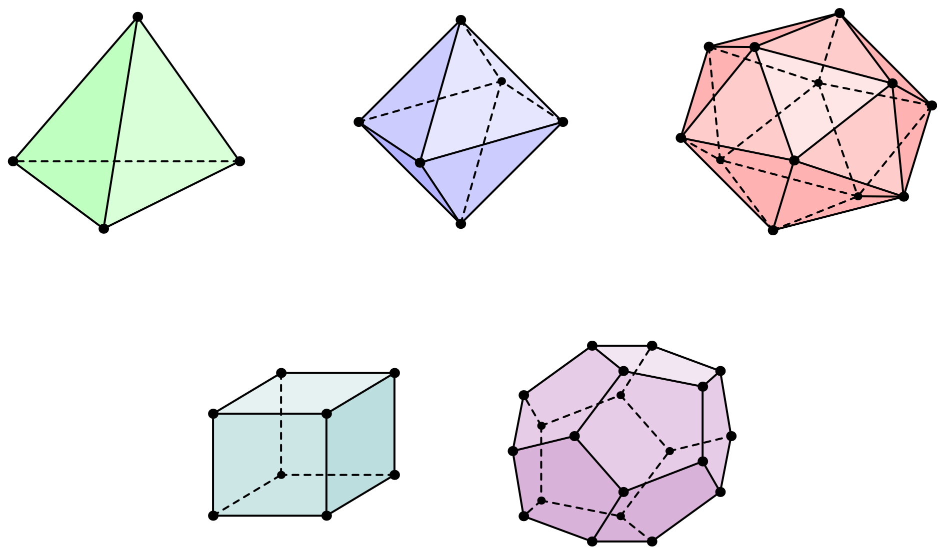
Coxeter groups and Artin groups

Day 1: Polytopes and Reflection Groups



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Overview

The **plan** is to spend

- two days on (topics related to) Coxeter groups, and
- two days on (topics related to) Artin groups.

The **theme** will be the close connections these groups have with other parts of mathematics (and the need to understand these connections in order to fully understand the groups).

For **Coxeter groups**, the list includes regular polytopes, Lie groups, symmetric spaces, and finite simple groups. All of these connections are well-known (but not to everyone).

For **Artin groups**, the associated objects are less well understood (and they include some interesting infinite continuous groups).

Where do Coxeter groups come from?

Although Coxeter groups (and Artin groups) can be easily defined via presentations, this fails to show why they are important. The motivation for the definition comes from two directions:

Platonic solids $\Rightarrow$ Regular polytopes $\Rightarrow$ Finite reflection groups

Lie groups $\Rightarrow$ Lie algebras $\Rightarrow$ Affine reflection groups

Both finite and affine reflection groups have simple presentations and Coxeter groups can be viewed as their natural generalization.

Polytopes

Def: A **polytope** P is the convex hull of a finite set of points in some Euclidean space, or, equivalently, it is a bounded, non-empty intersection of finite number of half-spaces.

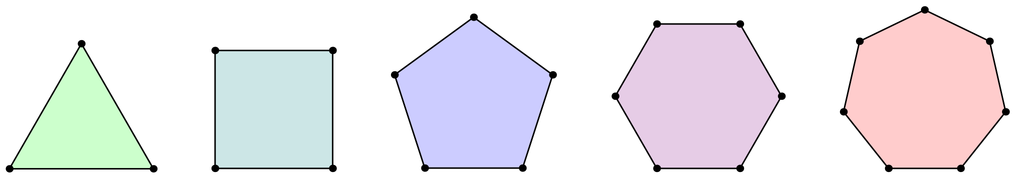
The dimension of the minimal affine subspace containing P is called the **dimension** of P.

If H is a half-space containing P and $Q = \partial H \cap P$ is non-empty, then Q is another polytope called a **face** of P. The faces of P are ordered by inclusion.

A face Q of d-dimensional polytope P is called a **vertex**, **edge**, or **facet** if the dimension of Q is 0,1, or $d-1$, respectively.

Regular Polytopes: Low Dimensions

The class of regular polytopes should include regular polygons:



