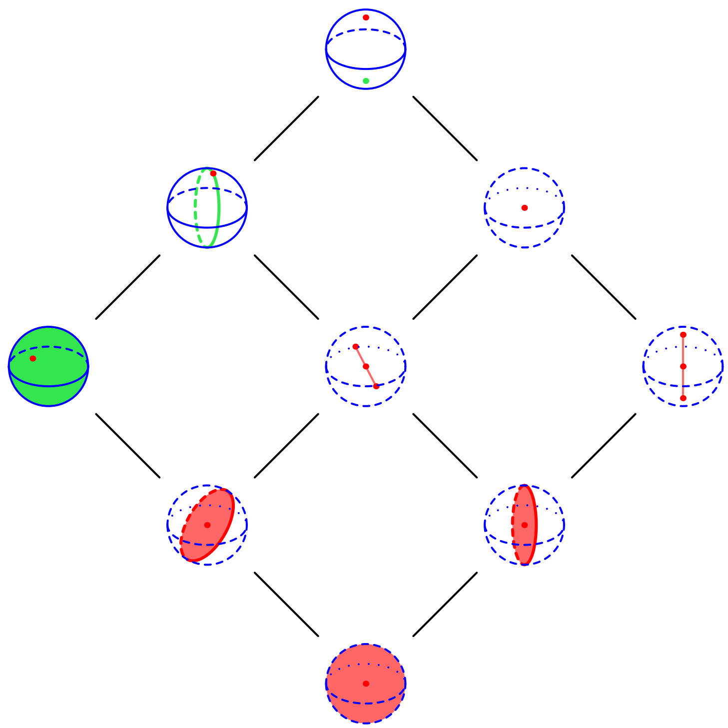
Coxeter groups and Artin groups

Day 4: Affine Isometries and Artin Groups



Jon McCammond (U.C. Santa Barbara)

Factoring Motions into Reflections

In this final talk, I'd like to analyze all possible ways to factor a spherical or affine isometry into a minimal number of reflections. There are many similarities between the two cases; the spherical case is essentially a warm-up for the affine case.

Main Question: What is the global structure of this poset and is it locally a lattice?

When an interval in this poset is a lattice, then pulling the isometry group apart at this element results in a Garside structure and a Garside group.

Bipartite Cayley Graphs

Because reflections are orientation reversing, for any symmetric space X^n, the Cayley graph of Isom(X^n) with respect to reflections is a bipartite graph. In particular, every edge has one endpoint closer to the origin and every edge occurs in a minimal length factorization of some element.

Said differently, if we pick up the Cayley graph at a point and shake, then the resulting graded graph is the poset we're interesting in.

Ex: The Cayley graph for $O(2)$ is an easy to visualize example.

[Blackboard]

Subspace posets

For convenience we first define several elementary posets.

Def: Let X^n be either $\mathbb{S}^n$, $\mathbb{R}^n$ or $\mathbb{H}^n$. There are many isometrically embedded copies of X^k inside X^n that we order by reverse inclusion ($X \leq Y$ iff $X \supset Y$). Call the resulting poset $\mathsf{Sub}(X^n)$. The space X^n is the minimal element in $\mathsf{Sub}(X^n)$.

The subspaces of $\mathbb{R}^n$ that go through the origin (i.e. the linear ones) form a subposet of $\mathsf{Sub}(\mathbb{R}^n)$ that we call $\mathsf{Lin}(\mathbb{R}^n)$. Notice that $\mathsf{Lin}(\mathbb{R}^n) = \mathsf{Sub}(\mathbb{S}^{n-1})$ (if we include $\mathbb{S}^{-1}$) and that the origin is its unique maximal element.

These posets are locally lattices. Joins are intersections and meets are spans of unions.

Up and Down

Let α be a motion and let r be a reflection.

Lem: If $r\alpha$ fixes a point not fixed by α, then $\mathrm{Fix}(\alpha) \subset \mathrm{Fix}(r)$.

Proof: [Picture]

Cor 1: If $\mathrm{Fix}(\alpha) \not\subset \mathrm{Fix}(r)$ then no new fixed points, $\mathrm{Fix}(r\alpha) = \mathrm{Fix}(r) \cap \mathrm{Fix}(\alpha)$, and $\mathrm{codim}(\mathrm{Fix}(r\alpha)) = \mathrm{codim}(\mathrm{Fix}(\alpha)) + 1$.

Cor 2: If $\mathrm{Fix}(\alpha) \subset \mathrm{Fix}(r)$ then there are new fixed points and $\mathrm{codim}(\mathrm{Fix}(r\alpha)) = \mathrm{codim}(\mathrm{Fix}(\alpha)) - 1$.

Reflection Length

From these corollaries, we find:

Prop: The word length of α with respect to reflections is equal to codim(Fix(α)).

Proof: $|\alpha| \geq$ codim(Fix(α)) since k reflections have at least $n-k$ dimensions that they all fix. Conversely, $|\alpha| \leq$ codim(Fix(α)) since given a motion that fixes $n-k$ dimensions we can find a length k path to the identity by fixing an additional direction each time.

