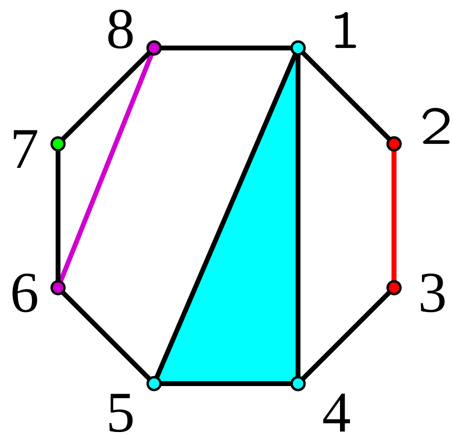
Finding parking when not commuting



$\{\{1,4,5\},\{2,3\},\{6,8\},\{7\}\}$

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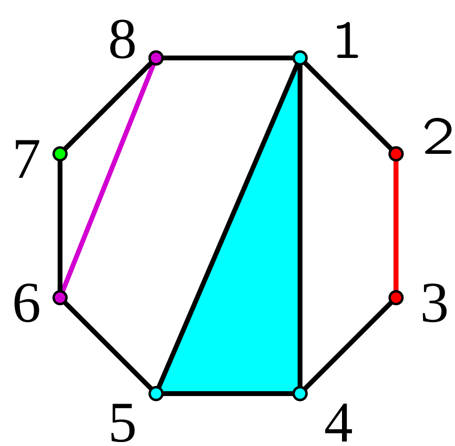
A common structure

The main goal of this talk will be to introduce you to a mathematical object which has a habit of appearing in a vast array of guises.

I. Non-crossing partitions
II. Symmetric groups
III. Braid groups
IV. Parking functions
V. Non-commutative geometry and free probability

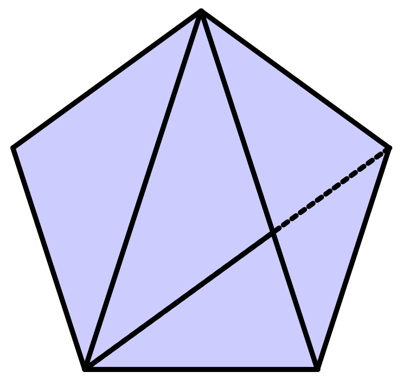


$$\{\{1,4,5\},\{2,3\},\{6,8\},\{7\}\}$$

Motivating example I: Fibonacci numbers

I. Dominoe tilings of a $2 \times n$ strip
II. Continued fractions and golden ratio
III. Diagonals of a regular pentagon
IV. Tridiagonal matrices

$$\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}}} = \frac{5}{8}$$

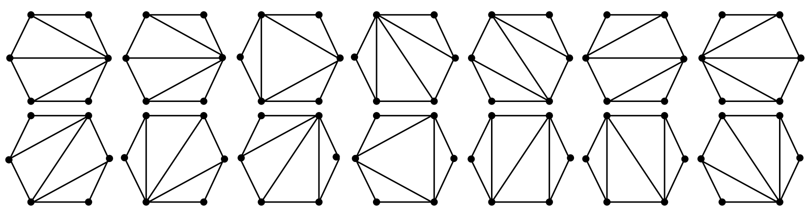


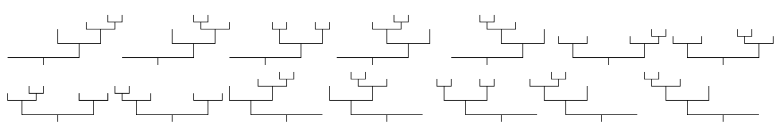
See Neal Sloane's online encyclopedia of integer sequences for more connections.

