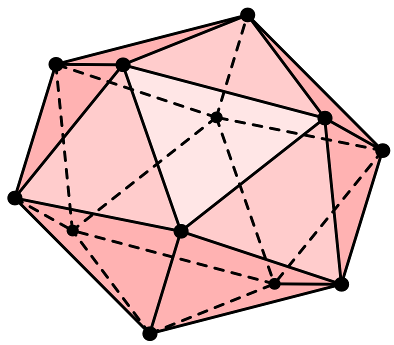
Möbius inversion and combinatorial curvature



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Outline

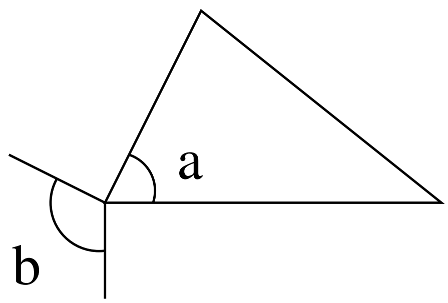
I. Two Theorems about Curvature

II. Angle Sums in Polytopes

III. A General Theorem

Normalized Angles

Let F be a face of a polytope P.



• The normalized *internal angle* $\alpha(F, P)$ is the proportion of unit vectors perpendicular to F which point into P (i.e. the measure of this set of vectors divided by the measure of the sphere of the appropriate dimension).

• The normalized *external angle* $\beta(F, P)$ is the proportion of unit vectors perpendicular to F so that there is a hyperplanes with this unit normal which contains F and the rest of P is on the other side.

Thm: $\sum_{v \in P} \beta(P, v) = 1.$

Curvature in PE complexes

Following Cheeger-Müller-Schrader (and Charney-Davis), we can think of the curvature of a piecewise Euclidean cell complex X as concentrated at its vertices.

$$\begin{aligned}
\chi(X) &= \sum_P (-1)^{\mathrm{dim} P} \\
&= \sum_P \sum_{v \in P} (-1)^{\mathrm{dim} P} \beta(v, P) \\
&= \sum_v \sum_{P \ni v} (-1)^{\mathrm{dim} P} \beta(v, P) \\
&= \sum_v \kappa(v)
\end{aligned}$$

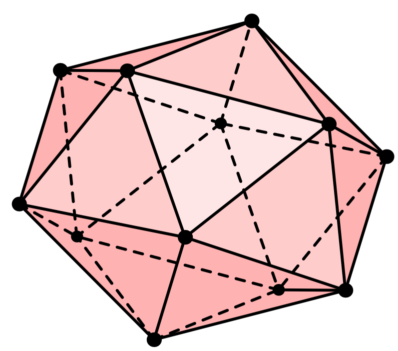
where $\kappa(v) := \sum_{P \ni v} (-1)^{\dim P} \beta(v, P)$.

Rem: This equation led to the Charney-Davis conjecture.

An Example

If X is the boundary of a dodecahedron, then

$$\begin{aligned}\kappa(v) &= \beta(v,v) - \sum_{e \ni v} \beta(v,e) + \sum_{f \ni v} \beta(v,f) \\ &= 1 - 5\left(\tfrac{1}{2}\right) + 5\left(\tfrac{1}{3}\right) = \tfrac{1}{6}\end{aligned}$$



Since $\chi(X) = \sum_v \kappa(v)$ there must be 12 vertices $(2 = V/6)$.

Combinatorial Gauss-Bonnet

An *angled 2-complex* is one where we arbitrarily assign normalized external angles $\beta(v, f)$ for each vertex-face pair.

Define $\kappa(v)$ as above. Define $\kappa(f)$ as a correction term which measures how far the external vertex angles are from 1.

$$\kappa(f) = 1 - \sum_{v \in f} \beta(v, f)$$

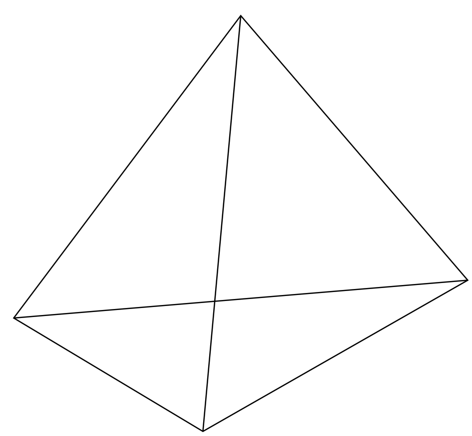
Thm(Gersten,Ballmann-Buyalo,M-Wise) If X is an angled 2-complex, then

$$\sum_{v} \kappa(v) + \sum_{f} \kappa(f) = \chi(X)$$

Rem: In all these papers the sum was $2\pi\chi(X)$ since the angles were not normalized. As we shall see normalization is crucial for the equations in higher dimensions.

