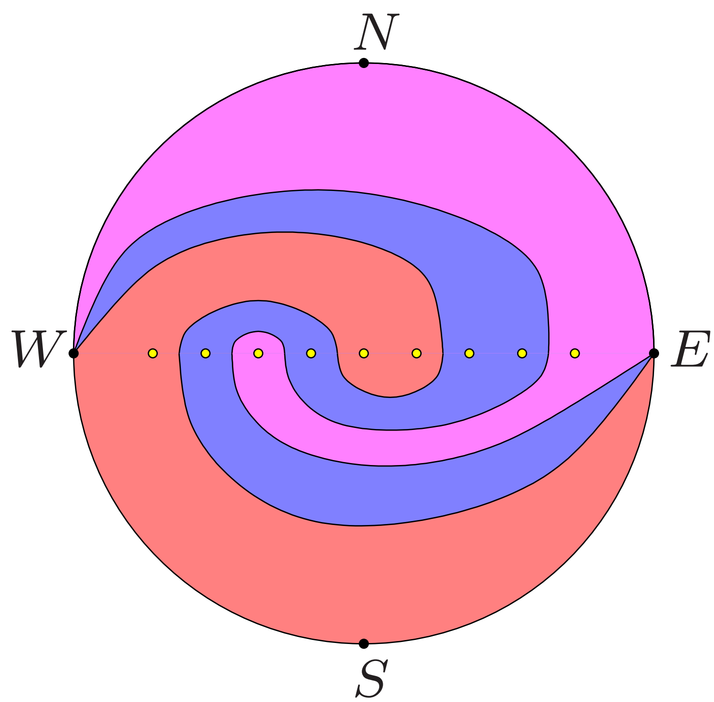
Garside structures for (some more) Artin groups



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(joint work with Noel Brady, John Crisp, and Anton Kaul)

Overview

I. Coxeter groups and Artin groups

II. Garside structures

III. Garside structures for free groups

IV. Garside structures for Artin groups

V. Other partial results

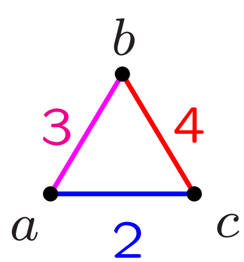
I. Coxeter groups and Artin groups

Let Γ be a finite graph with edges labeled by integers greater than 1, and let $(a, b)^n$ be the length n prefix of $(ab)^n$.

Def: The *Artin group* A_Γ is generated by its vertices with a relation $(a, b)^n = (b, a)^n$ whenever a and b are joined by an edge labeled n.

Def: The *Coxeter group* W_Γ is the Artin group A_Γ modulo the relations $a^2 = 1$ $\forall a \in$ **Vert**(Γ).

Graph



Artin presentation

$\langle a, b, c |\ aba = bab, ac = ca, bcbc = cbcb \rangle$

Coxeter presentation

$$\left\langle a, b, c \middle| \begin{array}{c} aba = bab, ac = ca, bcbc = cbcb \\ a^2 = b^2 = c^2 = 1 \end{array} \right\rangle$$

Coxeter groups are natural

Coxeter groups are a natural generalization of finite reflection groups and they are amazingly nice to work with.

1. They have a decidable word problem
2. They are virtually torsion-free
3. They have finite CAT(0) $K(\pi, 1)$s
4. They are linear
5. They are automatic

Artin groups are natural yet mysterious

Artin groups are "natural" in the sense that they are closely tied to the complexified version of the hyperplane arrangements for Coxeter groups.

But they are "mysterious" in the sense that it is unknown if

1. They have a decidable word problem
2. They are (virtually) torsion-free
3. They have finite (dimensional) $K(\pi, 1)$s
4. They are linear
5. The positive monoid injects into the group

Actually 5 was recently shown to be true by Luis Paris, but the proof is still mysterious.

II. Garside structures

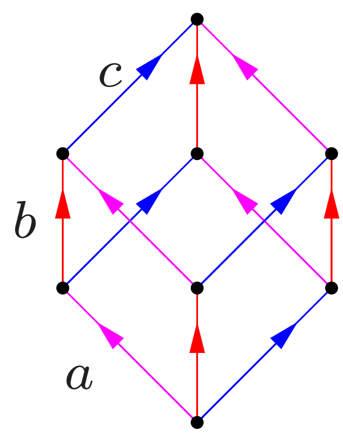
A *Garside structure* on a group G is given by a submonoid M and an element Δ in M. The necessary conditions are

1. M is an atomic monoid
2. M is the positive cone of a left-invariant lattice order $\leq$ on G.
3. M is generated by $x \in M$ with $x \leq \Delta$.
4. conjugation by Δ respects the lattice order.

