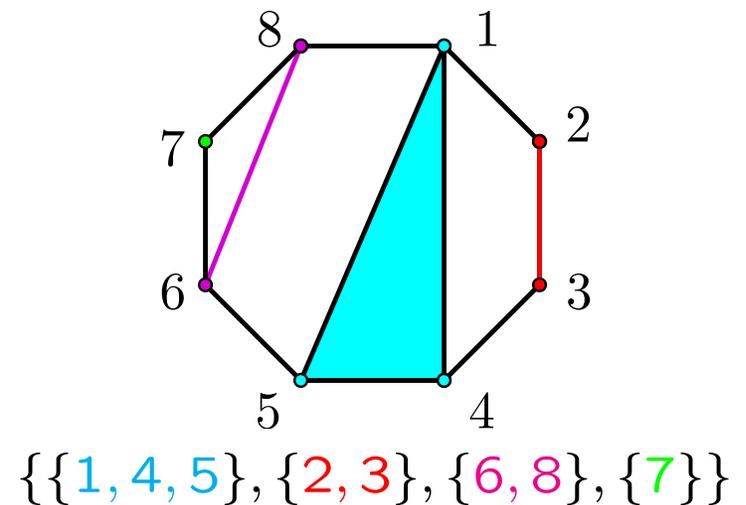
$\{\{1, 4, 5\}, \{2, 3\}, \{6, 8\}, \{7\}\}$

Jon McCammond  
U.C. Santa Barbara

## A common structure

The goal of this talk will be to introduce you to a mathematical object that has a habit of appearing in a vast array of guises.

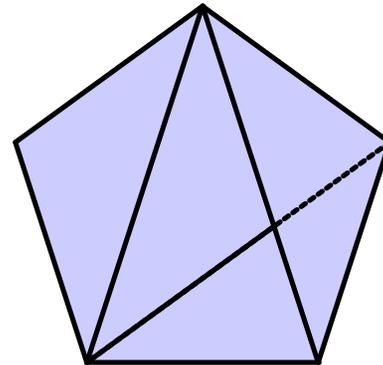
- I. Non-crossing partitions
- II. Symmetric groups
- III. Braid groups
- IV. Parking functions
- V. Non-commutative geometry and free probability



## Motivating example I: Fibonacci numbers

- I. Dominoe tilings of a  $2 \times n$  strip
- II. Continued fractions and golden ratio
- III. Diagonals of a regular pentagon
- IV. Tridiagonal matrices

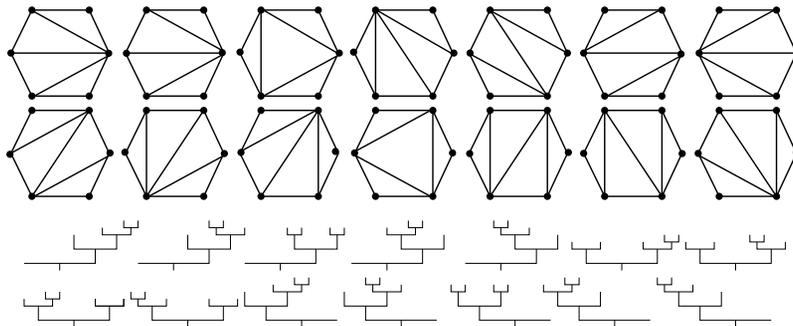
$$\frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}} = \frac{5}{8}$$



See Neal Sloane's online encyclopedia of integer sequences for more connections. <http://www.research.att.com/~njas/sequences/>

## Motivating example II: Catalan numbers

- I. Triangulations of an  $n$ -gon
- II. Rooted (planar) binary trees
- III. Ways to associate  $n$  letters
- IV. Dyck paths from  $(0, 0)$  to  $(n, n)$

