1) $\int \tan x \sec^2 x\,dx = \int u\,du = u^2/2 = \frac{\tan^2 x}{2} + C$

$u = \tan x$

$du = \sec^2 x\,dx$

A) $\int \underbrace{x}_{u} \underbrace{\sec^2 x \tan x\,dx}_{dv} = \frac{x\tan^2 x}{2} - \int \frac{\tan^2 x}{2} dx$

$du = dx \quad v = \tan^2 x/2 \quad = \frac{x\tan^2 x}{2} - \int \frac{\sec^2 x - 1}{2} dx = \frac{x\tan^2 x - \tan x + x}{2} + C$

B) $\int x \sin x \cos x\,dx = \frac{1}{2}\int \underbrace{x}_{u} \underbrace{\sin(2x)\,dx}_{dv} = \frac{-x\cos(2x)}{4} + \frac{\sin(2x)}{8} + C$

$du = dx \quad v = -\cos(2x)/2$

C) $\int x[\cos^2 x - \sin^2 x]dx = \int x \cos 2x\,dx = \frac{x\sin(2x)}{2} + \frac{\cos(2x)}{4} + C$

2) $\int_a^b \frac{\sqrt{x^2-1}}{x} dx = \int_{\sec^{-1} a}^{\sec^{-1} b} \frac{\tan\theta}{\sec\theta} \sec\theta \tan\theta\,d\theta = \int_{\sec^{-1} a}^{\sec^{-1} b} \tan^2\theta\,d\theta$

$x = \sec\theta$

$dx = \sec(\theta)\tan(\theta)d\theta$

$= \left[\tan\theta - \theta\right]_{\sec^{-1} a}^{\sec^{-1} b}$

$= \sqrt{b^2-1} - \sqrt{a^2-1} - \sec^{-1} b + \sec^{-1} a$

A) $\int_1^{2/\sqrt{3}} \frac{\sqrt{x^2-1}}{x} dx = \sqrt{4/3 - 1} - \sqrt{1^2-1} - \pi/6 + 0 = \sqrt{1/3} - \pi/6$ | Answer $a = \sqrt{1/3}$, $b = 6$

B) $\int_1^2 \cdots dx = \sqrt{3} - \pi/3$ | Answer: $a = \sqrt{3}$; $b = 3$

C) $\int_1^{\sqrt{2}} \cdots dx = 1 - \pi/4$ | Answer: $a = 1$, $b = 4$

3. A) $\int \frac{x^4+3}{x^3-x}\,dx = \int x + \frac{x^2+3}{x^3-x}\,dx$

$$\frac{x^2+3}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$x^2+3 = A(x^2-1) + B(x^2+x) + C(x^2-x)$$

$A = -3$
$B = C = 2$

$$\int = \int x + \frac{-3}{x} + \frac{2}{x-1} + \frac{2}{x+1}\,dx = \frac{x^2}{2} - 3\ln x + 2\ln(x^2-1) + C$$

B) $\int \frac{x^4+2}{x^3-x}\,dx = \cdots = \frac{x^2}{2} - 2\ln(x) + \frac{3}{2}\ln(x^2-1) + C$

C) $\int \frac{x^4+1}{x^3-x}\,dx = \cdots = \frac{x^2}{2} - \ln(x) + \ln(x^2-1) + C$

$$= \frac{x^2}{2} + \ln\left(\frac{x^2-1}{x}\right) + C$$

4a) A) $p(1)=p(2)=0 \quad p(3)=1$

$\downarrow$ symmetric around $x=3/2$

$$p(x) = A\,(x-3/2)^2 - B \quad \to \quad A = 1/2,\ B = -1/8$$

$$p(x) = \tfrac{1}{2}(x-3/2)^2 - 1/8$$

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or roots at 1 and 2

$$p(x) = k\,(x-1)(x-2) \to k = 1/2$$

$$p(x) = \frac{(x-1)(x-2)}{2}$$

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or $p(x) = Ax^2 + Bx + C$

$$\left.\begin{aligned} p(1) &= A+B+C = 0 \\ p(2) &= 4A+2B+C = 0 \end{aligned}\right\} \quad \left.\begin{aligned} 3A + B &= 0 \\ B &= -3A \end{aligned}\right\} \quad \begin{aligned} 8A+2B &= 1 \\ 8A-6A &= 1 \\ A &= 1/2 \\ B &= -3/2 \\ C &= 1 \end{aligned}$$

$$p(3) = 9A+3B+C = 1$$

$$p(x) = x^2/2 - 3/2\,x + 1$$

B) $p(1) = 1 \quad p(2) = 0 \quad p(3) = 0$

$$p(x) = \frac{(x-2)(x-3)}{2}$$

C) $p(1) = 0 \quad p(2) = 1 \quad p(3) = 0$

$$p(x) = -(x-1)(x-3) = 1-(x-2)^2$$

4b) $$\int_1^3 p(x)\,dx \underset{\text{Simpson's rule}}{=} \underbrace{\frac{3-1}{2}\cdot 1/3}_{1/3}\,\big(p(0) + 4p(1) + p(2)\big) = \begin{aligned} &A),B)\ 1/3 \\ &C)\ 4/3 \end{aligned}$$