Constructing Garside structures

One way to produce such a structure is to start with a bounded, graded, atomic, consistently edge-labeled lattice which is balanced.

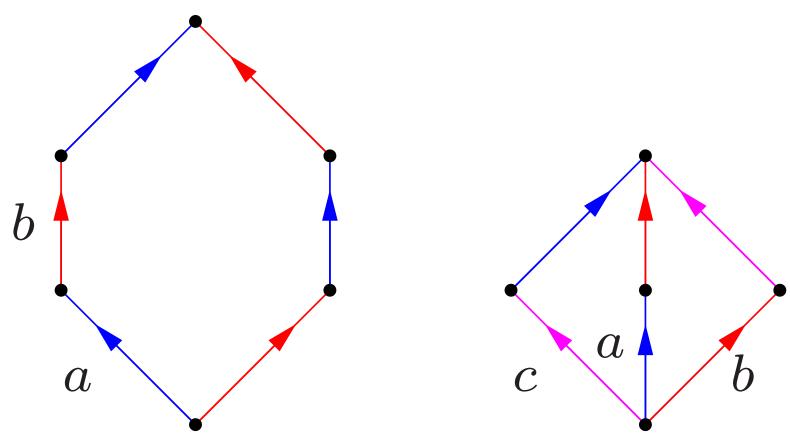
Balanced means that the words readable starting at the bottom are the words readable ending at the top.



A Garside structure for $\mathbb{Z}^3$ is shown.

Examples of Garside structures

Braid groups and other finite-type Artin groups each have two Garside structures. For the 3-string braid group the two posets are shown. The second one is the *dual* of the first.

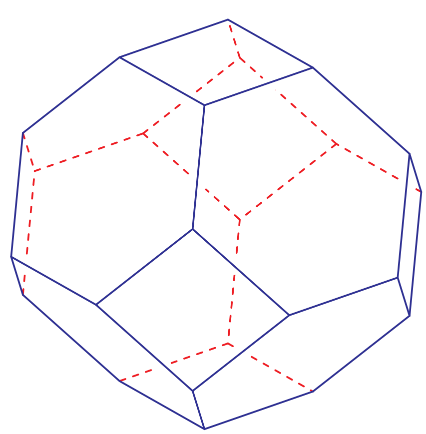


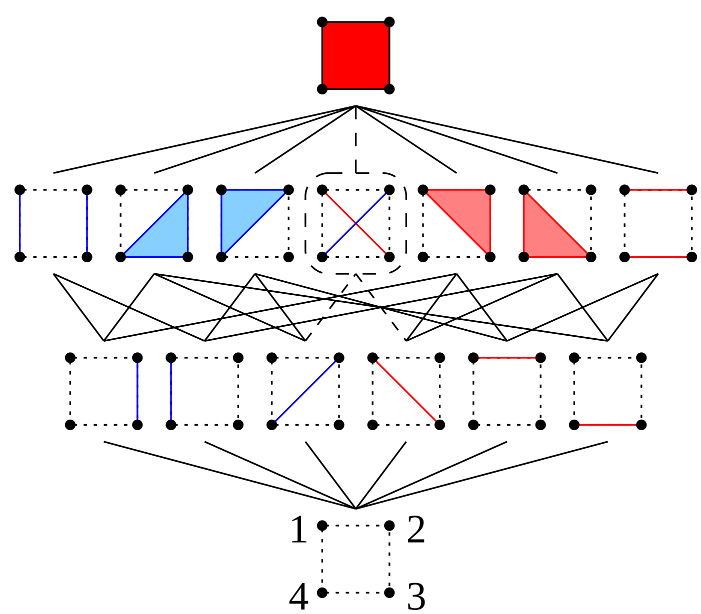
$$\langle a, b | aba = bab \rangle = \langle a, b, c | ab = bc = ca \rangle$$