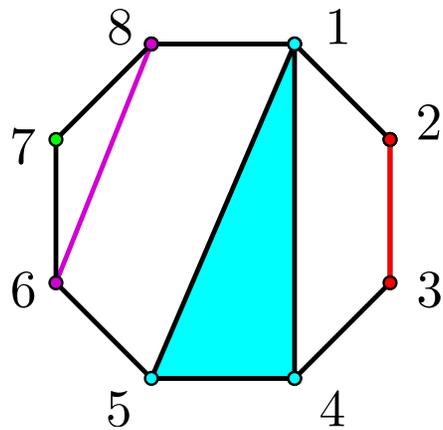


$(1(2(3(45))))$	$(1(2((34)5)))$
$(1((23)(45)))$	$(1((2(34))5))$
$(1(((23)4)5))$	$((12)(3(45)))$
$((12)((34)5))$	$((1(23))(45))$
$((1(2(34)))5)$	$((1((23)4))5)$
$((((12)3)(45)))$	$((((12)(34))5))$
$((((1(23))4)5))$	$(((((12)3)4)5))$

... and many more (See R. Stanley's list of 106 distinct combinatorial interpretations of these numbers)

## I. Non-crossing partitions

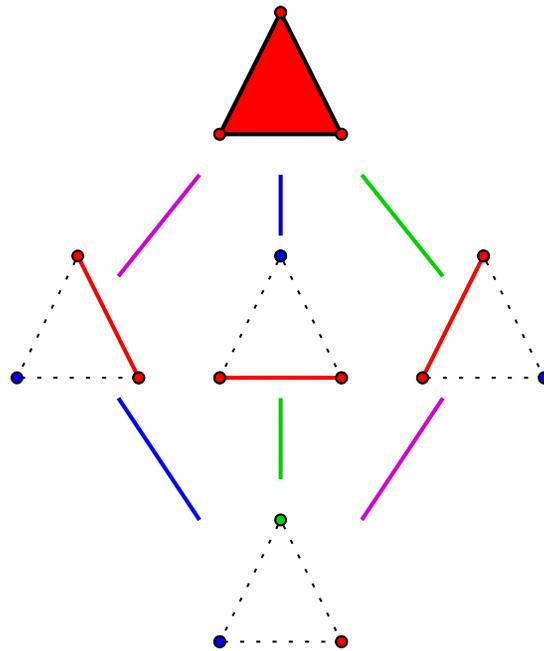
**Def:** A *noncrossing partition* is a partition of the vertices of a regular  $n$ -gon so that the convex hulls of the blocks are disjoint.



The figure illustrates the partition  $\{\{1, 4, 5\}, \{2, 3\}, \{6, 8\}, \{7\}\}$ .

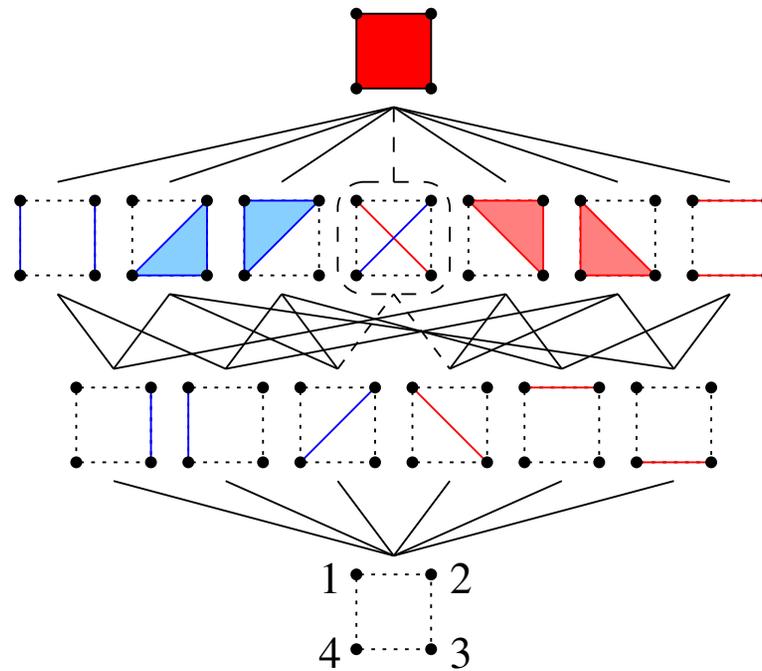
## Non-crossing partition lattice

One noncrossing partition  $\sigma$  is contained in another  $\tau$  if each block of  $\sigma$  is contained in a block of  $\tau$ . Let  $NC_n$  be the poset (partially ordered set) of all non-crossing partitions of an  $n$ -gon under this relation.