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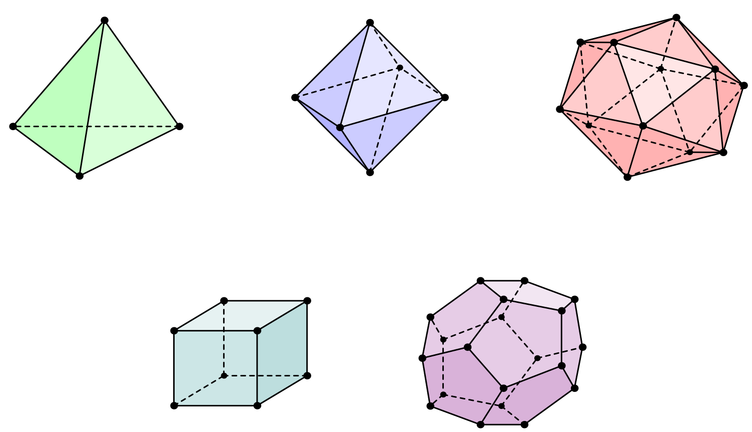
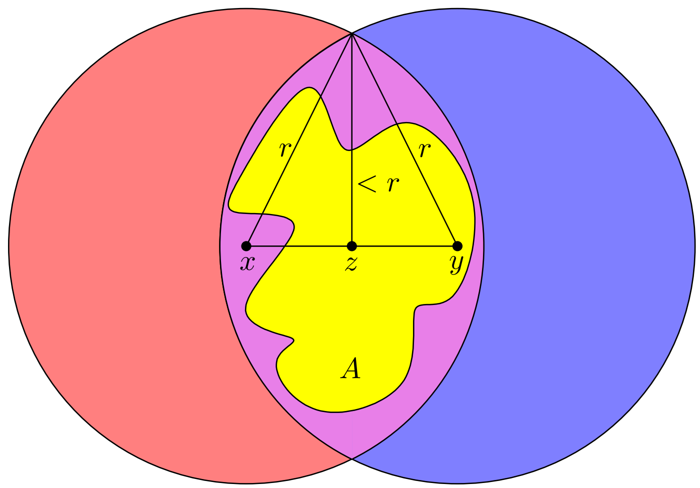
and the platonic solids:



Barycenters

Thm: If A is a bounded subset of $\mathbb{R}^n$ then there is a unique closed ball containing A of smallest possible radius.

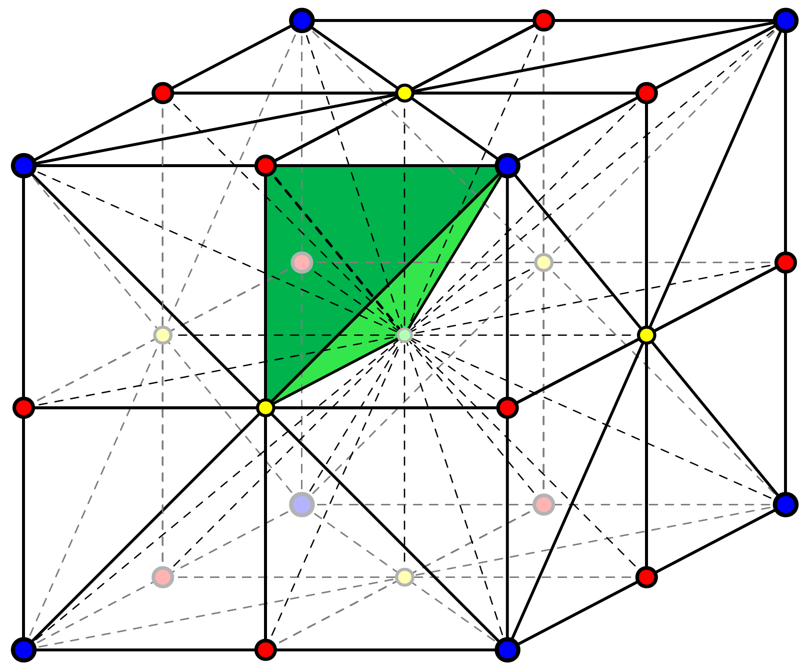


Cor: Every bounded subset of $\mathbb{R}^n$ has a unique center.

The barycenters of the faces can be used to subdivide a polytope.

Barycentric Subdivisions

A subdivided cube with one of its 48 tetrahedra shaded.

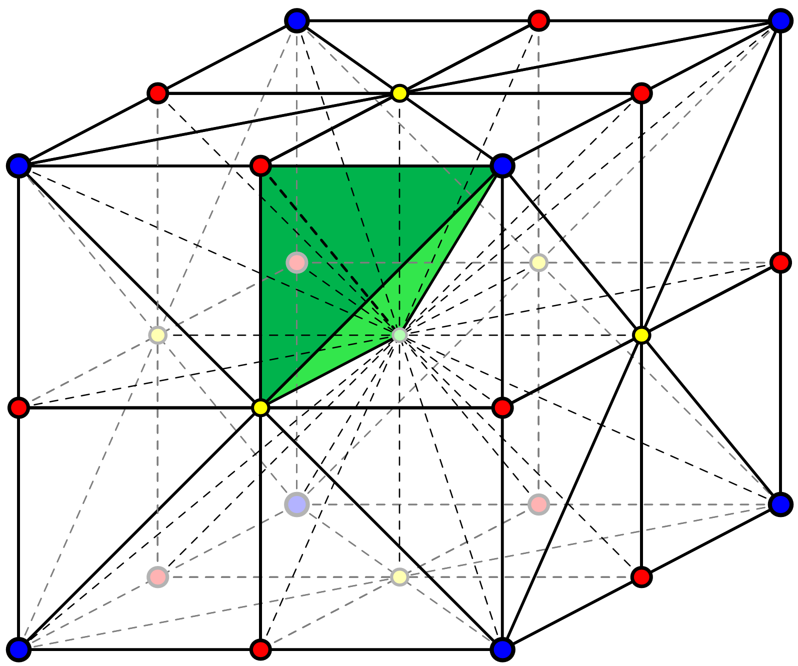


The vertices are color-coded to indicate the dimension of the face whose center is being marked: 0=●, 1=●, 2=●, and 3=●.