`http://www.research.att.com/~njas/sequences/`

Motivating example II: Catalan numbers

I. Triangulations of an n-gon
II. Rooted (planar) binary trees
III. Ways to associate n letters
IV. Dyck paths from $(0,0)$ to (n,n)





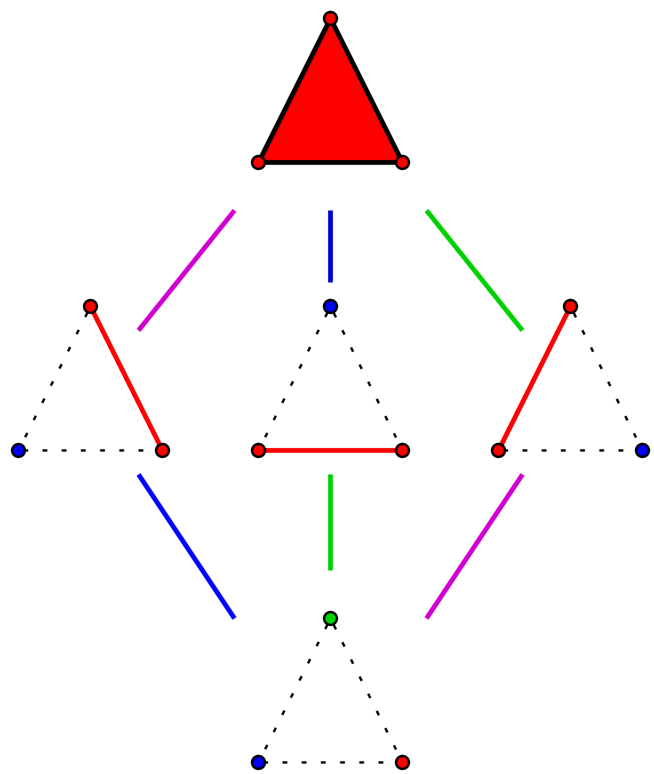
(1(2(3(45))))	(1(2((34)5)))
(1((23)(45)))	(1((2(34))5))
(1(((23)4)5))	((12)(3(45)))
((12)((34)5))	((1(23))(45))
((1(2(34)))5)	((1((23)4))5)
(((12)3)(45))	(((12)(34))5)
(((1(23))4)5)	((((12)3)4)5)

... and many more (See R. Stanley's list of 106 distinct combinatorial interpretations of these numbers)

I. Non-crossing partitions

Def: A *noncrossing partition* is a partition of the vertices of a regular n-gon so that the convex hulls of the blocks are disjoint.

One noncrossing partition σ is contained in another τ if each block of σ is contained in a block of τ. Let NC_n be the poset (partially ordered set) of all non-crossing partitions of an n-gon under this relation.



Properties of NC_n

Thm: NC_n is a graded, bounded lattice with Catalan many elements. It is also self-dual and Cohen-Macaulay.

• Lattice means that least upper bounds and greatest lower bounds always exist.

• Self-dual means there is a order-reversing bijection from NC_n to itself.

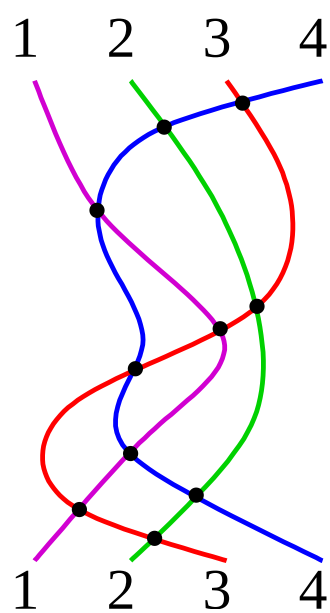
• Cohen-Macaulay is a strong restriction on the homology of various complexes derived from the poset.

In addition there is a “local action” of S_n on the maximal chains.

II. Symmetric groups

Def: S_n is the group of permutations of the set $\{1, \ldots, n\}$.

Thm: Every product of 2-cycles which equals the identity is of even length.