The A_3 Poset and its dual

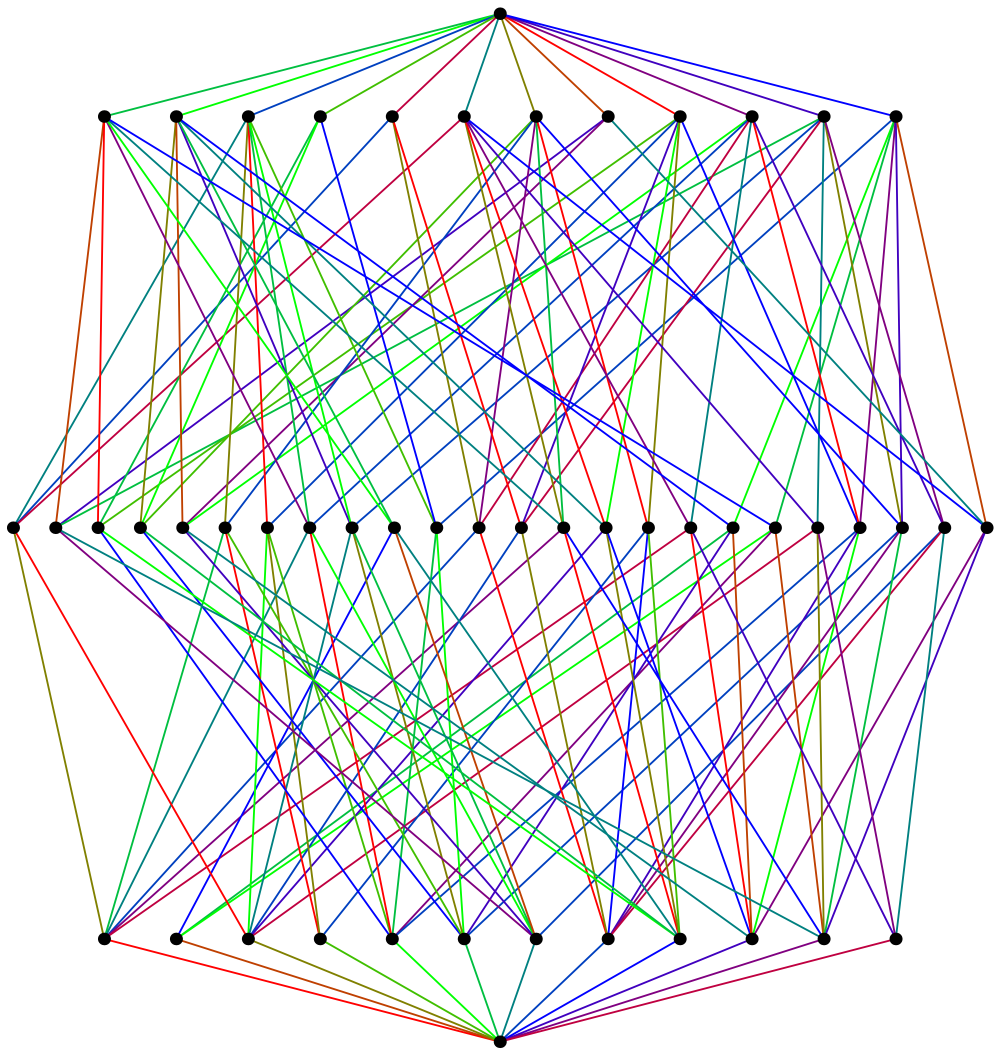
The standard Garside structure a braid group is a height function applied to the 1-skeleton of a permutahedron (which is the Cayley graph of S_4 with respect to the adjacent transpositions).

The dual structure is what combinatorialists call the "non-crossing partition lattice".

