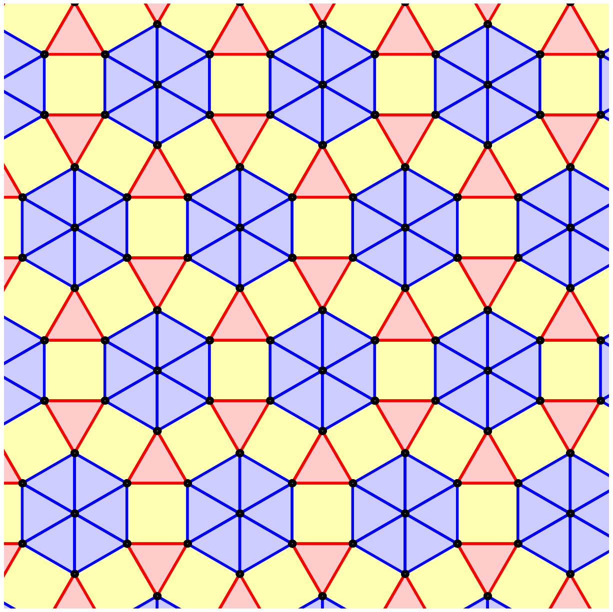
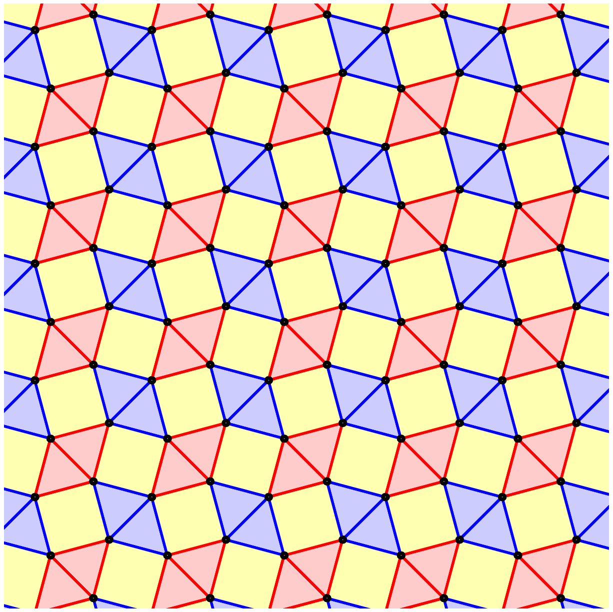
Triangles, Squares and Biautomaticity

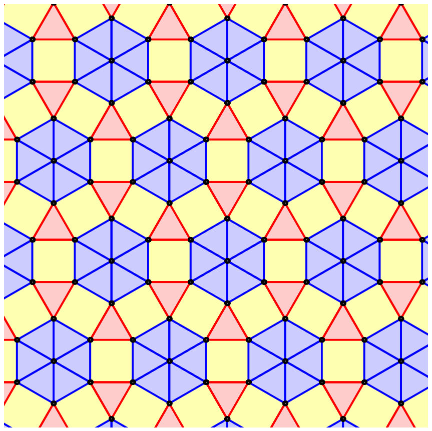
 

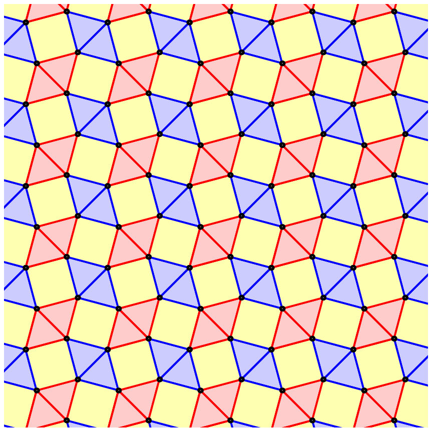
Jon McCammond (U.C. Santa Barbara)
Rena Levitt (U.C. Santa Barbara)

The Basic Problems

Consider a tiling of the plane by regular triangles and squares.

1) How many such tilings are there?
2) What does a 1-skeleton geodesic look like?
3) What does the set of all 1-skeleton geodesics look like?
4) Do 1-skeleton geodesics that start and end close remain close?
5) Given start and end points that are close can one choose 1-skeleton geodesics that remain close throughout?





The Shapes

Throughout this talk I will use the following shorthand notations:

$\triangle$	$=$	triangle or triangular	$\triangle^n$	$=$	simplex or simplicial
$\square$	$=$	square	$\square^n$	$=$	cube or cubical

Thus $\triangle^n$-cplx, and $\square^n$-cplx are simplicial and cubical complexes, respectively, with the PE regular simplex / cube metrics assigned to each cell. Every edge has length 1.

Finally, $\prod \triangle$ and $\prod \triangle^n$ refer to the Euclidean polytopes that are faces of direct products of equilateral triangles / regular simplices, respectively. We call these *triangle product polytopes* and *simplicial product polytopes*. More about these later.