Angle Sums

The sum of the internal angles in a triangle is π, but the sum of the dihedral angles in a tetrahedron can vary. The relations between the various internal and external angles in a Euclidean polytope are best described via incidence algebras.



Posets and Incidence Algebras

Let P be a finite poset on $[n]$ numbered according to some linearization of P, and let $I(P)$ be its *incidence algebra*.

Rem: The elements of $I(P)$ can also be thought of as functions from $P \times P \to \mathbb{R}$.

The identity matrix is the *delta function* where $\delta(x, y) = 1$ iff $x = y$.

The *zeta function* is the function $\zeta(x, y) = 1$ if $x \leq_P y$ and 0 otherwise (i.e. 1's wherever possible).

The *möbius function* is the matrix inverse of ζ. Note that $\mu\zeta = \zeta\mu = \delta$.

Incidence Algebras for Polytopes

The faces of a Euclidean polytope under inclusion (including the empty face) is its *face lattice*.

The set of all normalized internal (external) angles of a polytope P forms a single element α (β) of the incidence algebra of its face lattice—once we extend these notions so that $\alpha(\emptyset, F)$ and $\beta(\emptyset, F)$ have well-defined values.
One possibility is

$$\alpha(\emptyset, F) = \left\{ \begin{array}{l} 1 \text{ if } \dim F \leq 0 \\ 0 \text{ if } \dim F > 0 \end{array} \right\}$$

$$\beta(\emptyset, F) = \left\{ \begin{array}{l} 1 \text{ if } \dim F < 0 \\ 0 \text{ if } \dim F \geq 0 \end{array} \right\}$$

Equations for Angles

The most interesting of angle identity is the one discovered by Peter McMullen.

Thm(McMullen) $\alpha\beta = \zeta$, i.e.

$$\sum_{F \leq G \leq H} \alpha(F,G)\beta(G,H) = \zeta(F,H)$$

Proof Idea:

• Look at (a polytopal cone) $\times$ (its dual cone)

• Integrate $f(\vec{x}) = \exp(-||\vec{x}||^2)$ over this $\mathbb{R}^{2n}$ in two different ways.

Möbius Functions for Polytopes

Because the value of the möbius function is the reduced Euler characteristic of the geometric realization of interior of the interval, we have:

Lem: The möbius function of the face lattice of a polytope is $\mu(F,G) = (-1)^{\dim G - \dim F}$.

Proof: The geometric realization of the portion of the face lattice between F and G is a sphere.

Def: Let $\bar{\alpha}(F,G) = \mu(F,G)\alpha(F,G)$, [Hadamard product] (i.e. $\bar{\alpha}$ is a *signed* normalized internal angle.

Thm(Sommerville) $\mu\alpha = \bar{\alpha}$ i.e.

$$\sum_{F \leq G \leq H} \mu(F,G)\alpha(G,H) = \mu(F,H)\alpha(F,H)$$

Cor: $\bar{\alpha}\beta = \mu\alpha\beta = \mu\zeta = \delta$.

Combinatorial Gauss-Bonnet Revisited

General CGB Thm Every factorization $\alpha\beta = \zeta$, gives rise to a Gauss-Bonnet type formula.

In particular, $\widetilde{\chi}(X)$ is

$$
\begin{aligned}
&= \sum_{P \geq \emptyset} (-1)^{\dim P} = \sum_{P \geq \emptyset} (-1)^{\dim P} \; \zeta(\emptyset, P) \\
&= \sum_{P \geq \emptyset} (-1)^{\dim P} \left(\sum_{Q \in [\emptyset, P]} \alpha(\emptyset, Q) \beta(Q, P) \right) \\
&= \sum_{Q \geq \emptyset} (-1)^{\dim Q} \; \alpha(\emptyset, Q) \left(\sum_{P \geq Q} \bar{\beta}(Q, P) \right) \\
&= \sum_{Q \geq \emptyset} (-1)^{\dim Q} \; \alpha(\emptyset, Q) \; \kappa^{\uparrow}(Q)
\end{aligned}
$$

where $\kappa^{\uparrow}(Q)$ is defined as the obvious signed sum implicit in the final equality.

Rem 1: Factorizations with lots of 0s are best.
Rem 2: Both earlier theorems are special cases.