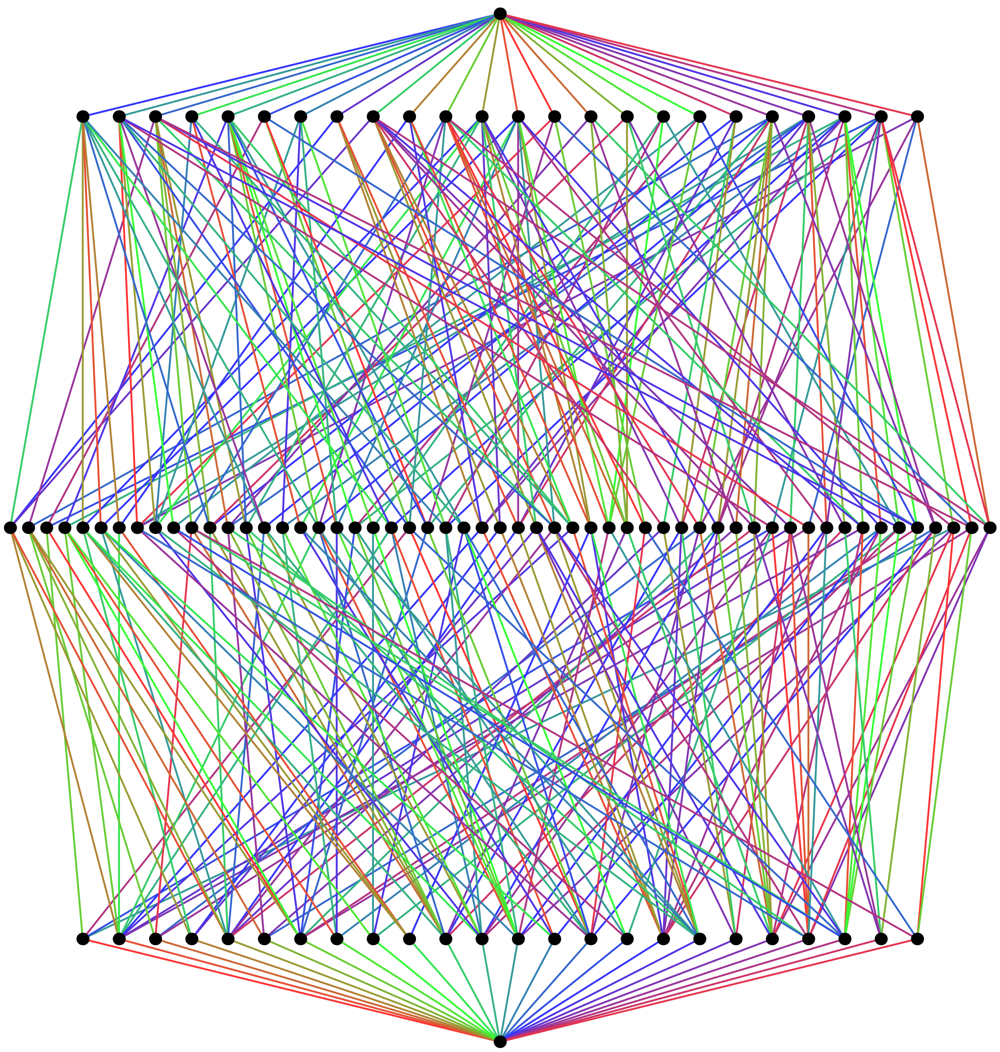




The dual D_4 Poset



The dual F_4 Poset



Why "dual"?

[Bessis - "The Dual Braid Monoid"]

S = standard generators
T = set of all "reflections"
c = a Coxeter element = $\prod s$
$\mathbf{w}_0$ = the longest element in W
n = the rank (dimension) of W
N = # reflections = # of positive roots
h = Coxeter number = order of **c**

	Classical monoid	Dual monoid
Set of atoms	**S**	**T**
Product of atoms Number of atoms Regular degree	**c** n h	$\mathbf{w}_0$ N 2
Δ Length of Δ Order of $p(\Delta)$	$\mathbf{w}_0$ N 2	**c** n h

Garside structures for non-finite type Artin groups

S = standard generators
T = set of all "reflections"
c = a Coxeter element = $\prod s$
$\mathbf{w}_0$ = the longest element in W
n = the rank (dimension) of W
N = # reflections = # of positive roots
h = Coxeter number = order of **c**

Extending the previous table we have:

	Classical monoid	Dual monoid
Set of atoms	**S**	**T**
Product of atoms	**c**	NA
Number of atoms	n	NA
Regular degree	∞	NA
$\boldsymbol{\Delta}$	NA	**c**
Length of Δ	NA	n
Order of $p(\Delta)$	NA	∞