The Curvature Theories

Consider the following theories of curvature:

- NPC=*non-positively curved* = locally CAT(0).
- SNPC=*simplicially non-positively curved* = locally systolic.

The first works well for 2-dim cplxes and for $\square^n$-cplxes.
The second is specifically designed for $\triangle^n$-cplxes.

Rem: For $\triangle$-cplxes, NPC = SNPC

The First Theorems

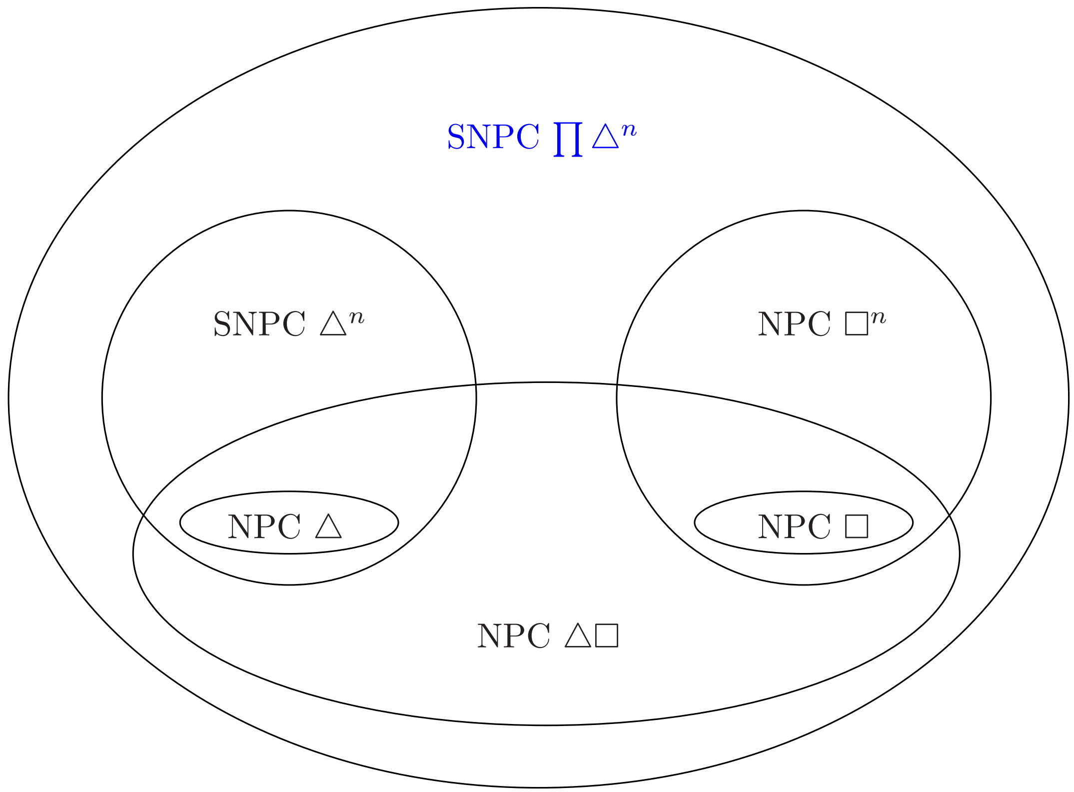
In 1991, Steve Gersten and Hamish Short proved several interesting biautomaticity results including the following:

Thm 1: π_1(cpt NPC $\square$-cplx) is biautomatic.
Thm 2: π_1(cpt NPC $\triangle$-cplx) is biautomatic.

Graham Niblo and Lawrence Reeves generalized Thm 1.
Tadeusz Januszkiewicz and Jacek Świątkowski extended Thm 2.

Thm 3: π_1(cpt NPC $\square^n$-cplx) is biautomatic.
Thm 4: π_1(cpt SNPC $\triangle^n$-cplx) is biautomatic.

The Landscape



Rem: The class of NPC $\prod\triangle$-cplxes (not shown) encompasses both NPC $\triangle\square$-cplxes and NPC $\triangle^n$-cplxes.

Some Conjectures

The parallels between the two theories leads one to wonder whether there is a common underlying theory.

Conj 1: π_1(cpt NPC $\triangle\Box$-cplx) is biautomatic.
Conj 2: π_1(cpt NPC $\prod\triangle$-cplx) is biautomatic.
Conj 3: π_1(cpt SNPC* $\prod\triangle^n$-cplx) is biautomatic.

*suitably defined

Today I'll mostly focus on Conj 1 with a few final comments about Conj 2.

Why Biautomatic?

There are many different major theories that (successfully) try to generalize the negatively-curved geometry of Gromov hyperbolic groups (CAT(0), SNPC, relative hyperbolicity, etc.).

But there has really only been one major theory that generalizes the computational aspects of Gromov hyperbolic groups: the theory of automatic and biautomatic groups.

Thus, when we wish to try and show that some class of groups is computationally well-behaved, it is natural to try and show that every group in that class is biautomatic.

