Coxeter groups and Artin groups

Day 3: Pulling Apart Orthogonal Groups

$$\begin{array}{ccccc} \textbf{?} & \hookleftarrow & \textbf{Artin} & \supset & \textbf{Braid}_n \\ \downarrow\!\!\!\!\downarrow & & \downarrow\!\!\!\!\downarrow & & \downarrow\!\!\!\!\downarrow \\ \textbf{Lie} & \hookleftarrow & \textbf{Coxeter} & \supset & \textbf{Sym}_n \end{array}$$

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Continuous Braid Groups

Today's goal is to describe one way to complete this diagram:

$$\begin{array}{ccccc} \textbf{?} & \hookleftarrow & \textbf{Artin} & \supset & \textbf{Braid}_n \\ \twoheaddownarrow & & \twoheaddownarrow & & \twoheaddownarrow \\ \textbf{Lie} & \hookleftarrow & \textbf{Coxeter} & \supset & \textbf{Sym}_n \end{array}$$

The mystery groups would complete the sentence, "Symmetric groups are to braid groups as Lie groups are to (blank)."

Rem: Lie groups are not even countably generated so neither are the mystery groups.

Q: Why should we try and construct such a thing?

Why

Every Coxeter group can be faithfully represented as a reflection group acting properly discontinuously by isometries on some symmetric space (finite=positive, infinite=non-positive curved).

It seems to me that this (directly and indirectly) leads to

- Tits' solution to the word problem
- the Davis Complex with with the Moussong metric
- ...all other nice properties of Coxeter groups.

In short, the good geometry is the ultimate source of the good computational and algorithmic properties of Coxeter groups.

A similar good geometry might help us understand Artin groups.

Continuous Groups

Even if G is a continuous group such as the Lie group $O(n)$, we can still treat G as though it were an infinite discrete group.

- What subsets generate G?
- How can it be presented?
- What cell complexes is it the fundamental group of?
- Can we put metrics on them and use the geometry of its universal cover to better understand the group?

Sample Question: How should we visualize the free product of $\mathbb{S}^1$ and $\mathbb{S}^1$?

Longitude metric on $\mathbb{S}^2$

Consider the 2-sphere with the longitude metric (or the Paris-New Zealand metric).

- What is its fundamental group?
- What is its universal cover?

The universal cover is related to the complex one would want to consider for the free product of $\mathbb{S}^1$ and $\mathbb{S}^1$.

Moreover, this group, with the extra restrictions deserves to be called $\mathsf{FC}^{+}_{\mathbb{S}^1}$.

[Blackboard]

Cayley Graph of $O(2)$

Everyone learns that the group of Euclidean motions are generated by reflections but have you ever wondered what the Cayley graph for $\mathrm{Isom}(\mathbb{R}^n)$ looks like with respect to this set?

Consider the Cayley graph of $O(2)$ with respect to the set of reflections. The result is a spherical join of two circles which looks like a 3-sphere with a strange metric structure. What is a presentation for $O(2)$?

$$O(2) = \langle r_\alpha \in \mathbb{R}P^2 \mid r_\alpha^2,\ r_\alpha r_{\alpha+\gamma} = r_\beta r_{\beta+\gamma}\ \forall \alpha, \beta, \gamma \in \mathbb{R}P^2 \rangle$$

It's not quite a continuous Coxeter group. [Blackboard]

Decision Problems and Computability

Q: Is the word problem decidable for the matrix group $\mathrm{GL}_n(\mathbb{K})$?

A: Of course. Regardless of what generating set you consider, we only need to multiply the matrices and check whether or not the product is equal to identity matrix.

When $\mathbb{K} = \mathbb{R}$ there is an issue of whether we can input, export, add, subtract, multiply and test equality of real numbers, but notice that this is the ONLY issue.

Thus the word problem is decidable modulo a black box that handles field operations such as these.