What Garside structures are good for

If G is a group with a Garside structure, then it

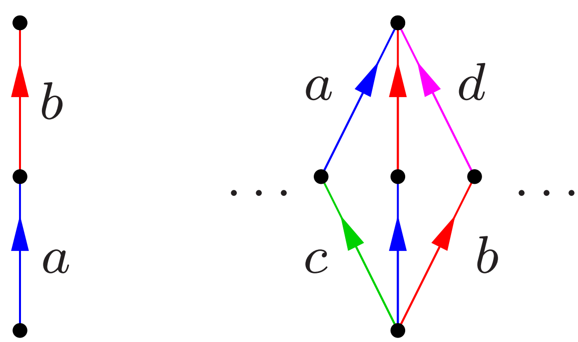
1. has a presentation derived from the poset
2. is the group of fractions of this presentation
3. has a decidable word problem*
4. has a finite (dimensional) $K(\pi, 1)$
5. is torsion-free.

Thus finding Garside structures for Artin groups would be a very good thing. The hardest part is almost always showing that the candidate poset is a lattice.

*(in the appropriate sense)

III. Garside structures for free groups

Let F_n be a free group with basis $x_1, x_2, \ldots, x_n$ and let $\Delta = x_1 x_2 \cdots x_n$. We can start building a Garside structure by continuing to add paths (and generators) to create a bounded graded, consistently edge-labeled poset which is balanced.



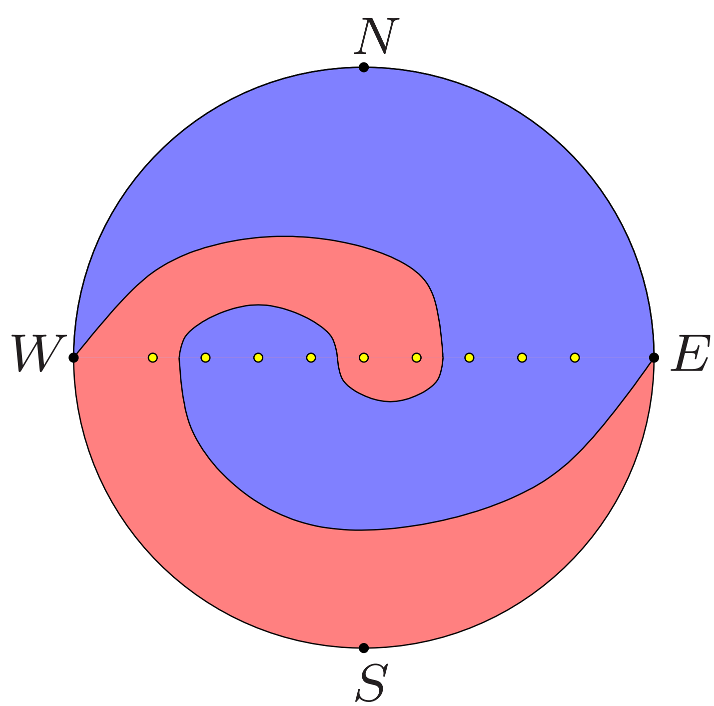
The construction in this case leads to a universal cover which is an infinitely branching tree cross the reals with a free $\mathbf{F}_2$ action.

$$\langle a_i | a_i a_{i+1} = a_j a_{j+1} \rangle$$

A more topological definition

Let $\mathbf{D}^*$ denote the unit disc with n puntures and 4 distinguished boundary points, N, S, E and W.

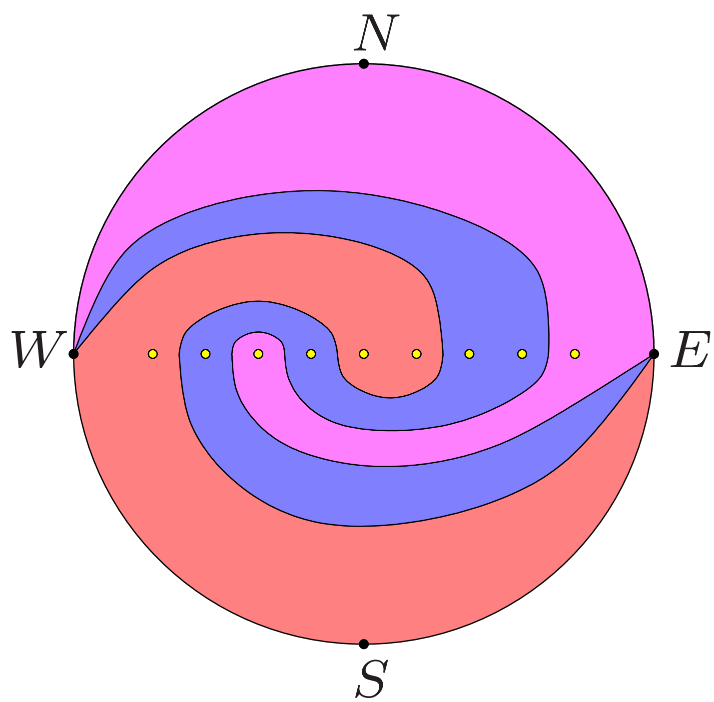
Def: A *cut-curve* is an isotopy class (in $\mathbf{D}^*$) of a path from E to W (rel endpoints, of course).