Flats are Crucial

Thm: CAT(0) spaces with no flats are δ-hyperbolic.

Thm: Every Gromov hyperbolic group is biautomatic.

Thm: CAT(0) spaces with isolated flats are hyperbolic relative to their flats.

Thm (Rebecchi) Every relatively hyperbolic group whose peripheral subgroups have prefix closed biautomatic structures is itself biautomatic.

A Revelant Counter-Example

Dani Wise has an example of a CAT(0) group that experts believe is neither automatic nor biautomatic.

The group is the fundamental group of a NPC 2-complex built out of triangles and squares, but the metric used to produce local CAT(0) structure is of necessity not the regular one.

Murray Elder has shown that there does not exist an automatic or biautomatic structure that uses paths that are geodesics in the 1-skeleton metric.

Language Requirements

In order to find a biautomatic structure, it is necessary to select a set of paths that

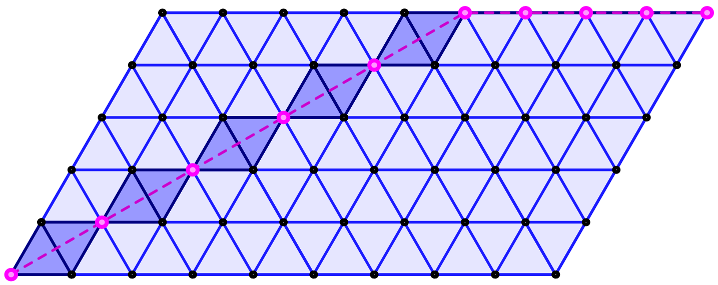
1) can be described by a regular language, and
2) K-fellow travels for some fixed K.

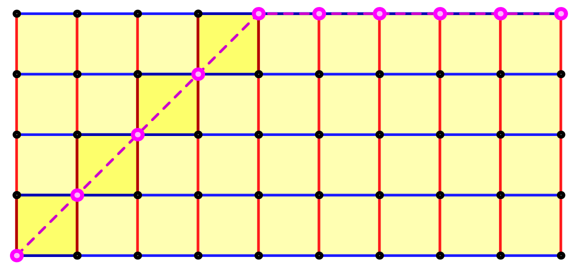
One natural place to look for such paths is among the collection of 1-skeleton geodesics.

Def: For fixed vertices u and v let $E(u,v)$ be the smallest full subcomplex containing u, v and every 1-skeleton geodesic connecting them. We call this an *envelope of geodesics*.

Envelopes and Chosen Paths according to Gersten-Short

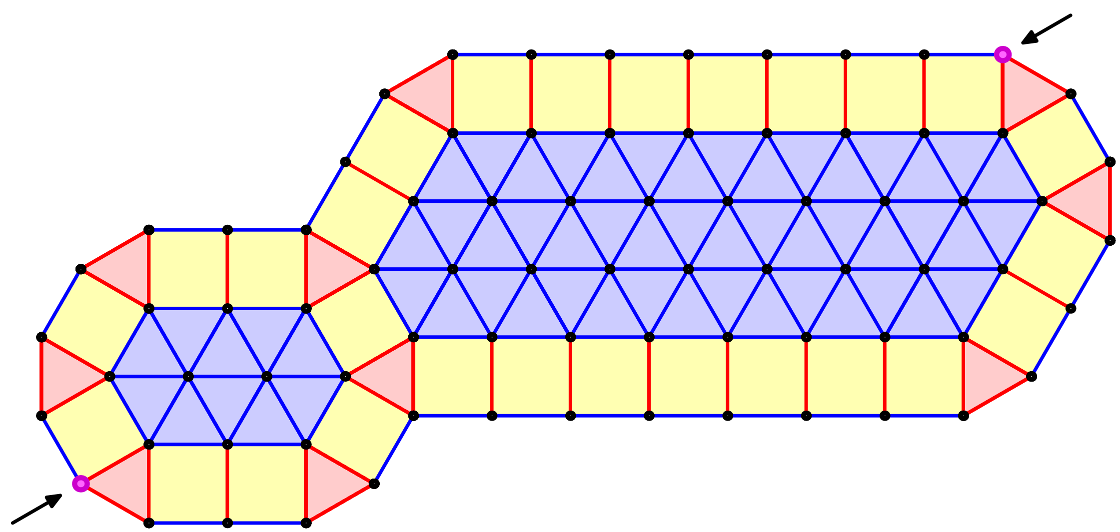
Piecewise Euclidean CAT(0) cplxes without flats are δ-hyperbolic. Thus, the interesting parts of the Gersten and Short paths take place within the flats. Recall that a *flat* is an isometrically embedded copy of $\mathbb{R}^n$ with $n > 1$.





$\triangle$-cplxes and $\square$-cplxes each have only one type of flat plane. A typical envelope of geodesics and the corresponding chosen paths are shown above.