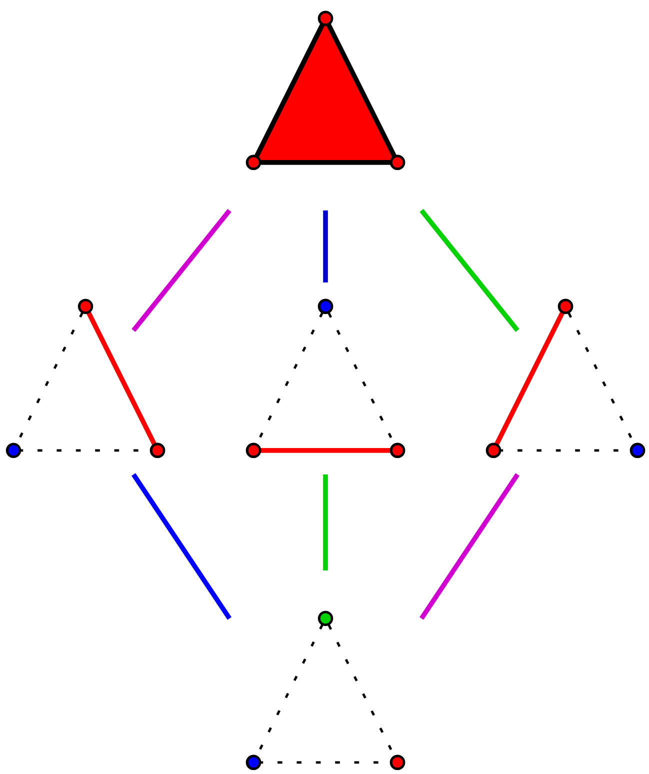
(34)
(23)
(12)
(34)
(23)
(12)
(23)
(34)
(12)
(23)

Cor: The parity of a factorization of an element is independent of the factorization.

Factorization into 2-cycles

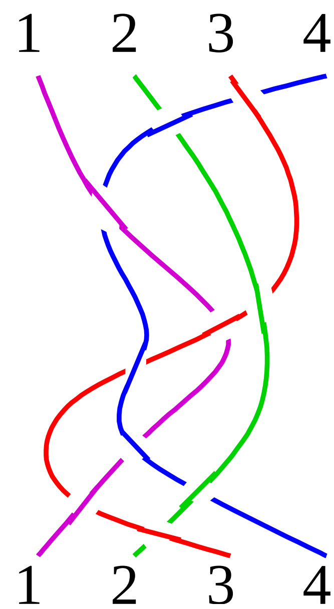
Lem: The poset of prefixes of minimal factorizations of the n-cycle $(123\cdots n)$ into 2-cycles is exactly NC_n.

Rem: This makes for an easy proof of self-duality and for the S_n action on the edge-labels (and it explains the colors assigned to the edges of the Hasse diagram).



III. Braid groups

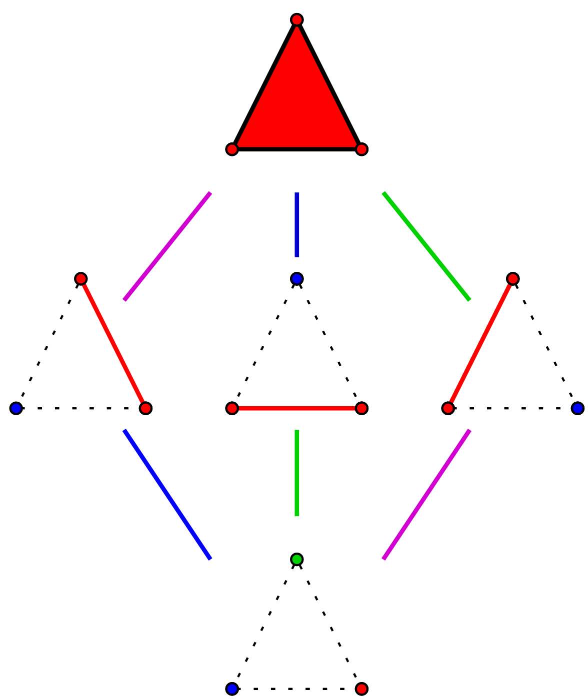
Def: The braid group B_n keeps track of how n strings can be twisted.



Clearly B_n maps onto S_n. Braid groups are related to many, many areas of mathematics, including mathematical physics, quantum groups, and, not surprisingly, 3-manifold topology. They are also intimately related to noncrossing partitions.