Computations in Coxeter Groups

In addition to the proof using the decidability of matrices over the algebraic closure of $\mathbb{Z}$, Tits gave a much simpler solution to the Word Problem.

Thm: Every element in a Coxeter group W can be shortened to a geodesic in a non-length increasing way using only rewriting rules of the form $s^2 \to 1$ and $(sts\cdots) \leftrightarrow (tst\cdots)$ where both sides have length $m = m(s,t)$.

The linear representation is crucial in the early stages of the proof. In particular, without the linear representation there is no way to see that there exists a group where the order of st is really m (when $(st)^m$ is a relation) and not some proper divisor.

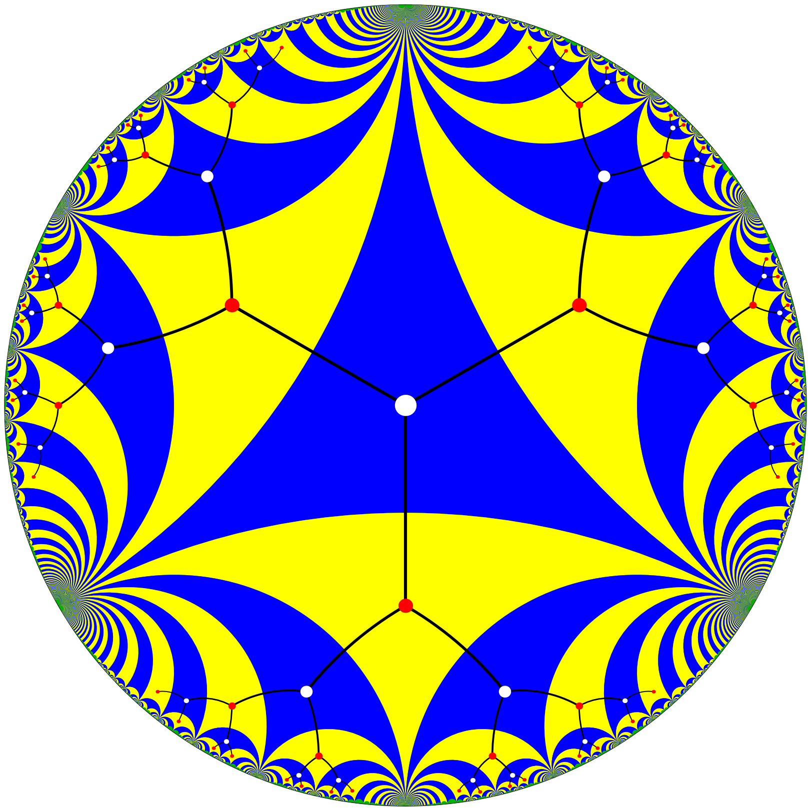
Analyzing a Coxeter Group

Steps towards understanding a Coxeter group geometrically:

- Diagram
- Presentation
- Matrix
- Representation
- Type
- Symmetric Space
- Reflecting Hyperplanes
- Cayley Graph
- Davis Complex
- Moussong Metric

[Blackboard]

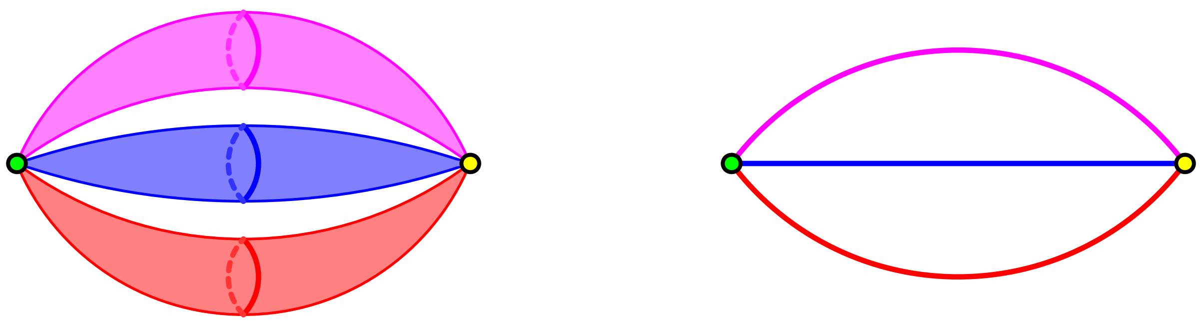
Davis Complex with the Moussong Metric



(This isn't the right picture but it's the one I have at hand)