Cor: A product of reflections is a minimal factorization of their product iff their normal vectors are linearly independent.

The Combinatorial Model: Spherical version

Thm (Sherk,T.Brady-Watt) If α is a fixed point free isometry of the n-sphere, then the poset of minimal factorizations of α into reflections is isomorphic to Lin($\mathbb{R}^{n+1}$). In particular, the isomorphism is given by sending each $\beta \leq \alpha$ to Fix(β).

Rem: For each linear subspace Y in $\mathbb{R}^n$ there are several motions β in Isom($\mathbb{S}^n$) with Fix(β) $= Y$. The point is that there is one and only one of these β's that lies under α.

[Blackboard]

Lattice and Labels

Q: Is the interval below α in Isom($\mathbb{S}^n$) a lattice?

A: Yes. It is isomorphic to some Lin($\mathbb{R}^k$) and Lin($\mathbb{R}^k$) is a lattice.

Rem: Notice that for every α with $|\alpha| = k$, the structure of the factorization poset is the same. The differences only become apparent once we remember the edge labels.

Cor: For every α, the orthogonal group $\widehat{O}(n)$ pulled apart at α has a Garside structure.

Next, we turn to understanding Isom($\mathbb{R}^n$) and affine Artin groups which is joint work with Noel Brady and John Crisp.

Classifying Isometries

The **translation length** of an isometry α is the infimum of the distances that points are moved. If it is positive, then α is called **hyperbolic**. If α has a fixed point it is called **elliptic**. If α does not fix a point but its translation length is zero, then it is called **parabolic**.

Rem: Parabolic isometries can only occur in hyperbolic and higher rank symmetric spaces. All spherical and affine isometries are either elliptic or hyperbolic.

Sometimes elliptic and hyperbolic isometries are lumped together under the heading *semisimple*.

Minsets

If α is a hyperbolic isometry on a complete non-positively curved space, then its translation length is achieved by some point and the **Minset** of α is the set of all such points.

This is the hyperbolic isometry analog of the fixed set of an elliptic isometry.

Rem: In $\mathbb{R}^n$ the Minset of a hyperbolic isometry is always an affine subspace on which α acts by translation. [Bridson-Haefliger]

Pure Translations

Def: A **pure translation** is one that is the product of two distinct parallel reflections.

Prop: Every hyperbolic isometry of $\mathbb{R}^n$ has a **unique** factorization into an elliptic isometry and a commuting pure translation. Moreover, the fixed set of the elliptic is equal to the minset of the hyperbolic.

[Picture]

(this is a special case of a general Lie group factorization result)

Cor: A hyperbolic isometry α of $\mathbb{R}^n$ can be factored into $\text{codim}(\text{Minset}(\alpha)) + 2$ reflections.

Sphere at Infinity

One of the ways to utilize the spherical results in the affine setting is to notice that every affine isometry has a well-defined motion of the sphere at infinity.

From this perspective we can show that the reflection length of a hyperbolic isometry is exactly equal to codim(Minset(α)) $+ 2$.

Prop: If α is an isometry of maximal length (hyperbolic and minset is a line), then $r\alpha$ is elliptic iff the normal vector of α is not perpendicular to the direction of the minset.

[Blackboard]

Nearly independent

Def: Call a set of k vectors **nearly independent** if its span has dimension $k-1$.

Prop: A set of k distinct affine reflections form a minimal factorization of a hyperbolic isometry iff they have no common fixed point and their normal vectors form a nearly independent set.

The Grid

We think of the factorization of δ as ordered pairs (α, β) and we separate out the pairs of the form (hyp,ell) (ell,ell), and (ell,hyp). The form (hyp,hyp) is ruled out since it is easily seen to not be a minimal factorization.

Within each type stratify by dimenison of the min or fix set.

[Blackboard]

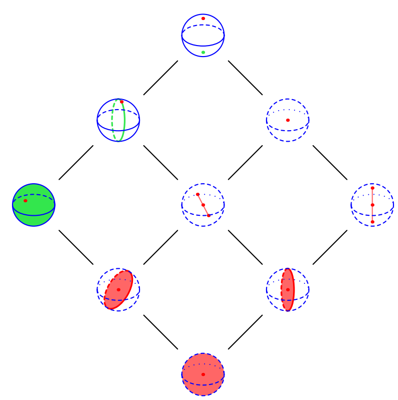
Once we have a combinatorial model, focusing on the second coordinate instead of the first will produce an upside down version of our model - thus proving that the poset is self dual.

The Combinatorial Model: Affine version

• For each elliptic isometry, record its fixed set (or equivalently the boundary sphere of its fixed set at infinity plus a choice of coset inside R^n).

• For each hyperbolic isometry α record the boundary sphere of its minset at infinity, with a red dot added indicating the translation direction on the min set.

Rem: This data alone determines whether $\alpha \leq \beta$.