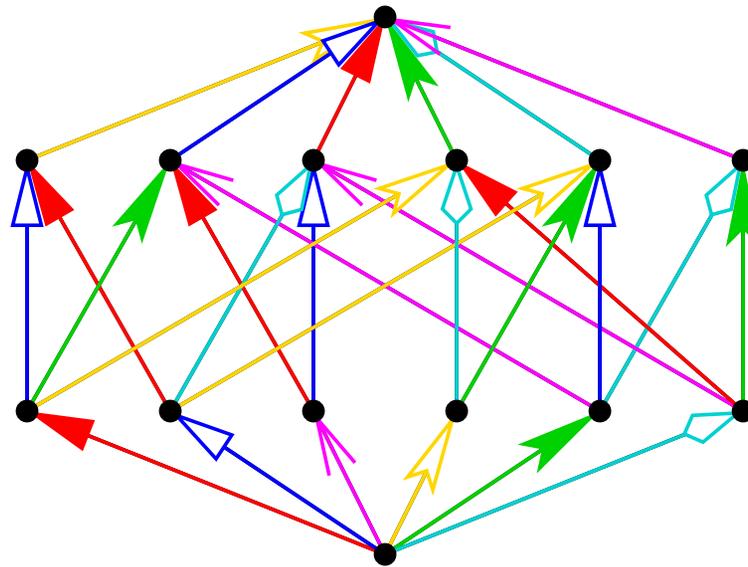
# Non-crossing partition lattice on 4 elements

For  $n = 4$  we get the following:



## Non-crossing partition lattice on 4 elements

The abstracted version of  $NC_4$  looks like this:



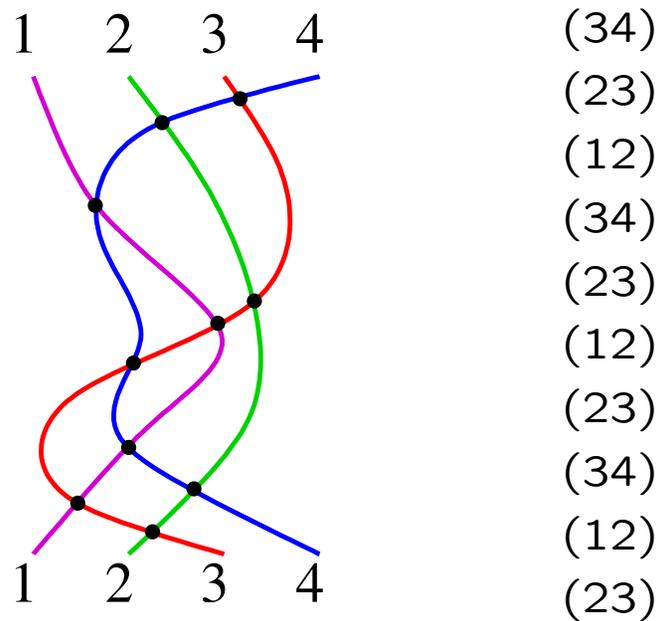
## Properties of $NC_n$

**Thm:**  $NC_n$  is a graded, bounded lattice with Catalan many elements. It is also self-dual.

- Lattice means that least upper bounds and greatest lower bounds always exist.
- Self-dual means that it looks the same when we flip it over