Cayley Graphs

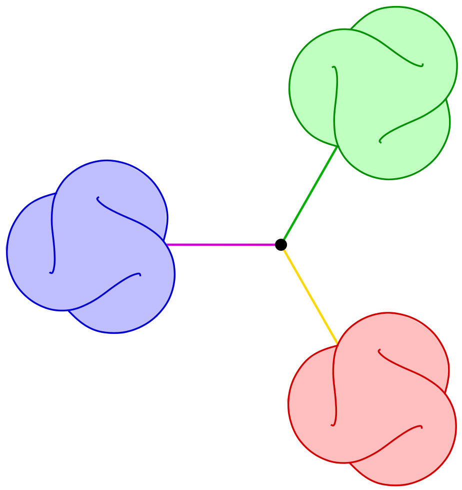
If G is a group, then any connected graph Γ with a free and vertex-transitive G-action is called a **Cayley graph for** G. For Coxeter groups W we typically do not draw the *real* Cayley graph on which W acts *freely*. Rather, we draw a graph that only has a *proper* action. Define $W^+ := \mathrm{Ker}(W \to \mathbb{Z}_2)$.



If FC_n denotes the free Coxeter group with n generators, then the fundamental group of the graph on the left is FC_3^+.

Free Coxeter Group: Base Complex

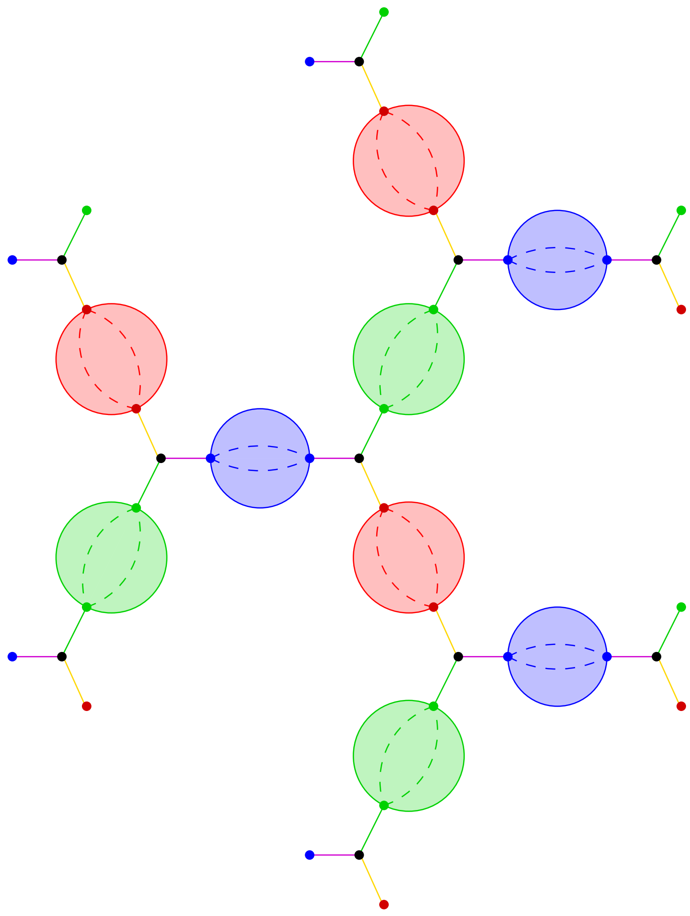
The free Coxeter group is a free product of $\mathbb{Z}_2$'s so it is the fundamental domain of a wedge product of $\mathbb{R}P^2$'s.