Regular Polytopes: Definition

Def: A polytope is **regular** if its isometry group acts transitively on the maximal simplices in its barycentric subdivision.

Ex: A cube is regular.

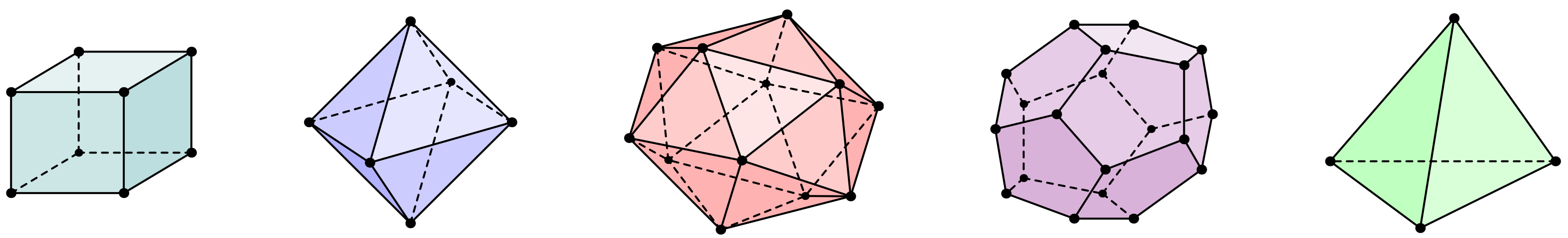


Dual Regular Polytopes

Prop: If P is a regular polytope and Q is the convex hull of the barycenters of the facets of P, then Q is another regular polytope called the **dual** of P.

Rem: The dual of the dual is a rescaled version of the original.

Ex: The cube and octahedron are dual. The icosahedron and dodecahedron are dual. The tetrahedron is self-dual.



Regular Polytopes: High Dimensions

In every dimension there are regular polytopes that are analogs of the tetrahedron, octahedron and cube.

The n-dimensional **simplex** is the convex hull of an orthonormal basis in $\mathbb{R}^{n+1}$, i.e. $\triangle_n :=$**Conv**$(\{e_i\})$.

The n-dimensional **orthoplex** is the convex hull of an orthonormal basis and its negative in $\mathbb{R}^n$, i.e. $\Diamond_n :=$**Conv**$(\{\pm e_i\})$.

The n-dimensional **cube** is the subspace $\square_n := [-1, 1]^n \subset \mathbb{R}^n$.

$\square_n$ and $\Diamond_n$ are dual; $\triangle_n$ is self dual.

Regular 4-Polytopes: 3 More Examples

Ex: The Poincaré homology 3-sphere has a piecewise spherical geometric structure. The preimage of a point in its universal cover is a collection of 120 symmetrically placed points in $\mathbb{S}^3$. The convex hull of these points in $\mathbb{R}^4$ is a regular 4-polytope with 120 vertices and 600 tetrahedral facets called the **600-cell**. Its dual is a regular 4-polytope with 600 vertices and 120 dodecahedral facets called the **120-cell**.

Ex: There are exactly 24 lattice points in $\mathbb{Z}^4 \subset \mathbb{R}^4$ that are distance 2 from the origin: 8 with shape $(\pm 2, 0^3)$ and 16 with shape $(\pm 1^4)$. The convex hull of these 24 points is a regular 4-polytope with 24 vertices and 24 octahedral facets called the **24-cell**. It is self-dual.

The Classification Theorem

Perhaps surprisingly, these examples form a complete list.

Theorem: Every regular polytope is

1. a closed interval,
2. a regular m-gon with $m \geq 3$,
3. one of the 5 platonic solids,
4. one of the 6 regular 4-polytopes, or
5. an n-dimensional simplex, orthoplex or cube with $n > 4$.