## II. Symmetric groups

**Def:**  $S_n$  is the group of permutations of the set  $[n] := \{1, \dots, n\}$ .



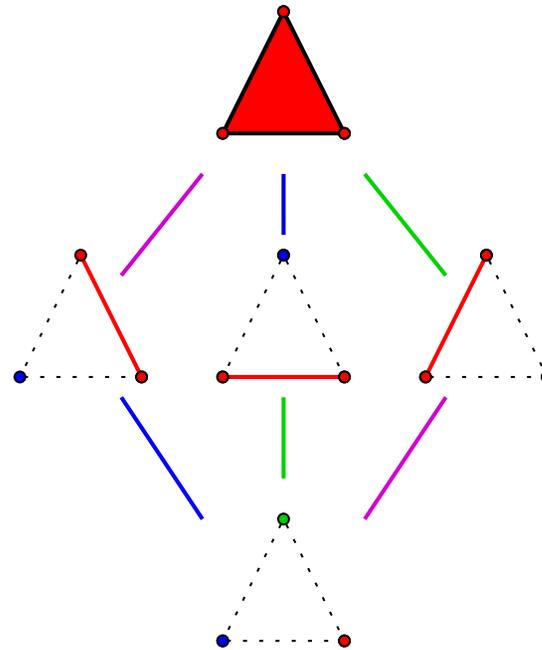
**Thm:** Every trivial product of 2-cycles is of even length.

**Cor:** The parity of a factorization is well-defined.

## Factorization into 2-cycles

**Lem:** The poset of suffixes of minimal factorizations of the  $n$ -cycle  $(123 \dots n)$  into 2-cycles is exactly  $NC_n$ .

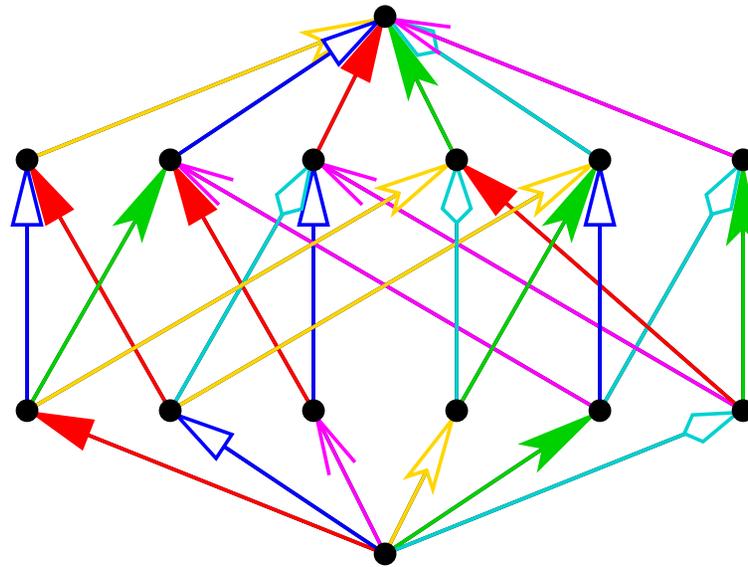
$$\begin{aligned} [(123) &= (12)(23) \\ &= (23)(13) \\ &= (13)(12)] \end{aligned}$$



**Rem:** This makes for an easy proof of self-duality (and it explains the colors assigned to the edges of the Hasse diagram).

## Factorization into 2-cycles

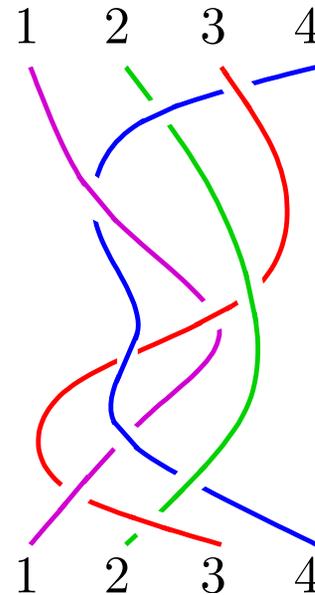
The colors here describe all factorizations of  $(1234)$  into three 2-cycles.



### III. Braid groups

**Def:** The braid group  $B_n$  keeps track of how  $n$  strings can be twisted.

Clearly  $B_n$  maps onto  $S_n$ .



Braid groups are related to many, many areas of mathematics, including mathematical physics, quantum groups, and, not surprisingly, 3-manifold topology. They are also intimately related to non-crossing partitions.