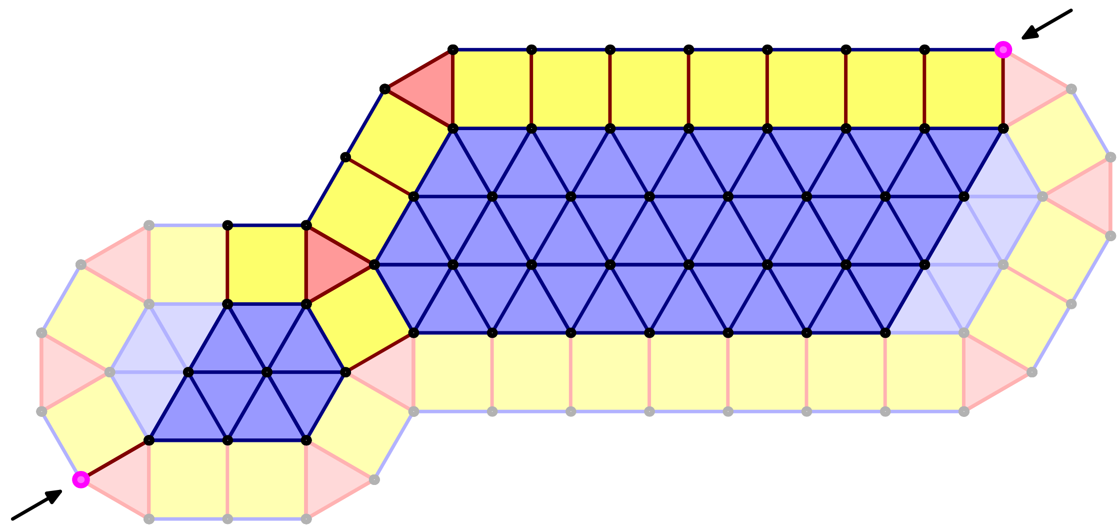
Envelopes and Chosen Paths according to Levitt, I



Consider the △□-cplx shown above.

- What is the shortest path from left to right?
- What is the envelope of all such short paths?
- Which path best generalizes the Gersten and Short path?

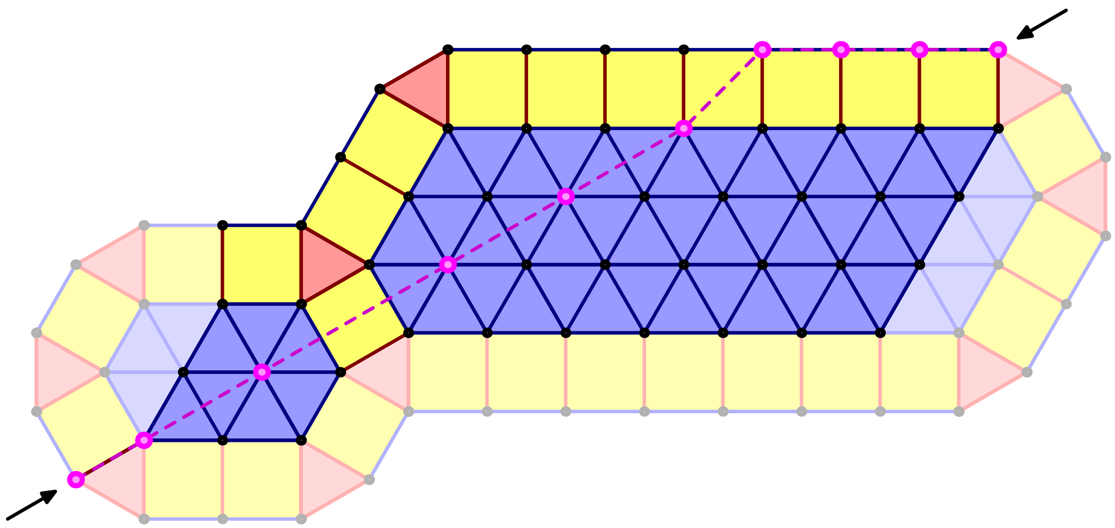
Envelopes and Chosen Paths according to Levitt, II



Consider the △□-cplx shown above.

- What is the shortest path from left to right?
- What is the envelope of all such short paths?
- Which path best generalizes the Gersten and Short path?

Envelopes and Chosen Paths according to Levitt, III

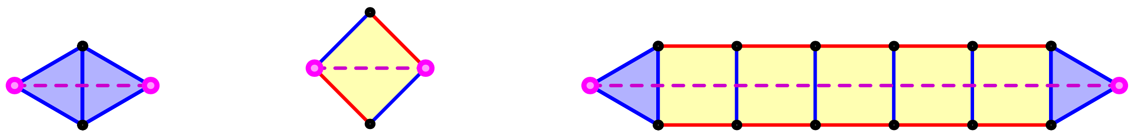


Consider the $\triangle\square$-cplx shown above.

- What is the shortest path from left to right?
- What is the envelope of all such short paths?
- Which path best generalizes the Gersten and Short path?