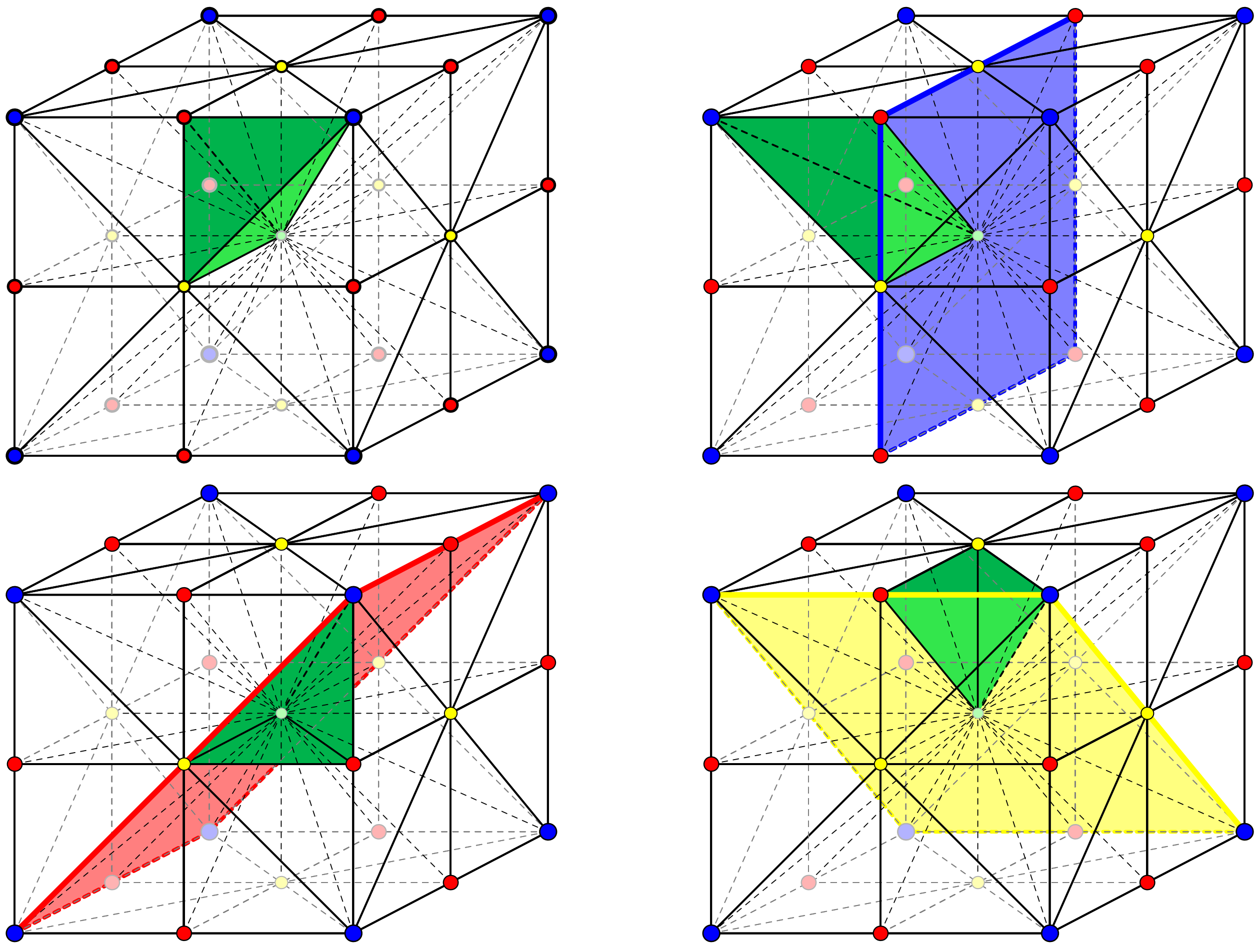
Basic Reflections

Rem: Maximal simplices have one vertex of each color, and isometries must preserve the colors. As a consequence, the reflection through an interior facet of a maximal simplex in a regular polytope must be an isometry.

Prop: If P is a regular polytope then any set of basic reflections generates the isometry group.

Def: Let $v_0, \ldots, v_d$ be the vertices of a fixed maximal simplex;
Let r_i be the reflection through the facet opposite v_i $(i < d)$;
And let $\vec{n}_i$ be the vector from v_i to $r_i(v_i)$.