(These are supposed to be three copies of Boys' surface)

Boys' Surface

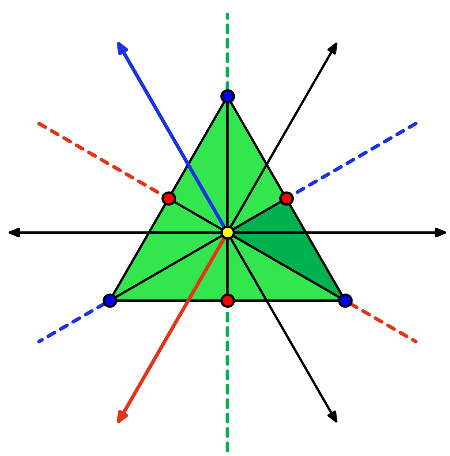
Free Coxeter Group: Universal Cover

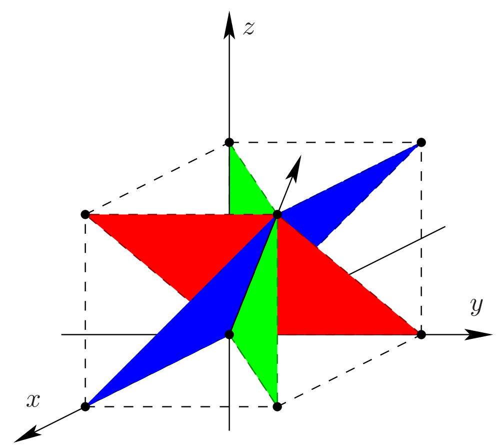


Real Braid Arrangement

The (real) **braid arrangement** is the space of all n-tuples of distinct real numbers $(x_1, x_2, \ldots, x_n)$.

Ex: In $\mathbb{R}^2$ this consists of the stuff above the line $y = x$ and the stuff below the line $y = x$. In $\mathbb{R}^3$ it consists of 6 connected pieces separated by the planes $x = y$, $x = z$ and $y = z$. In $\mathbb{R}^n$ it has $n!$ connected pieces separated by the hyperplanes $\{x_i = x_j\}$.



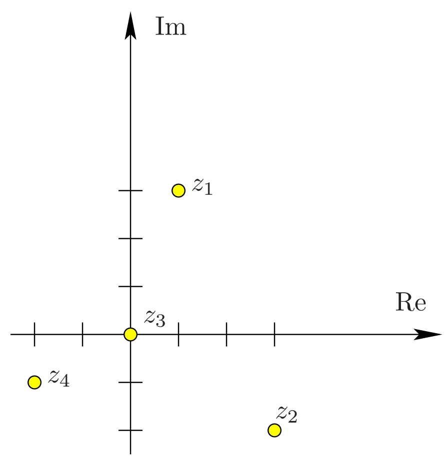


Complexified Braid Arrangement

The **complexified braid arrangement** is the space of all n-tuples of distinct complex numbers $(z_1, z_2, \ldots, z_n)$. There's a trick to visualizing this space.