Stacks and Moves

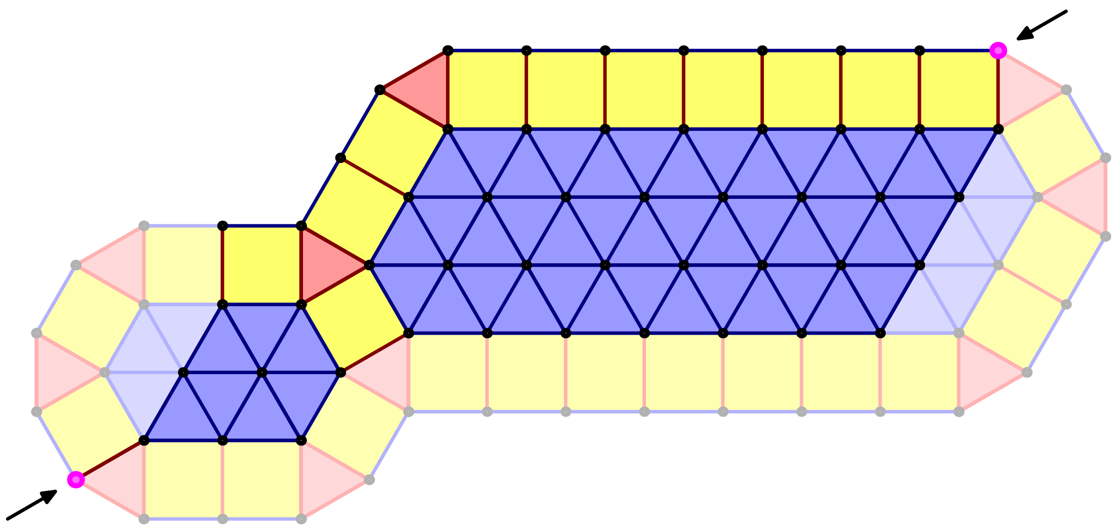
Every face of a triangle or a square has an antipodal face. A *stack* is an alternating sequence of 2-cells and faces where (1) the first and last face are vertices, and (2) the faces before and after the 2-cells are antipodal.



A *move* is the replacement of one geodesic from vertex to vertex with the other one. In addition to the three types of moves shown above, there are two shortening moves.

Key Property: Monotonicity

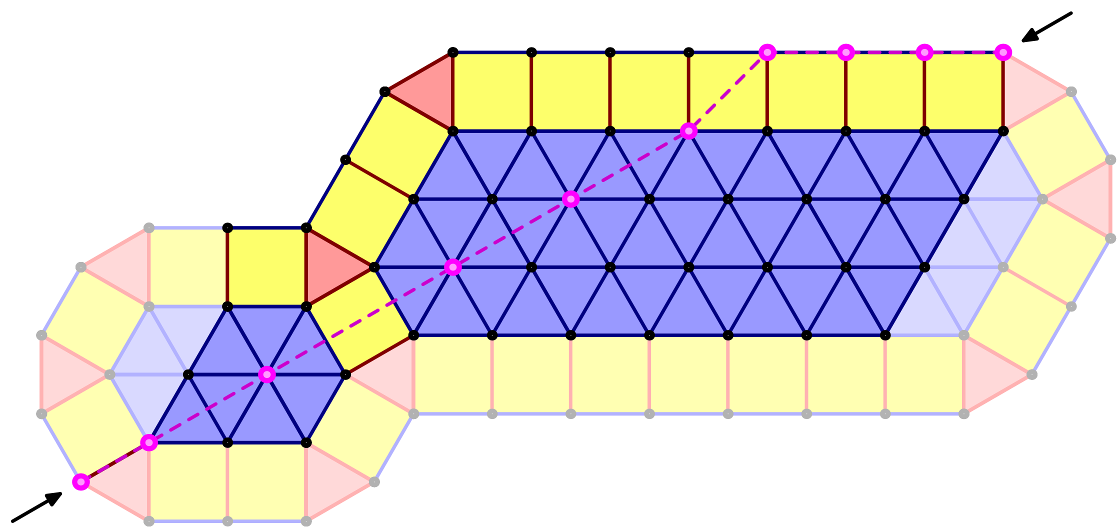
Thm: In a CAT(0) △□-cplx, every path from u to v can be reduced to a geodesic in a monotonic way using moves and shortening moves. Moreover, any two geodesics connecting u and v are equivalent via moves through stacks.



The proof uses combinatorial Gauss-Bonnet.