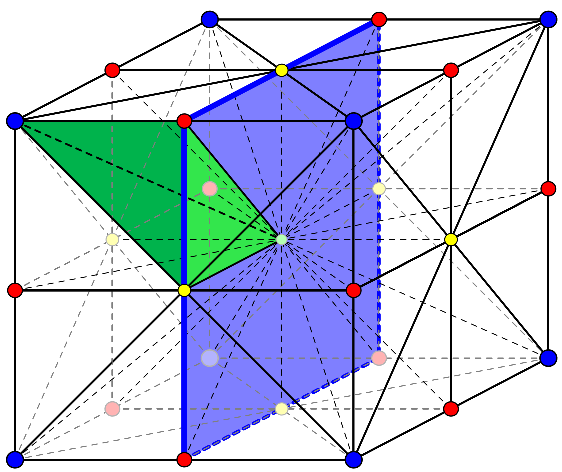
Basic Reflections in the Cube

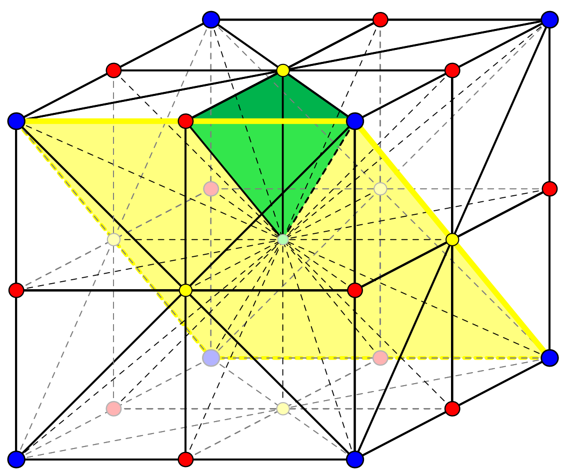


Commuting Reflections

Rem: If there is a k with $i < k < j$ then $\vec{n}_i$ and $\vec{n}_j$ are perpendicular and r_i and r_j commute.

Proof: $\vec{n}_i$ is parallel to the k-cell v_k represents and $\vec{n}_j$ is perpendicular to it.

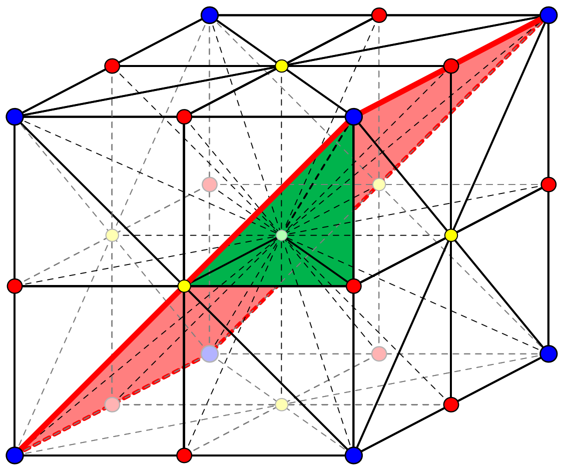


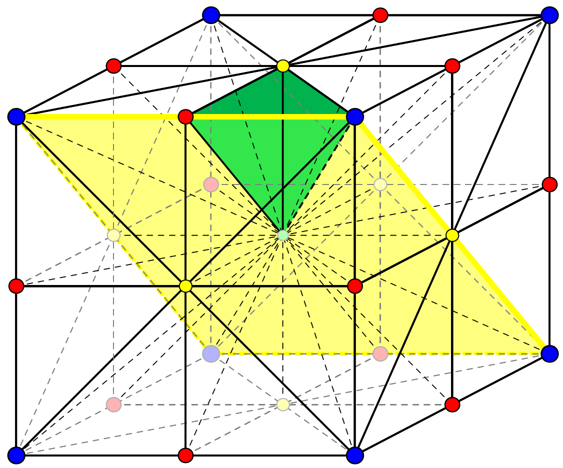


Non-Commuting Reflections

Rem: If $i + 1 = j$, then the angle between $\vec{n}_i$ and $\vec{n}_j$ is $\pi - \pi/n$ for some $n \geq 3$. In particular, r_i and r_j do not commute.

Proof: Look at maximal simplices surrounding the co-dimension 2 face that excludes v_i and v_j.

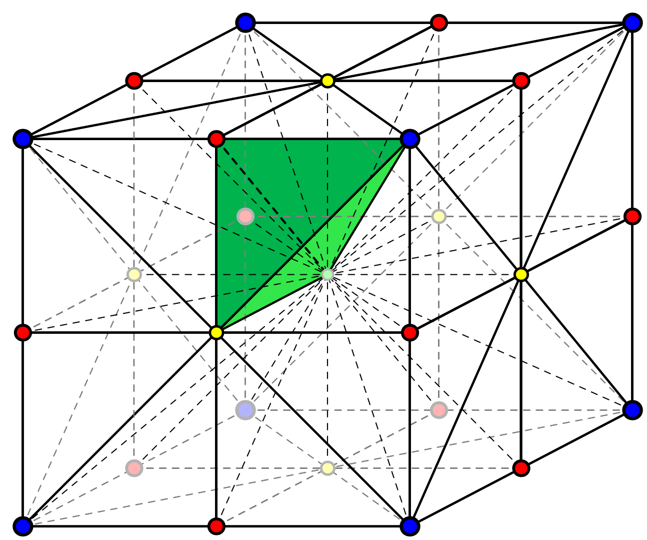




Schläfli Symbols

Cor: The vector arrangement $\{\vec{n_i}\}$ can be summarized with a list of numbers called its **Schläfli symbol**.

Ex: The cube is described by the list $\{4, 3\}$ since $\vec{n}_0$ and $\vec{n}_1$ form a $3\pi/4$ angle and $\vec{n}_1$ and $\vec{n}_2$ form a $2\pi/3$ angle.



