2, 4, 6

Name:
GSI's Name: $\boxed{\text{Solution}}$
Section:

Midterm 2
Math 1B, Fall 2008
Wilkening

0	1	
1	3	
2	3	
3	3	
4	7	
5	8	
6	5	
7	6	
total	36	

0. (1 point) write your name, your GSI's name, and your section number at the top of your exam.

1. (3 points or 0 points) Suppose $|\sin x| \neq 1$. Evaluate $\sum_{n=0}^{\infty} (\sin x)^{2n}$.

a. $\tan x$

b. $\sec^2 x$

c. $\sinh x$

d. $\frac{x}{\sqrt{1-x^2}}$

e. none of the above

$\sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2}$ when $|x|<1$. When $|\sin x| \neq 1$, we must have $|\sin x|<1$, so

$$\sum_{n=0}^{\infty} (\sin x)^{2n} = \frac{1}{1-\sin^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

2. (3 points or 0 points) Describe the behavior of the sequence $a_1 = 2, \quad a_{n+1} = \frac{a_n^2+3}{4}$

a. a_n increases monotonically and converges to 3

b. a_n decreases monotonically and converges to 1

c. a_n increases monotonically to ∞

d. a_n decreases monotonically to $-\infty$

e. a_n is not monotonic

3. (3 points or 0 points) Supppose $0 \leq a_n < 1$, $a_n < b_n$, and $\sum b_n$ is convergent. Circle all the statements that are necessarily true:

a. $\sum a_n^2$ converges and $\sum a_n^2 < \sum a_n$

b. $\sum \sqrt{a_n}$ converges and $\sum \sqrt{a_n} < \sum a_n$

c. $\sum b_n^2$ converges and $\sum b_n^2 < \sum b_n$

d. $\sum \sqrt{b_n}$ converges and $\sum \sqrt{b_n} < \sum b_n$

e. if $p > 0$ then $\sum (-1)^n a_n^p$ is convergent

4a. (3 points) Let $f(x) = e^{-x^3}$.
Write down the Maclaurin series for $f(x)$ and evaluate $f^{(99)}(0)$ and $f^{(100)}(0)$.

$$e^x = \sum_0^\infty \frac{x^n}{n!}$$

$$e^{-x^3} = \sum_0^\infty \frac{(-1)^n x^{3n}}{n!}$$

$$\sum_0^\infty \frac{(-1)^n x^{3n}}{n!} = \sum_0^\infty \frac{f^{(k)}(0)}{k!} x^k$$

match like coefficients:

$$\frac{f^{(99)}(0)}{99!} = \frac{(-1)^{33}}{33!}, \quad \frac{f^{(100)}(0)}{100!} = 0$$

4b. (4 points) Find all x that satisfy the equation $\sum_{n=1}^{\infty} nx^n = 1$.

$$\sum_0^\infty x^n = \frac{1}{1-x}$$

$$\sum_1^\infty nx^{n-1} = \frac{1}{(1-x)^2} \Rightarrow \sum_1^\infty nx^n = \frac{x}{(1-x)^2}$$

Solve $\frac{x}{(1-x)^2} = 1$

$$x = (1-x)^2$$
$$x = 1 - 2x + x^2$$
$$x^2 - 3x + 1 = 0$$
$$x = \frac{3 \pm \sqrt{5}}{2}$$

only $\frac{3-\sqrt{5}}{2}$ has $|x|<1$,

so $x = \frac{3-\sqrt{5}}{2}$

5. (2 points each) For each of the following series, determine whether the series is absolutely convergent (AC), conditionally convergent (CC), or divergent (D). Show some work, but do not spend excessive time justifying all your steps.

$$\sum_{n=1}^{\infty} [\sin(1/n^2)]^{1/3}$$

$\sin 1/n^2 \sim 1/n^2$ for large n.