Brady-Krammer complex

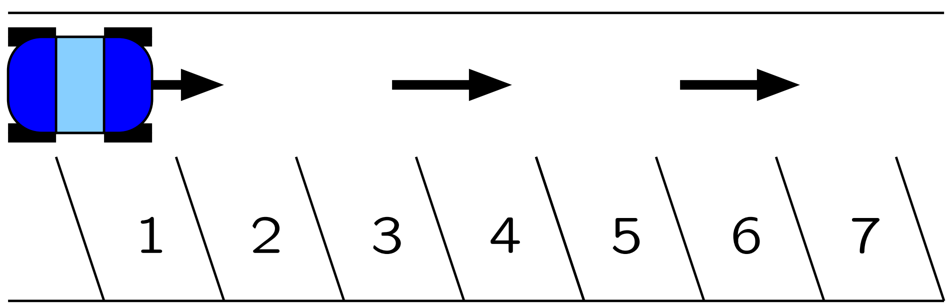
• For each maximal chain in NC_{n+1}, there is a n-simplex.
• Subchains in common lead to faces in common.
• Finally, the simplices with the same S_n labels are identified.



Thm(T.Brady,Krammer) The complex described is an Eilenberg-MacLane space for the braid group B_n.

IV. Parking functions

Def: A parking function of length n is a sequence $(a_1, \ldots, a_n)$ of positive integers whose nondecreasing rearrangement $b_1 \le b_2 \le \ldots \le b_n$ satisfies $b_i \le i$.



The number of parking functions is $(n+1)^{n-1}$.

There are a number of bijections with labeled rooted trees on $[n]$ (or equivalently the number of acyclic functions on $[n]$).

Parking functions and NC_{n+1}

Let $\sigma - \tau$ be an edge of covering relation in NC_{n+1} and let B and B' be the blocks which are joined. If $\min B < \min B'$ we define the label as the largest element of B which is below all of B'.

Lem: The labels on the maximal chains in NC_{n+1} are the parking functions.

Rem: There are also connections along these lines with the theory of symmetric functions and Hopf algebras.

$$f(x, y) = x^2 + xy + y$$

$$g(x, y) = x^2 + xy + y^2$$

V. Classical Probability

Let X be a random variable having a probability density function $f(x)$ (discrete or continuous). The *expectation* of $u(x)$ is
$E(u(X)) = \int_{-\infty}^{\infty} u(x)f(x)dx$.

Ex: mean $\mu = E(X)$

Ex: variance $\sigma^2 = E((X-\mu)^2)$

Ex: moment generating function
$M(t) = E(e^{tX}) = \sum_{n \geq 0} E(X^n)\frac{t^n}{n!}$.

Moments and cumulants

The coefficents in the moment generating function are called the *moments* of $f(X)$. The coefficients of $\log M(t)$ are called the (classical) *cumulants* of X.

The main advantage of the cumulants is that they contain the same information as the moments but the cumulant of the sum of two random variables X and Y are the sum of their cumulants – so long as they are independent.

Non-commutative geometry

Non-commuatative geometry is a philosophy whereby standard geometric arguments on topological spaces are converted into algebraic arguments on their commutative $C*$-algebras of functions.

The goal is then to find analogous arguments on non-commutative $C*$-algebras which reduce to the standard results in the commutative case.

Free Probability

A non-commutative probability space is a pair $(\mathcal{A}, \phi)$ where $\mathcal{A}$ is a complex unital algebra equipped with a unital linear functional E (expectation). The non-commutative version of independence is "freeness".

The combinatorics of non-crossing partitions is very closely involved in the non-commutative version of cumulants. In fact, some researchers who study free probability describe the passage from the commutative to the non-commutative version as a transition from the combinatorics of the partition lattice to the combinatorics of non-crossing partitions.

Summary

The lattice of non-crossing partitions might play a role in any situation which involves

- the symmetric groups
- the braid groups
- free probablity

or

anywhere where the Catalan numbers can be found (including the combinatorics of trees and the combinatorics of parking functions).

They also show up in real hyperplane arrangements, Prüfer codes, quasisymmetric functions,...

Final illustration: Associahedra

(1(2(3(45))))	(1(2((34)5)))
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• The Catalan numbers count the ways to assoicate a list of numbers.

• If we include partial assoications we get an ordering.

• This ordering is the face lattice of a polytope called the associahedron.

• The “Morse theory” of this polytope can be described using the non-crossing partition lattice.