Notice that cut-curves divide $\mathbf{D}^*$ into two pieces, one containing S and the other containing N. Its *height* is the number of puncture in the lower piece.

Poset of cut-curves

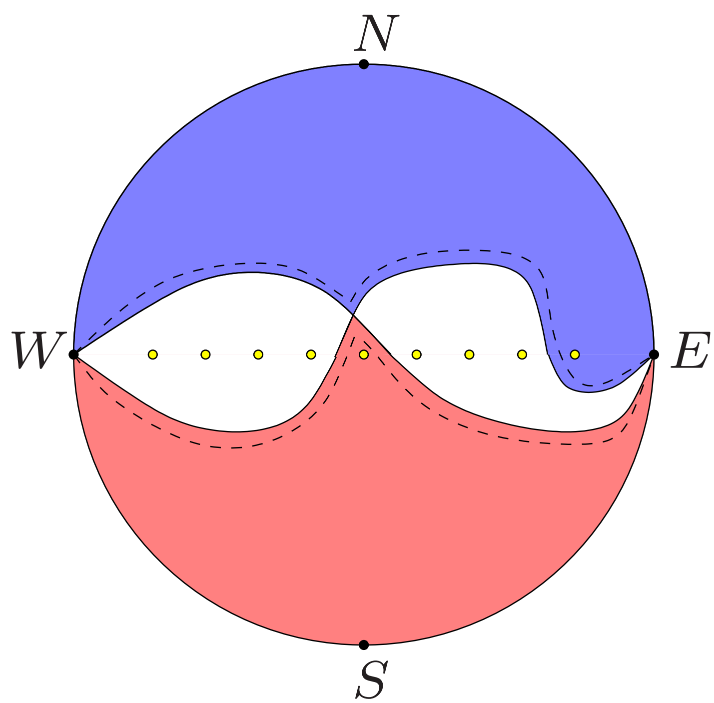
Let $[c]$ and $[c']$ be cut-curves. We write $[c] < [c']$ if there are representatives c and c' which are disjoint (except at their endpoints) and c is "below" c'.



Notice that if representative c is given, then we can tell whether $[c] < [c']$ by keeping c fixed and isotoping c' into a "minimal position" with respect to c (i.e. no football shaped regions with no punctures).

Proving the lattice property

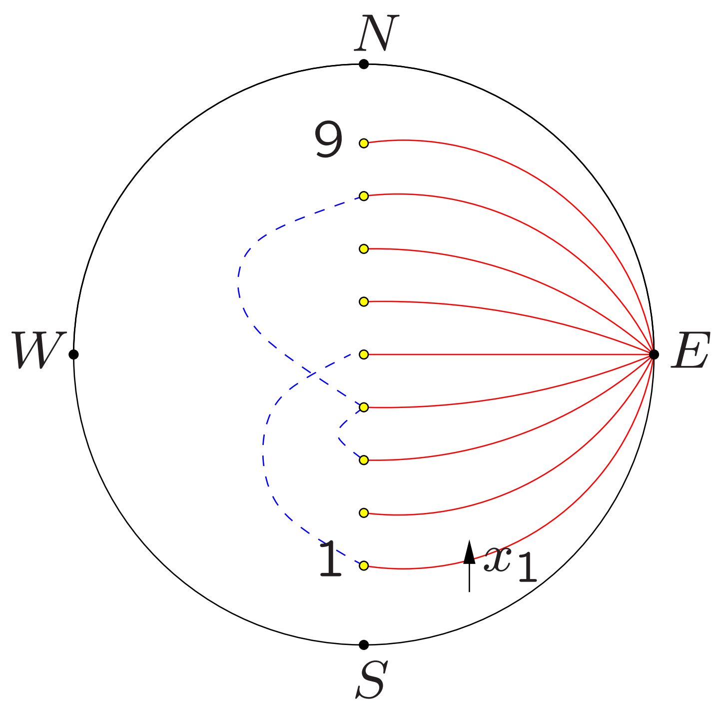
Lemma The poset of cut-curves is a lattice.