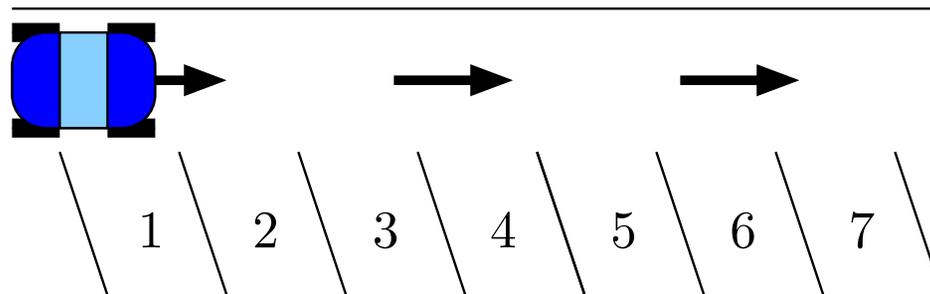
## **A Space for the Braid Groups**

The noncrossing partition lattice can be used to build a space whose “fundamental group” is the braid group. In fact, it is something called an Eilenberg-MacLane space for these groups because when it is completely unwrapped it is contractible.

The procedure involves turning chains into simplices and then gluing according to labels and orientations.

## IV. Parking functions

**Def:** A parking function of length  $n$  is a sequence  $(a_1, \dots, a_n)$  of positive integers whose nondecreasing rearrangement  $b_1 \leq b_2 \leq \dots \leq b_n$  satisfies  $b_i \leq i$ .

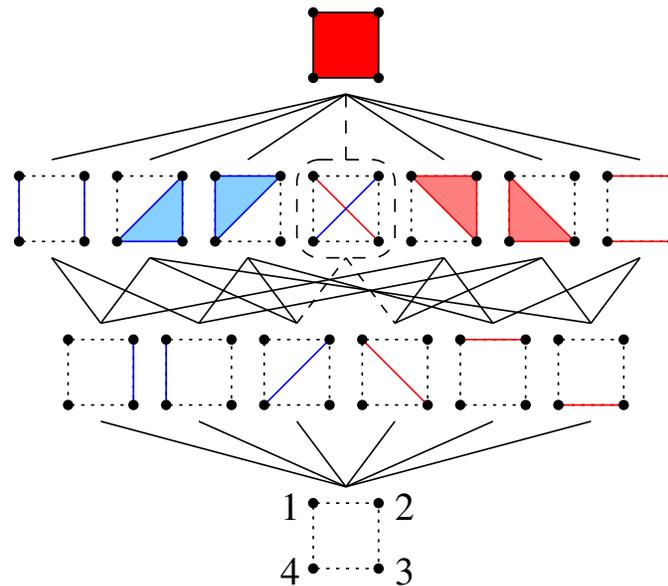


The number of parking functions is  $(n + 1)^{n-1}$ .

There are a number of bijections with labeled rooted trees on  $[n]$  (or equivalently the number of acyclic functions on  $[n]$ ).

## Parking functions and $NC_{n+1}$

Let  $\sigma - \tau$  be an edge of covering relation in  $NC_{n+1}$  and let  $B$  and  $B'$  be the blocks which are joined. If  $\min B < \min B'$  we define the label as the largest element of  $B$  which is below all of  $B'$ .



## Parking functions and $NC_{n+1}$

**Lem:** The labels on the maximal chains in  $NC_{n+1}$  are the parking functions.

**Rem:** There are also connections along these lines with the theory of symmetric functions and Hopf algebras.

$$f(x, y) = x^2 + xy + y$$

$$g(x, y) = x^2 + xy + y^2$$

## V. Classical Probability

Let  $X$  be a random variable having a probability density function  $f(x)$  (discrete or continuous). The *expectation* of  $u(x)$  is

$$E(u(X)) = \int_{-\infty}^{\infty} u(x)f(x)dx.$$

**Ex:** mean  $\mu = E(X)$

**Ex:** variance  $\sigma^2 = E((X - \mu)^2)$

**Ex:** moment generating function

$$M(t) = E(e^{tX}) = \sum_{n \geq 0} E(X^n) \frac{t^n}{n!}.$$

## Moments and cumulants

The coefficients in the moment generating function are called the *moments* of  $f(X)$ . The coefficients of  $\log M(t)$  are called the (classical) *cumulants* of  $X$ .