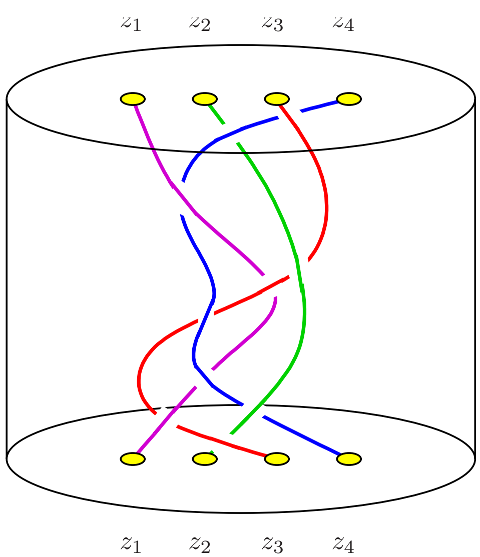
Ex: The figure below encodes the point

$$(z_1, z_2, z_3, z_4) = (1 + 3i, 3 - 2i, 0, -2 - i)$$



Braids

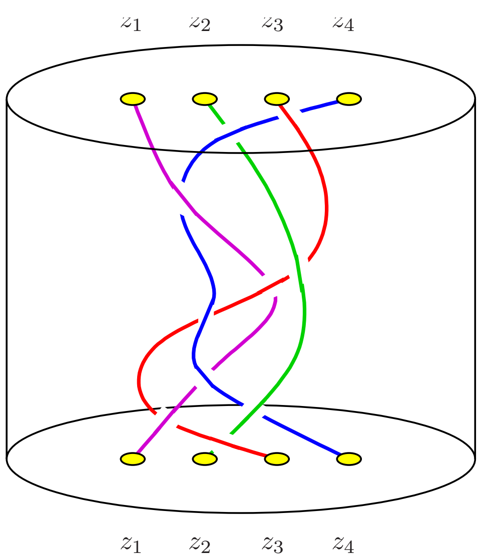
Paths in the complexified arrangement move the labeled points in the complex plane without letting them collide.



If we keep track of this movement by tracing out what happens over time we see actual braided strings—hence the name. The path shown is a non-trivial closed loop.

(Pure) Braid Groups

The fundamental group of the complexified braid arrangement is the **pure** braid group.



To get the ordinary braid group we quotient by the (free) action of the symmetric group on the hyperplane complement.
Exer: Do the same thing for type B_n.

(Pure) Artin Groups

The same idea works for all finite Coxeter groups and can be extended to all Coxeter groups by analogy. Every resulting presentations fits the following pattern: for every pair of distinct generators $s, t \in S$ there is at most one relation of the form $(sts \cdots) = (tst \cdots)$ where both sides have m letters.

Ex: $\langle a, b, c \mid aba = bab, ac = ca \rangle$

The groups with these types of presentations are called **Artin groups** (or **Artin-Tits groups**).

Despite their simple presentations, they are generically quite mysterious groups. Note that this does NOT extend to the continuous setting.

Pulling Apart Groups

Let G be a group, let S be a generating set and let $g \in G$ be represented by a word in S^+.

Set $M := \{w \in S^+ \mid [w] = g \text{ and } |w| \text{ is minimal}\}$.

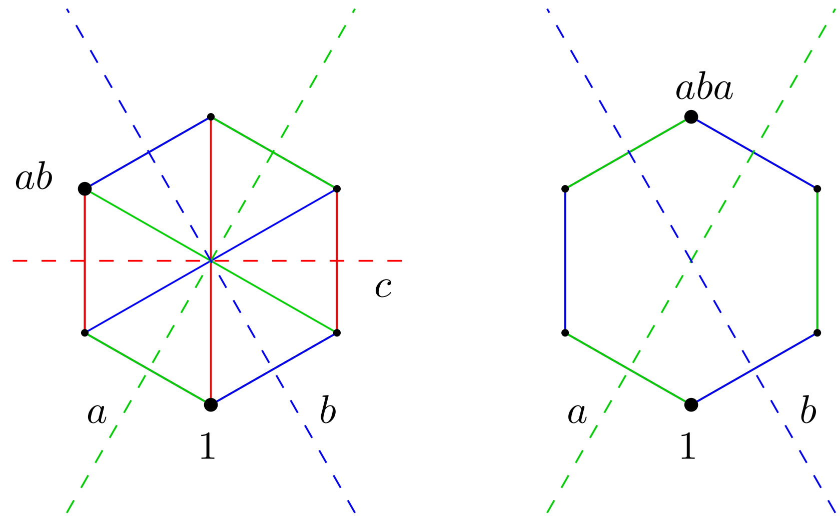
Define $\widehat{G}$ by the presentation $\langle S \mid u = v,\ \forall u, v \in M\ \rangle$.