4c) A) $f(x)=(x+1)e^{-x}$ $\quad 0\leq x\leq 1$

$f''(x)=(x-1)e^{-x}$ $\quad |x-1|\leq 1 \quad |e^{-x}|\leq 1$

$|f''(x)|\leq 1 = K_2$

B) $f(x)=(x+1)e^{-x}$ $\quad 1\leq x\leq 3$

$|x-1|\leq 2 \quad |e^{-x}|\leq 1/e$

$K_2 = 2/e$

C) $f(x)=(x+2)e^{-x}$ $\quad 0\leq x\leq 3$

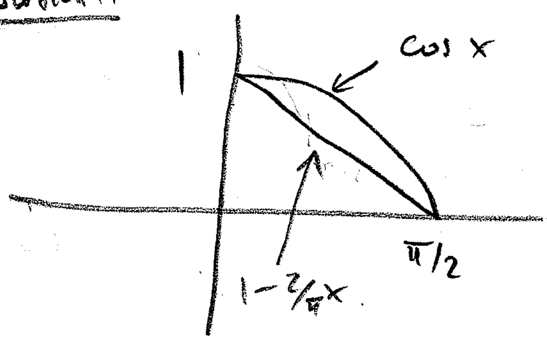
$f''(x)=xe^{-x}$

$|f''(x)|\leq |3|\,|1| = 3 = K_2$

5a) $\cos x \geq 1 - \frac{2}{\pi} x$ for $0 \leq x \leq \pi/2$

Notice $(\cos x)'' = -\cos x \leq 0$ for $0 \leq x \leq \pi/2$ $\cos x$ concave downward

Solution 1:



The graph of $1 - \frac{2}{\pi}x$ is the secant line segment from 0 to $\pi/2$.

$\cos x$ is concave, hence the graph of $\cos x$ is lying above the secant line

Solution 2: Look at $h(x) = \cos x - \left(1 - \frac{2}{\pi} x\right)$.

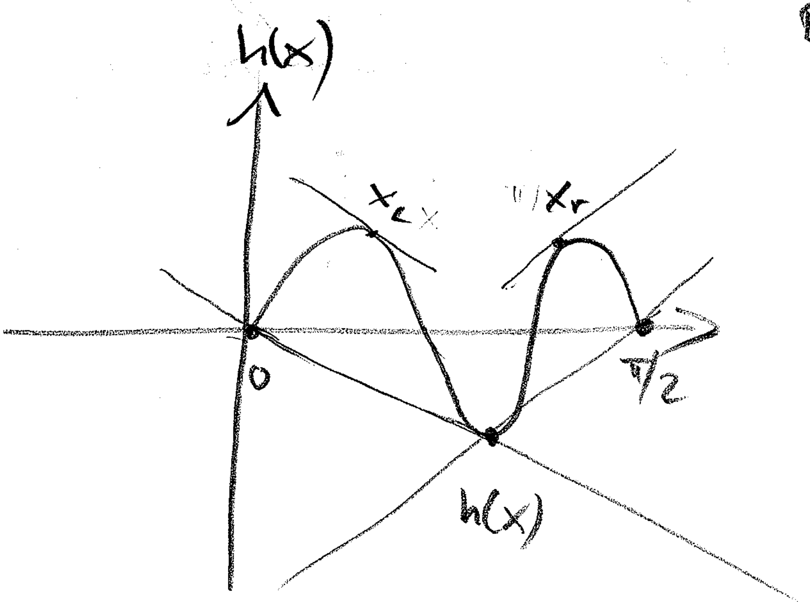
Notice $h(0) = h(\pi/2) = 0$ $h''(x) < 0$ for $0 < x < \pi/2$

$\downarrow$

h has no minimum between 0 and $\pi/2$

Minimum$^{=0}$ archived at 0 and $\pi/2$, hence $h(x) \geq 0$, so $\cos x \geq 1 - \frac{2}{\pi} x$.

Solution 3: Take h as above. Assume there is an x s.t. $h(x) < 0$.



By the mean value thm, there is $0 \leq x_l \leq x$ such that

$$h'(x_l) = \frac{h(x) - h(0)}{x} < 0$$

and $x \leq x_r \leq \pi/2$ such that

$$h'(x_r) = \frac{h(\pi/2) - h(x)}{\pi/2 - x} > 0.$$

But $h'(x_r) > h'(x_l)$ contradicts the fact that h' is decreasing because $h'' < 0$.

$x_r > x_l$

5b) $$\int_0^{\pi/2} \frac{dx}{\sqrt{\cos x}} \leq \int_0^{\pi/2} \frac{dx}{\sqrt{\underbrace{1-\tfrac{2}{\pi}x}_{u}}} = \tfrac{\pi}{2}\int_0^1 \frac{du}{\sqrt{u}} = \tfrac{\pi}{2}\left[2\sqrt{u}\right]_0^1 = \pi$$

convergent!

$$0 \leq \frac{1}{\sqrt{\cos x}} \overset{\text{by 5a}}{\leq} \frac{1}{\sqrt{1-\tfrac{2}{\pi}x}}$$

6. A) $F(x) = \int_0^x e^{-t^2}\,dt$ $\qquad F(1) = \int_0^1 e^{-t^2}\,dt \geq 1/e$

$F(x) \geq 1/e$ for $x \geq 1$

$$\int_1^t F(t)\,dt \geq \frac{t-1}{e} \longrightarrow \infty \quad \text{hence} \quad \int_0^\infty F(t)\,dt = \infty$$

B) $F(x) = \int_x^\infty e^{-t^2}\,dt$

$$\int_0^\infty \underbrace{F(x)}_{u}\,\underbrace{dx}_{dv} = \underbrace{x\,F(x)\Big]_0^\infty}_{\overset{\parallel}{0}} + \int_0^\infty x e^{-x^2}\,dx = \frac{-e^{-x^2}}{2}\Big]_0^\infty = \tfrac{1}{2}$$

c) $F(x) = \int_x^\infty e^{-t^2/2}\,dt$

$$\int_0^\infty F(x)\,dx = 0 + \int_0^\infty x e^{-x^2/2}\,dx = -e^{-x^2/2}\Big]_0^\infty = 1$$