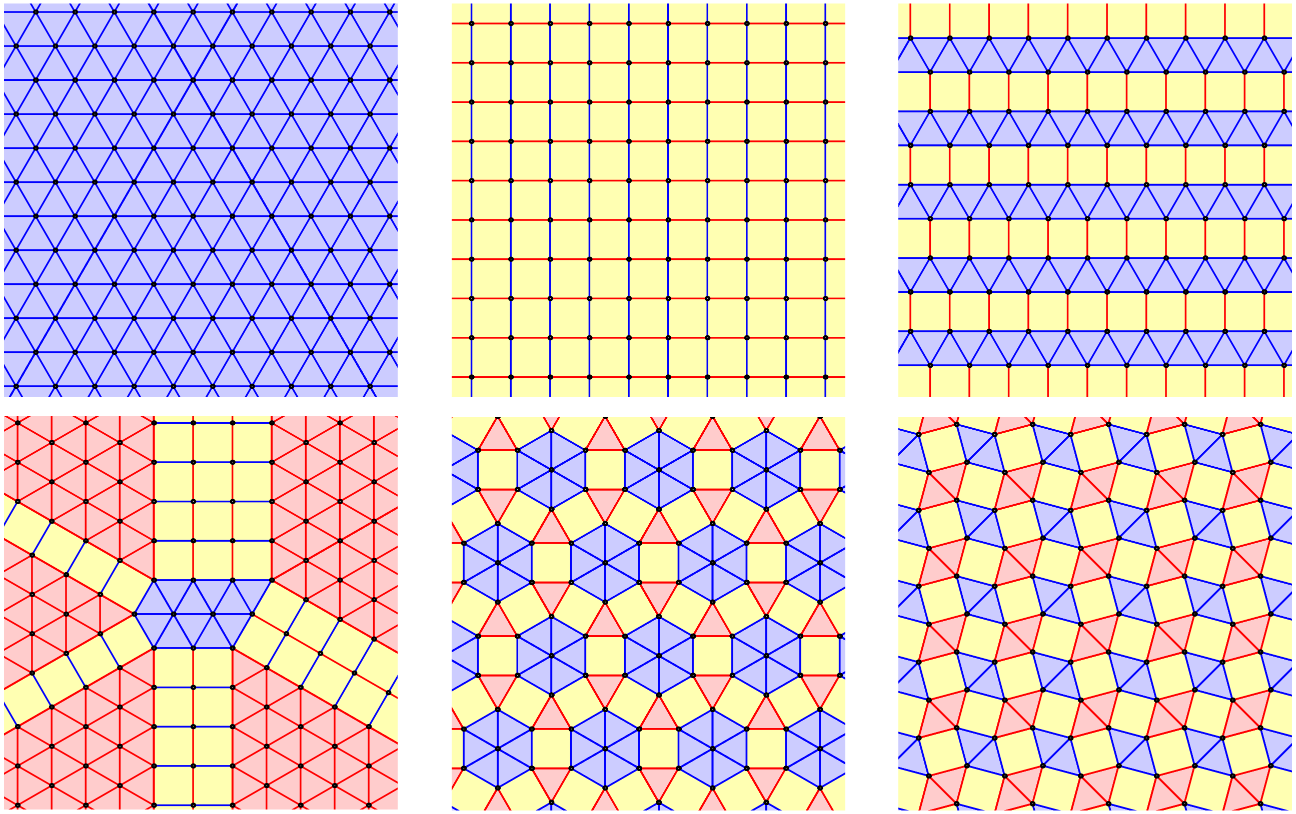
Key Property: Canonical Stacks

Thm: If $E(u, v)$ is an envelope in a CAT(0) $\triangle\square$-cplx, then the star of u in E consists of a single edge or a single 2-cell. Moreover, in the latter case, this 2-cell is part of a canonical stack in E.



The proof uses the asphericity of CAT(0) complexes.

Flats in △□-cplxes

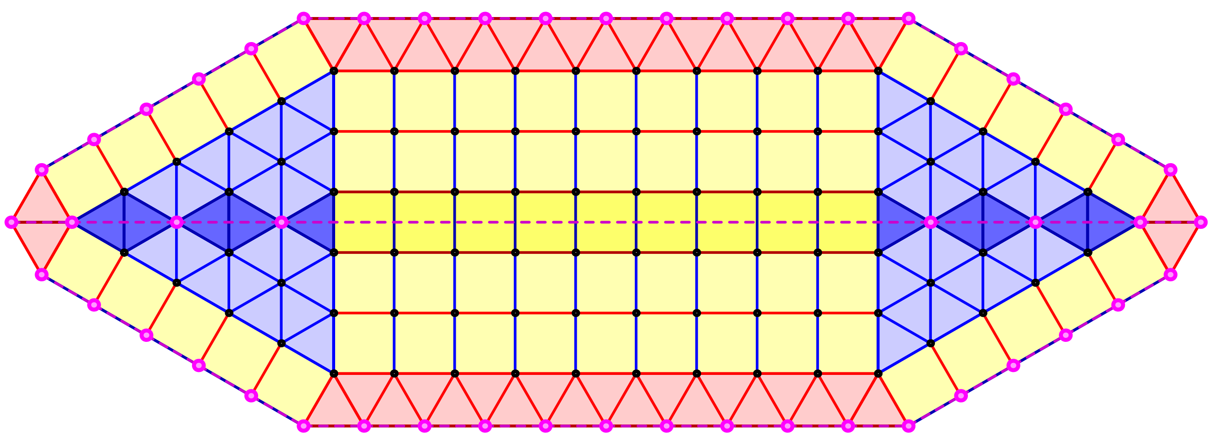


Flats in △□-cplxes can be pure, striped, radial, or crumpled.