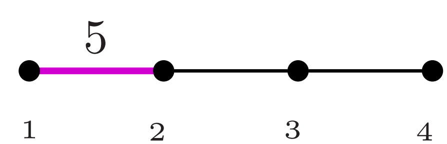
Common name	**Schläfli symbol**	**Cartan-Killing type**
n-simplex	$\{3^{n-1}\}$	A_n
n-orthoplex	$\{3^{n-2}, 4\}$	B_n
n-cube	$\{4, 3^{n-2}\}$	B_n
4-simplex	$\{3, 3, 3\}$	A_4
4-orthoplex	$\{3, 3, 4\}$	B_4
4-cube	$\{4, 3, 3\}$	B_4
24-cell	$\{3, 4, 3\}$	F_4
600-cell	$\{3, 3, 5\}$	H_4
120-cell	$\{5, 3, 3\}$	H_4
tetrahedron	$\{3, 3\}$	A_3
octohedron	$\{3, 4\}$	B_3
cube	$\{4, 3\}$	B_3
icosahedron	$\{3, 5\}$	H_3
dodecahedron	$\{5, 3\}$	H_3
m-gon	$\{m\}$	$I_2(m)$

Dynkin Diagrams

Dynkin Diagrams contain the same information as Schläfli's lists, but in a graphical form.

Draw a row of dots that represent the basic reflections r_0, r_1, etc. Connect the adjacent dots and label them by Schläfli numbers, omitting all the 3s.

Ex: For example, the 120-cell has Schläfli symbol $\{5, 3, 3\}$ and Dynkin diagram:



Vector Arrangements and Positive Definite Matrices

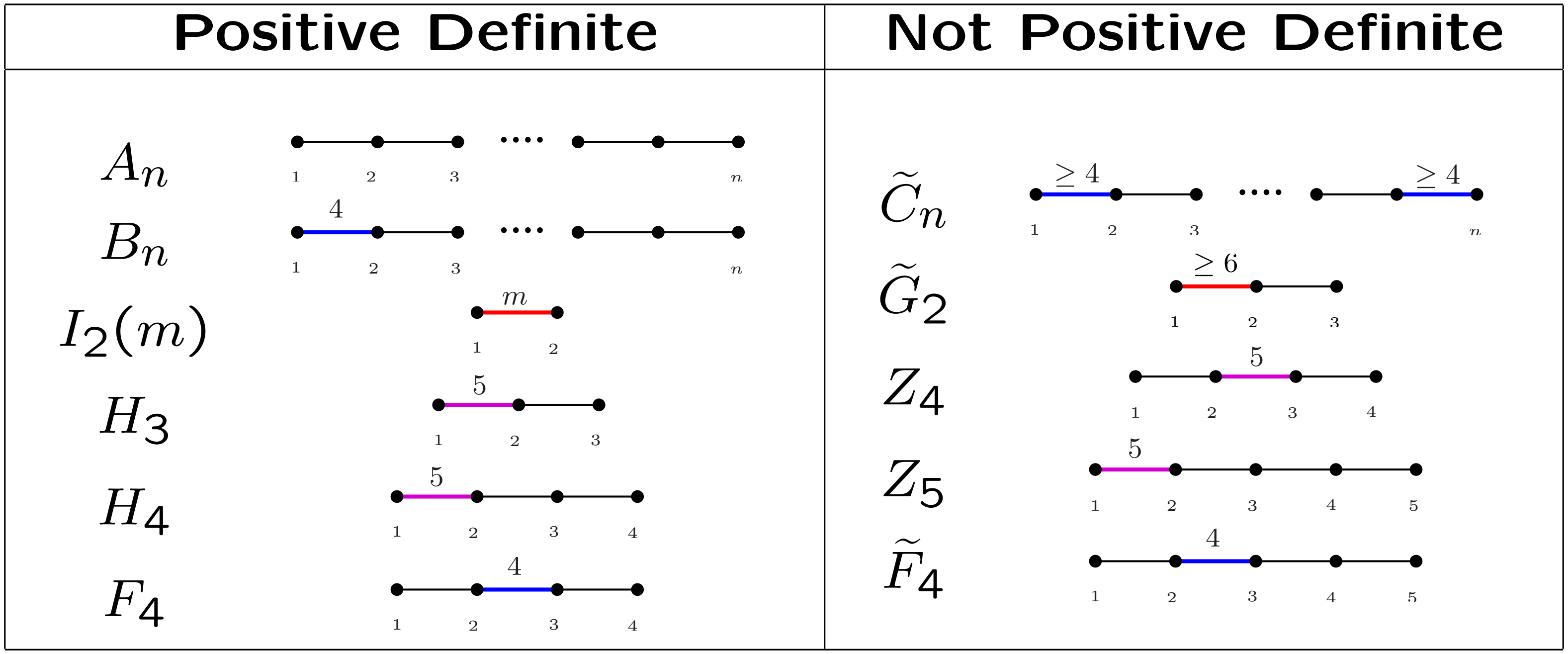
Thm: If $\{\vec{n}_i\}_{i \in [n]}$, is a set of linearly independent vectors in $\mathbb{R}^n$, then the real symmetric matrix $M = [\vec{n}_i \cdot \vec{n}_j]_{(i,j)}$ is positive definite. Conversely, if M is a real symmetric positive definite matrix, then there is an ordered n-tuple of linearly independent vectors in $\mathbb{R}^n$ (unique up to isometry) whose dot products are described by M.