The Intervals

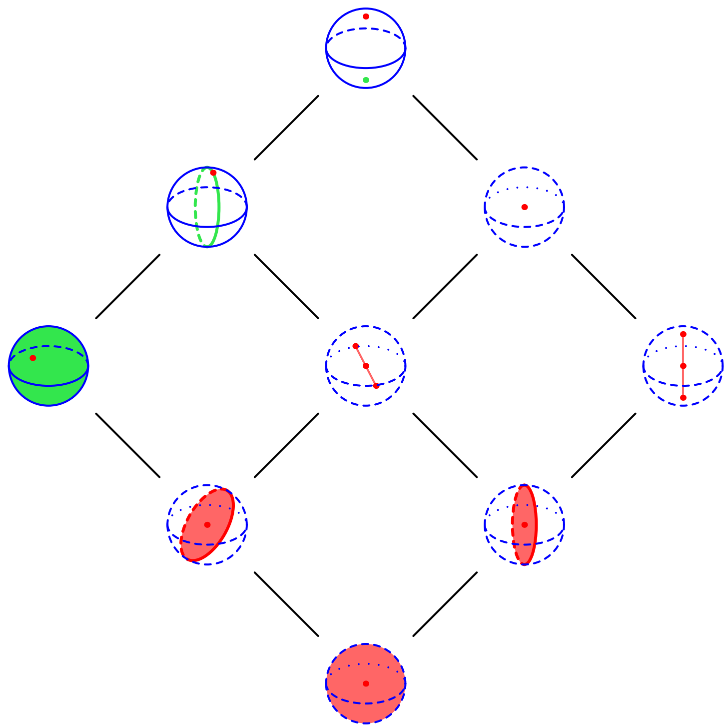
Thm: If δ is a maximal length element of Isom($\mathbb{R}^n$) (hyperbolic with a minset that is a line) and it is arranged so that its translation direction is the north pole, then

- for every possible fixed set Y there exists a unique $\alpha \leq \delta$ with Fix(α) $= Y$, and
- for every possible fixed sphere at infinity S containing the north pole and every possible northern hemisphere red dot x, there exists a unique $\alpha \leq \delta$ with ∂(Minset(α)) $= S$ and translating in the x direction.

[Blackboard]

Structure of the Poset

The structure of the an interval in the affine factorization poset is summarized by the following diagram.
[Extensive live explanations of the structure]



Lattice

Q: Is it a lattice?

A: We can split the proof into 9 cases. In 8 out of 9 the answer is yes, but the final case doesn't work.

The only problem is that there are bowties when we have two distinct hyperbolics and two distinct elliptics below δ with the same fixed sphere at infinity.

[Blackboard]

Dedekind-MacNeille completion

The method of Dedekind cuts that completes the rationals to the reals has been generalized to the context of arbitrary posets.

Every poset P can be uniquely minimally completed to a complete lattice. This result is called the **Dedekind-MacNeille** completion of P.

The continuous affine interval poset has a very simple Dedekind-MacNeille completion. The problem is that the new edges arrive without labels.

Affine Coxeter Groups

When we restrict our attention to only those reflections that occur in an affine Coxeter group, every reflection has only a finite number of possible fixed sphere at infinity. And the translation direction must be a root direction.

Combining these two facts, shows that both the (hyp,ell) row and the (ell,hyp) row are finite.

Cor: There are only a finite number of pairs below δ that do not have a well defined meet or join. In other words, it is nearly a lattice.

Affine Artin Groups

Obs: The factorization poset of affine type is a lattice iff the root system perpendicular the translation direction of the Coxeter element is irreducible.

Not a lattice: A_n (sometimes), B_n, D_n, F_4, E_6, E_7, E_8
Lattce: A_n (sometimes), C_n, G_2.

Moreover, when it is reducible there is a way to extend the generating set to pure translations of 1/2 (or 1/3) the translation length so that the factorization poset over this larger generating set is a lattice and very close to the original generating set.

From this it should follow that we can understand every affine Artin group.

Hyperbolic and Higher Rank: Crawl, Walk, Run

The obvious next step is to investigate the structure of the factorization poset when α is an isometry of hyperbolic space. We have made preliminary steps in this direction, but a number of our previous lemmas break down in the new context.

We have learned to crawl (spherical) and we're learning to walk (affiine) but we still need to learn to run (hyperbolic and above).

Eventually, there should be a uniform proof which establishes the structure of the factorization poset over an arbitrary generalized orthogonal group.

References

Coxeter Groups:

- **Groves-Benson** Finite Reflection Groups (GTM, Springer)
- **Humphreys** Coxeter Groups (CUP)
- **Kane** Coxeter Groups (CMS)
- **Björner-Brenti** Combinatorics of Coxeter Groups (Springer)
- **Davis** Geometry and Topology of Coxeter Groups (webpage)
- **Bourbaki** Lie Groups and Lie Algebras: Ch.4-6

Artin Groups:

- Currently no books on Artin groups are available.
- **Charney** Right-angled Artin groups (recent lecture notes)