More precisely, $\lim_{n\to\infty} \frac{(\sin 1/n^2)^{1/3}}{(1/n^2)^{1/3}} = 1$

Since $\sum_0^{\infty} \frac{1}{n^{2/3}}$ diverges, $\sum_1^{\infty} \left(\sin \frac{1}{n^2}\right)^{1/3}$ is D

$$\sum_{n=1}^{\infty} (-1)^n \ln \cos \frac{1}{n}$$

$\ln \cos \frac{1}{n} \sim \ln(1 - 1/n^2)$

$\sim -1/n^2$, so

$\left|(-1)^n \ln \cos \frac{1}{n}\right| \sim \frac{1}{n^2}$ in the sense that

$\lim_{n\to\infty} \frac{|(-1)^n \ln \cos 1/n|}{1/n^2} = 1.$ Since $\sum_1^{\infty} 1/n^2$ converges, $\sum_1^{\infty} (-1)^n \ln \cos \frac{1}{n}$ is AC

$$\sum_{n=1}^{\infty} \frac{(2n)!}{5^n (n!)^2}$$

ratio test: $\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)}{5(n+1)^2} \longrightarrow \frac{4}{5} < 1$

AC

$$\sum_{n=0}^{\infty} \binom{5}{n} (-3)^n$$

$\binom{5}{n} = 0$ for $n \geq 5$

$\therefore$ the sum is <u>finite</u>

AC

6a. (2 pts) Is the following statement True or False? Justify your answer with a proof or counterexample. (Obviously it's true if $a_n \geq 0$ and $b_n \geq 0$, so don't assume this).

If $\sum a_n$ is divergent and $\sum b_n$ is divergent, then $\sum(a_n + b_n)$ is also divergent.

False. If $a_n = \frac{1}{n}$ and $b_n = -\frac{1}{n}$, then both $\sum a_n$ and $\sum b_n$ diverge, but $a_n + b_n = 0$ so $\sum (a_n + b_n)$ converges.

6b. (3 points) Suppose $\sum c_n x^n$ has radius of convergence 2 while $\sum d_n x^n$ has radius of convergence 1. What is the radius of convergence of the series $\sum(c_n + d_n)x^n$? Explain.

If $|x| < 1$, then $\sum c_n x^n$ and $\sum d_n x^n$ converge, so $\sum (c_n + d_n) x^n$ converges.

If $1 < |x| < 2$, then $\sum c_n x^n$ converges and $\sum d_n x^n$ diverges, so $\sum (c_n + d_n) x^n$ diverges.

Because $\sum (c_n + d_n) x^n$ is a power series, and it diverges for $1 < |x| < 2$, it must diverge for $1 < |x|$.

So $R = 1$.

7a. (3 points) Prove that $e \geq \left(1+\frac{1}{k}\right)^k$ for $k \geq 1$. $\left(\textit{Hint: } \ln(1+x) = x - \frac{x^2}{2} + \cdots\right)$

$$e \geq \left(1+\frac{1}{k}\right)^k$$

$$\Leftrightarrow \quad \ln e = 1 \geq k \ln(1+1/k)$$

$$= k \left[1/k - 1/2k^2 + 1/3k^3 - \cdots \right]$$

$$= 1 - 1/2k + 1/3k^2 - \cdots$$

Since $1 - 1/2k + 1/3k^2 - \cdots$ is an alternating series with decreasing abs value of terms,

$1 \geq 1 - 1/2k + 1/3k^2 - \cdots$, completing the proof.

7b. (3 points) Use part (a) and mathematical induction to prove the following crude version of Stirling's approximation:

$$e^n n! \geq n^n \qquad \text{for all } n \geq 1.$$

Base case $n=1$: $\quad e^1\, 1! \overset{?}{\geq} 1^1$

$(e \geq 1)$ ✓

Induction step: suppose $e^k k! \geq k^k$.

Then $e^{k+1}(k+1)! = e^k \cdot e \cdot k! \cdot (k+1)$

(by induction hyp) $\geq k^k \cdot e \cdot (k+1)$

(by (a)) $\geq k^k (1+1/k)^k (k+1)$

$= (k+1)^k (k+1)$

$= (k+1)^{k+1}$