Rem: There are natural maps $\widehat{G} \twoheadrightarrow G$ and $\widehat{G} \twoheadrightarrow \mathbb{Z}$ that extend identity:$S \to S$ or constant:$S \to 1$.

Rem: Since this is such a general construction, we expect it to be useless. The surprising thing is how often it gives an interesting answer.

An Easy Example

Let $G = \text{Sym}_3$ and set $a = (12)$, $b = (23)$, and $c = (13)$.
$\widehat{G}(\{a,b\}, aba) = \langle a, b \mid aba = bab \rangle$
$\widehat{G}(\{a,b,c\}, ab) = \langle a, b, c \mid ab = bc = ca \rangle$



In both cases, $\widehat{G} = \text{Braid}_3$, the 3-string braid group.

The Right Set of Generators

Let G, S, g and M be as above.

Def: Call S **weakly closed** if $a, b \in S$ and ab a subword of an element of M, implies $c = aba^{-1}$ and $d = b^{-1}ab$ are also in S.

When S is weakly closed we can replace ab with ca or bd and find another word in M (i.e. we can move letters around).

Ex: A generating set closed under conjugacy is weakly closed, but smaller sets can also be weakly closed.

Given G, S, and g we can try and find the weak closure of S. This works so long as the length of g does not go down.

More Examples

G	S	g	$\widehat{G}$
$(\mathbb{Z}_2)^n$	basis	$(1, 1, \ldots, 1)$	$\mathbb{Z}^n$
Sym_n	basic transpositions	full flip	Braid_n
Sym_n	all transpositions	n-cycle	Braid_n
finite Coxeter	basic reflections	longest elt.	finite-type Artin
finite Coxeter	all reflections	Coxeter elt.	finite-type Artin
$FC_n = *\mathbb{Z}_2$	weak closure	$x_1 x_2 \cdots x_n$	$\mathbb{F}_n$

"**Thm:**" If W is a Coxeter group generated by its full set of reflections and we pull W apart at one of its Coxeter elements, then the resulting group is the corresponding Artin group.

[quotes have been added due to a glitch in the proof]

Functorality

Prop: Let (G, S, g) and (H, T, h) be groups, generating sets and elements. If $\phi : G \to H$ is a group homomorphism such that $\phi(S) = T$, $\phi(g) = h$ and $|g|_{S+} = |h|_{T+}$, then there is a homomorphism $\widehat{\phi} : \widehat{G} \to \widehat{H}$.

Cor: Pulling twice is the same as pulling once.

This is the key idea used in our attempt to show that the pulled Coxeter group $\widehat{W}$ is always equal to the corresponding Artin group A.

Having this equality makes more of the Artin group amenable to computation since we can compute in W.

Pulling Apart $O(2)$

- The set of all factors of a rotation into a pair of reflections looks like a suspension of a circle.

- The universal cover of the complex for $\widehat{O}(2)$ is an $\mathbb{S}^1$-branching tree cross the reals (an $\mathbb{R}$-tree).

- The action on the cross section has cyclic stabilizers.

- There may or may not be central elements depending on whether or not the original rotation is a rational multiple of π.

Returning to $O(n)$

The group $O(n)$ can also be generated by reflections and the poset of factors below a maximal length element can be analyzed. In this case the poset is a lattice and the result is a continuous Garside structure.

The resulting group is new, rather strange, but well-behaved. The corresponding complex admits a metric of non-positive curvature, is a $K(\widehat{G},1)$ and its universal cover is a topologically an $\widetilde{A}_{n-1}$-building cross the reals. The word problem is decidable and the elements have computable Garside normal forms.

Moreover, if we choose as our α the image of the Coxeter element of the symmetric group under the standard embedding, then this Garside structure contains the dual Garside structure for the braiid group thereby embedding Braid_n into $\widehat{G}$.