It is easy to determines whether a matrix is positive definite.

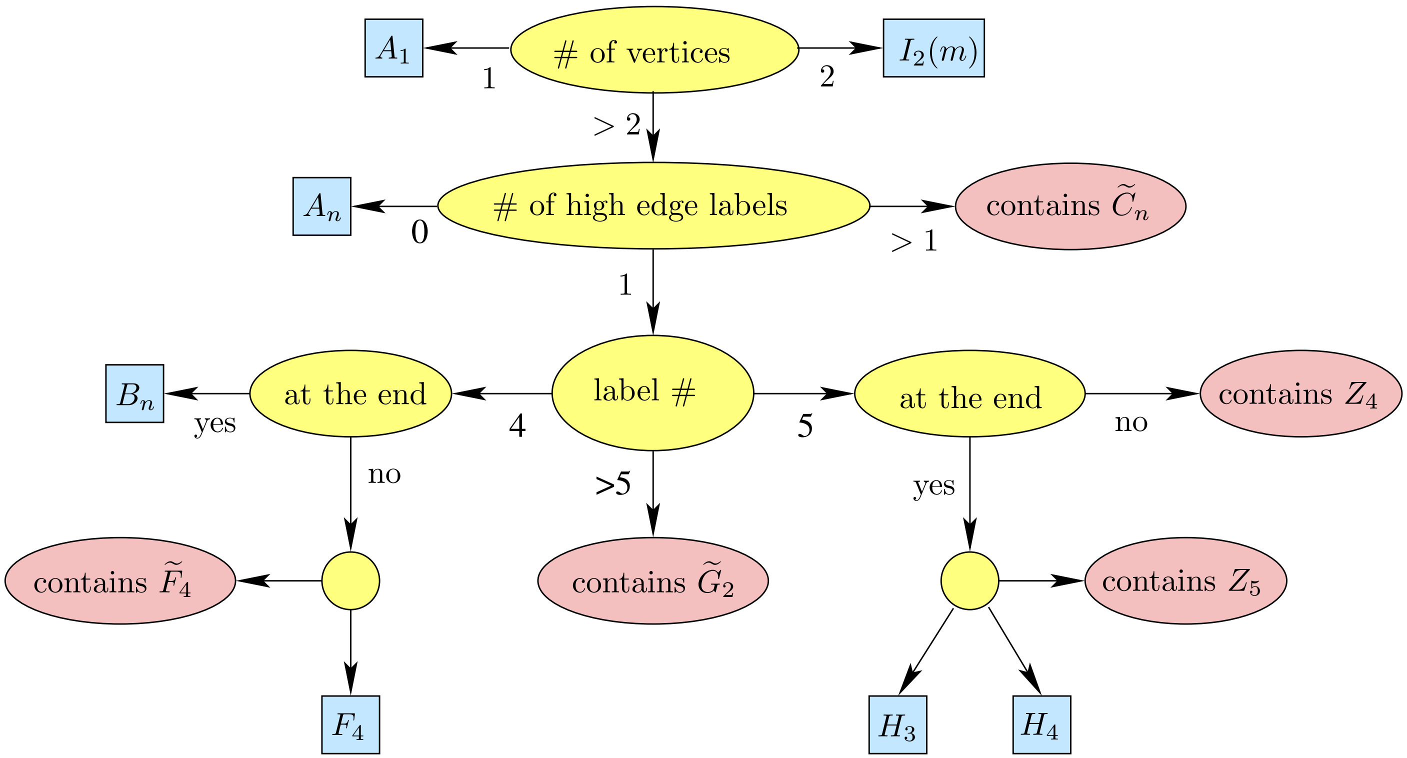
Prop: An $n \times n$ matrix is positive definite if and only if each of its principal minors has a positive determinant.

Cor: Dynkin diagrams of regular polytopes cannot contain Dynkin diagrams of non-examples.