Fellow Traveling Constant

Rem: Unlike △-cplxes or □-cplxes, there does not exists a uniform K such that paths in every △□-flat K-fellow travel.

Ex:



Thm: In every periodic flat, Rena's paths K-fellow travel, but the value of K depends on the flat.

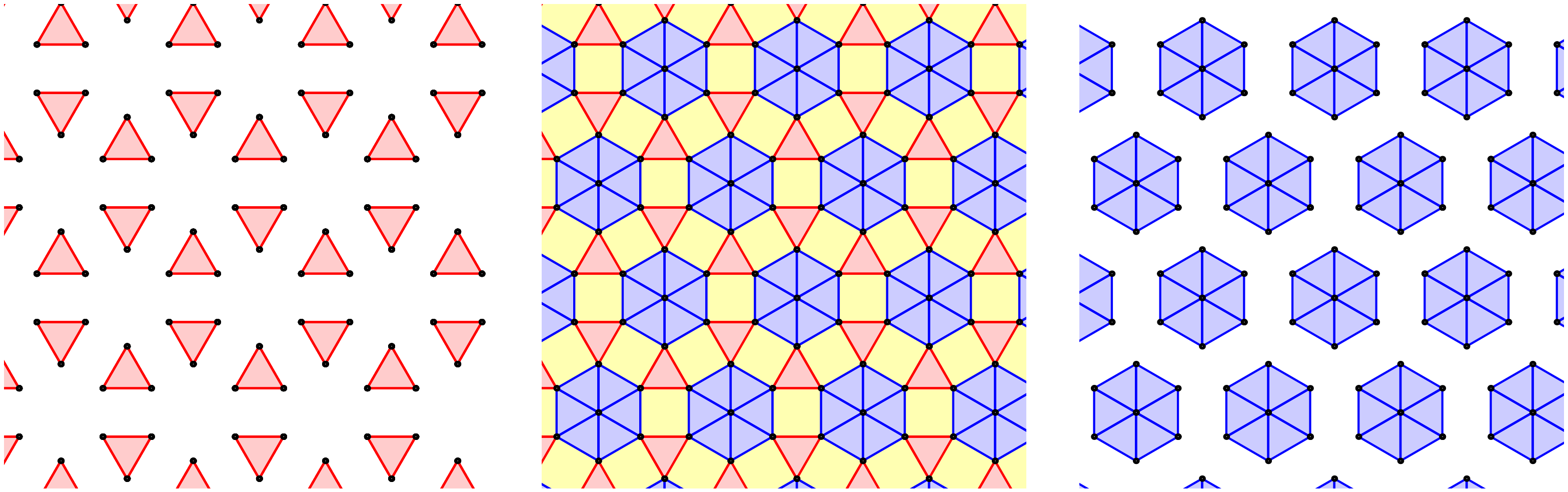
$\triangle\square$-Flats and Eisenstein Planes

Def: A pure $\triangle$-Flat is called an *Eisenstein plane* $\mathcal{E}$ because its vertex set can be identified with the Eisenstein integers $\mathbb{Z}[\omega]$ where ω is a primitive cube root of 1.

Lem: Every $\triangle\square$-Flat embeds into the 2-skeleton of the direct product of two Eisenstein planes $\mathcal{E} \times \mathcal{E}$. Moreover, the map on vertices is an isometry onto its image in the 1-skeleton metric.

Pf: (1) The triangles can be 2-colored based on slopes.
(2) Square regions are convex.
(3) From (1) and (2) we get nice projections to $\mathcal{E}$ and $\mathcal{E}$.
(4) The projections define the embedding and illuminate its structure.

Projections

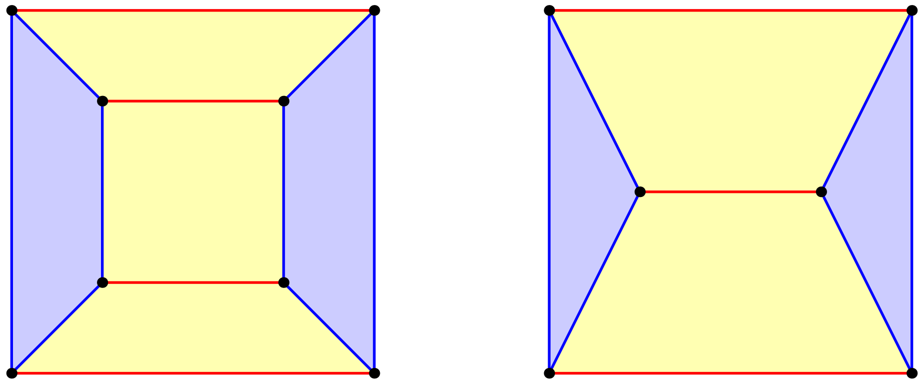


Because the yellow regions are convex, the red regions and the blue regions always slide together nicely.