The main advantage of the cumulants is that they contain the same information as the moments but the cumulant of the sum of two random variables  $X$  and  $Y$  are the sum of their cumulants – so long as they are independent.

## Non-commutative geometry

Non-commutative geometry is a philosophy whereby standard geometric arguments on (topological) spaces are converted into algebraic arguments on their commutative  $C^*$ -algebras of functions.

The goal is then to find analogous arguments on non-commutative  $C^*$ -algebras which reduce to the standard results in the commutative case.

## Free Probability

There is a definition of a non-commutative version of probability and a non-commutative version of independence is “freeness”. The combinatorics of non-crossing partitions is very closely involved in this non-commutative version of cumulants. In fact, some researchers who study free probability describe the passage from the commutative to the non-commutative version as a transition from the combinatorics of partitions to the combinatorics of non-crossing partitions.

## Summary

The lattice of non-crossing partitions might play a role in any situation which involves

- the symmetric groups
- the braid groups
- free probability

or anywhere where the Catalan numbers can be found (including the combinatorics of trees and the combinatorics of parking functions).

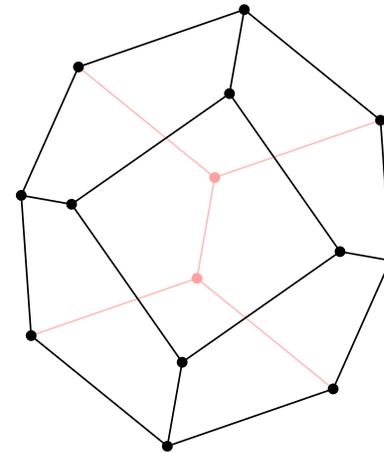
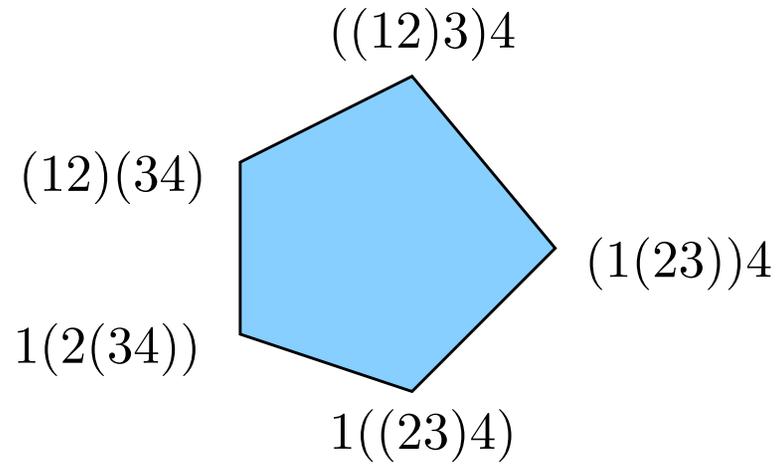
They also show up in lots of other mathematics: real hyperplane arrangements, Prüfer codes, quasisymmetric functions,...

## **Final illustration: Associahedra**

The Catalan numbers count the ways to associate a list of numbers. If we include partial associations we get an ordering.

This ordering can be used to construct a polytope called the associahedron, and the “Morse theory” of this polytope has intimate connections with the non-crossing partition lattice.

# Associahedra in Low Dimensions



- |                  |                  |
|------------------|------------------|
| $(1(2(3(45))))$  | $(1(2((34)5)))$  |
| $(1((23)(45)))$  | $(1((2(34))5))$  |
| $(1(((23)4)5))$  | $((12)(3(45)))$  |
| $((12)((34)5))$  | $((1(23))(45))$  |
| $((1(2(34)))5)$  | $((1((23)4))5)$  |
| $((((12)3)(45))$ | $((((12)(34))5)$ |
| $((((1(23))4)5)$ | $((((12)3)4)5)$  |