Examples and Non-examples



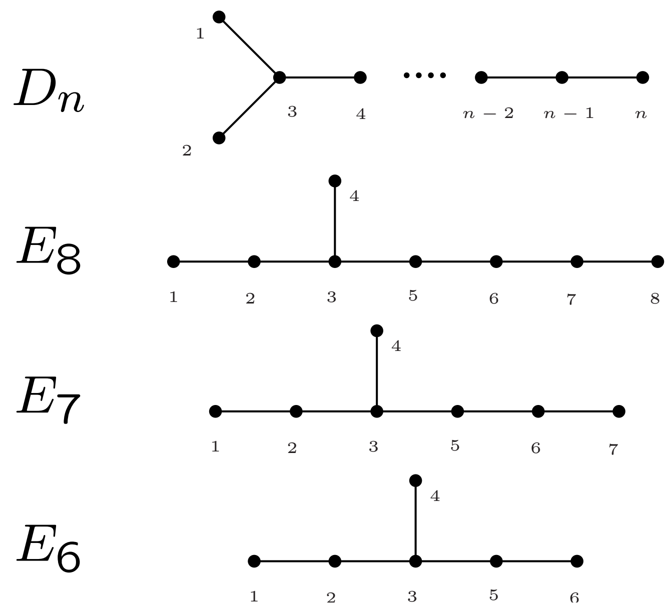
The Proof



# of vertices
A_1
1
2
$I_2(m)$
> 2
# of high edge labels
A_n
0
> 1
contains $\widetilde{C}_n$
1
label #
at the end
B_n
yes
4
no
5
at the end
no
contains Z_4
yes
>5
contains $\widetilde{F}_4$
contains $\widetilde{G}_2$
contains Z_5
F_4
H_3
H_4

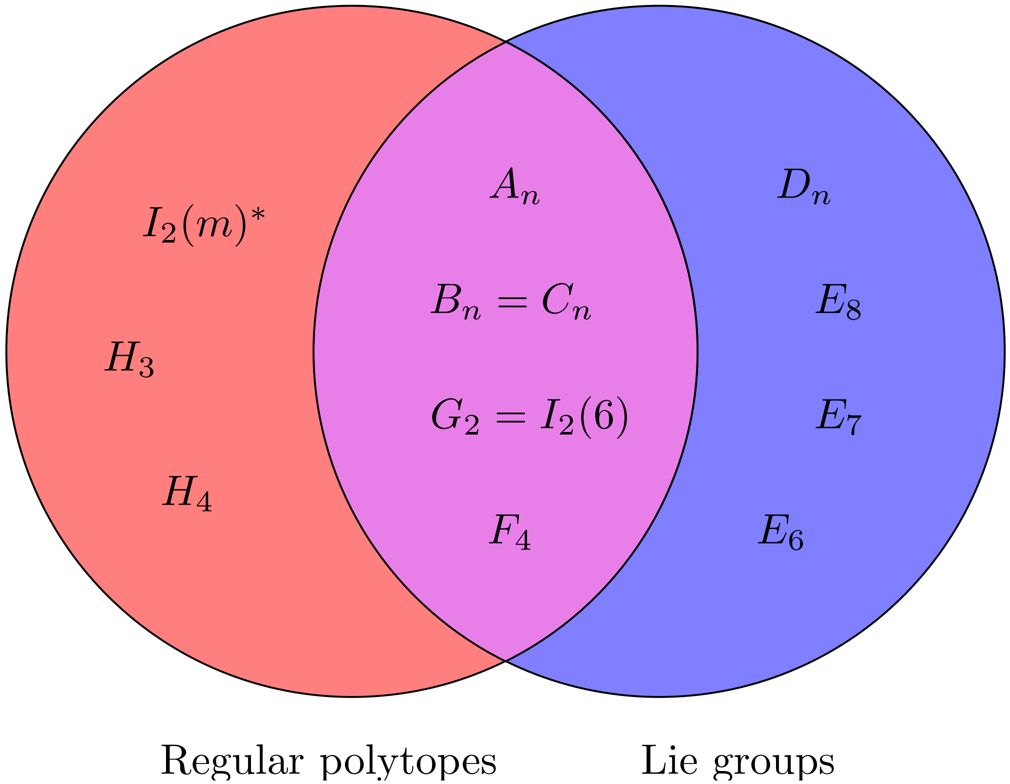
Other Finite Reflection Groups

If we broaden our perspective to study **all** finite groups generated by reflections, then there are additional examples.



The new examples are clearly not from regular polytopes since their Dynkin diagrams branch. The classification proof is similar.

Finite Coxeter Groups = Finite Reflections Groups



(The $*$ means that $m \neq 3, 4, 6$)

Quaternions and Octonions

Many of the finite reflection groups are closely tied to the quaternions and octonions. [Conway-Smith] [Baez]

• $I_2(m)$, H_3 and H_4 are finite subgroups of the quaternions.
• the Lie group of type G_2 is Aut($\mathbb{O}$).
• the Lie group of type F_4 is Isom($\mathbb{O}P^2$).
• the affine reflection groups of type D_4 and E_8 are closely related to the ring of integers in the quaternions and octonions, respectively, and E_6 and E_7 correspond to important subrings.