Rem: Notice that the product space $\mathcal{E} \times \mathcal{E}$ is isometric to $\mathbb{R}^4$ and regularly tiled by copies of $\triangle \times \triangle$. There is a trick that makes it possible to visualize this 4-polytope.

2-**Dimensional Schlegel Diagrams**

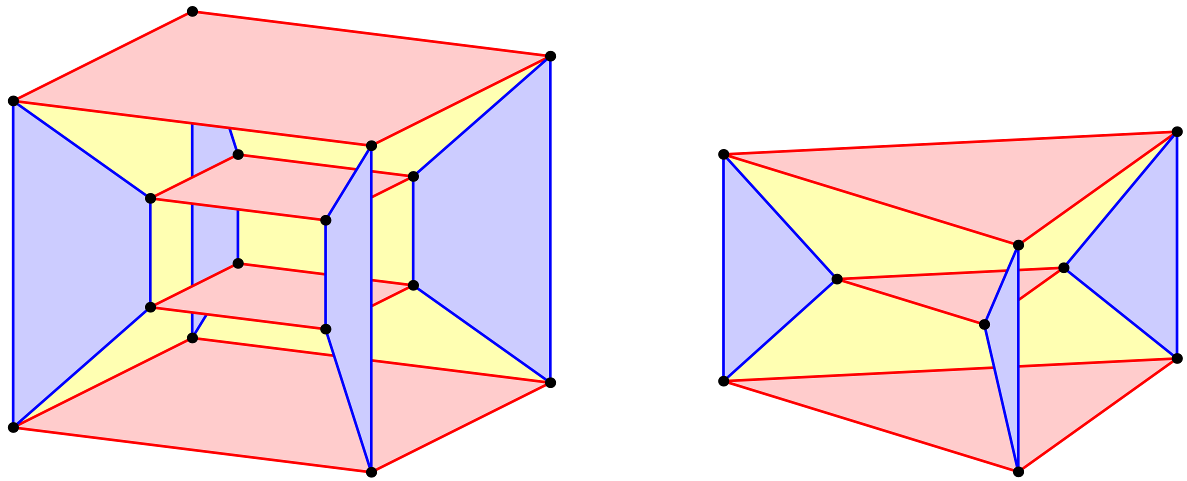
A *Schlegel diagram* is a way to visual a d-polytope in dimension $d-1$. For example, here are Schlegel diagrams for a cube and a triangular prism.



The idea is to project the boundary minus a facet into the removed facet that you are "looking" through along sightlines.

3-**Dimensional Schlegel Diagrams**

Here are Schlegel diagrams for the 4-polytopes $\square \times \square$ and $\triangle \times \triangle$, the direct product of two squares, and two triangles, respectively.



The first is, of course, the 4-cube; the other is nearly as fundamental, but much less widely known.

Fellow Traveling

Thm: In every periodic $\triangle\square$-flat F, Rena's paths K-fellow travel, but the value of K depends on the flat.

Pf: (1) F embedded in $\mathcal{E} \times \mathcal{E}$ is QI to an $\mathbb{R}^2$ in this $\mathbb{R}^4$.
(2) The envelope in F is the envelope in $(\mathcal{E} \times \mathcal{E}) \cap F$.
(3) The envelope in $\mathcal{E} \times \mathcal{E}$ is a 4-dim'l parallelopiped.
(4) The envelope in F is a quasi-zonotope.
(5) Rena's paths are quasi-lines until they near the boundary.
(6) Rena's paths K-fellow travel in F for some K.

A Flat Closing Lemma

Prop: The only crumpled planes that can immerse into a cpt NPC $\triangle\square$-cplx are the periodic ones.

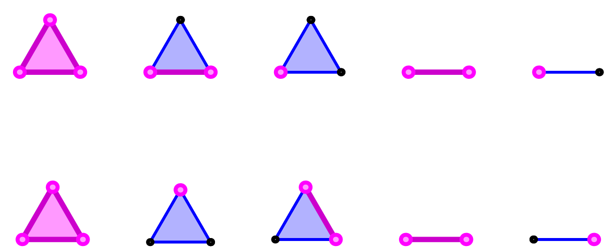
The proof essentially analyzes the convex subcomplexes inside crumpled planes and shows that they are in short supply.

Cor: For every immersed flat there is a K that works.

The main issue at this point is to prove that there is a global K that works. In some sense we are very close since the worst flats have been controlled.

Higher Dimensions: Stacks

Lem: If a polytope P is a product of simplices, then all of its faces are products of simplices and every proper face has an "antipodal" face in the 1-skeleton metric.



Based on this, we can define higher dimensional stacks as before.

Conj: Every d-dimensional flat in a $\prod \triangle$-cplx embeds into a product of d Eisenstein planes, and this map is an isometry on vertices in the 1-skeleton metric.