Proof: Suppose $[c]$ is above $[c_1]$ and $[c_2]$. Place representatives c_1 and c_2 in minimal position with respect to each other (i.e. no football regions) and then isotope c so that it is disjoint from both. This c is above the dotted line. Thus the dotted line represents a least upper bound for $[c_1]$ and $[c_2]$.

IV. Garside structures for Artin groups

For a general Artin group, we start with a specific marking of $\mathbb{D}^*$ (in the form of cuts) and draw arcs connecting the punctures which avoid the cuts.



From the graph Γ we define a subgroup H of the braid group which is generated by powers of half-twists along the arcs with the powers determined by the labels on the edges.

Topological Version of P_Γ

Define a graded poset P_Γ^{Top} as equivalence classes of cut curves $[c]_H$ where two cut curves are equivalent if they differ by an element of H acting on the disc.

The ordering is $[c]_H < [c']_H$ iff there are representatives which are disjoint.

When trying to convert this to a purely algebraic definition there is an issue of left vs. right actions of the braid group on the disc.

Algebraic Version of P_Γ

Let Γ be an ordered Dynkin diagram and let $H = H_\Gamma$ be the twist subgroup of B_n.

Let $B_{(i)}$ be the subgroup of the braid group B_n which never crosses the i and $i+1$ strands (isomorphic to $B_i \times B_{n-i}$).

Define a graded poset P_Γ^{Alg} by using the double cosets $H \backslash B_n / B_{(i)}$ as the set of vertices at level i. The ordering is given by

$$H\alpha B_{(i)} < H\beta B_{(j)} \quad (\alpha, \beta \in B_n)$$

if and only if $i < j$ and the double coset intersection is non-empty.

Coxeter Version of P_Γ

Define P_Γ^{Cox} be pushing the free group version into the Coxeter group W_Γ using the natural map.

More specifically, the free group Garside structure can be viewed as "residing" in the Cayley graph of the free group with respect to an infinite generating set C indexed by the braid group.

The image of C in W_Γ gives a generating set C_Γ and the poset P^{Cox} is determined by the image of the free structure in Cayley(W_Γ, C_Γ).

The P_Γ Theorem

Thm(BCKM): $\forall$ ordered Dynkin diagrams Γ,

$$P_\Gamma^{\mathsf{Top}} \cong P_\Gamma^{\mathsf{Alg}} \twoheadrightarrow P_\Gamma^{\mathsf{Cox}}$$

The edge-labeled poset $\overline{P_\Gamma^{\mathsf{Top}}} \cong \overline{P_\Gamma^{\mathsf{Alg}}}$ is called P_Γ.

Moreover, we can prove the following:

Thm(BCKM): $\forall$ ordered Dynkin diagrams Γ,

$$\overline{P_\Gamma^{\mathsf{Top}}} \cong \overline{P_\Gamma^{\mathsf{Alg}}} \cong P_\Gamma^{\mathsf{Cox}}$$

The bars indicate a quotient which uses images in A_Γ.

A space for A_Γ

Using standard techniques from the theory of Garside structures, we can turn P_Γ into a topological space K_Γ.

Thm(BCKM): $\forall$ ordered Dynkin diagrams Γ,

$$\pi_1(K_\Gamma, *) \cong A_\Gamma$$

Thus, we are presenting the right group. The (currently missing) lattice is crucial to showing that the universal cover of this space is contractible.

An idea in the air

Here are some partial results to date:

[BCKM] (October 03, Talks, Slides posted)
Free groups / 3-generator

[D. Bessis] (January 04, Preprint posted)
Free groups

[F. Digne] (February 04, Preprint posted)
Type $\widetilde{A}_n$