Math 1B, Fall 2008
Section 108

Quiz 5 Solutions

(1) *Determine whether the given sequence converges or diverges. If it converges, find the limit.*

$$a_n = \arctan(2n).$$

Since $\lim_{x\to\infty} \arctan(x)$ exists, we have

$$\lim_{n\to\infty} \arctan(2n) = \lim_{x\to\infty} \arctan(x) = \pi/2.$$

In the first step, we used that $2n \to \infty$ when $n \to \infty$.

(2) *Determine whether the given series converges or diverges.*

$$\sum_{n=1}^{\infty} \frac{1+3^n}{2^n}.$$

Solution 1: We can write

$$\sum_{n=1}^{\infty} \frac{1+3^n}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \frac{3^n}{2^n}.$$

Both $\sum_{n=1}^{\infty} \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{3^n}{2^n}$ are geometric series with ratios $r_1 = 1/2$ and $r_2 = 3/2$, respectively. Thus $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges and $\sum_{n=1}^{\infty} \frac{3^n}{2^n}$ diverges. Since $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$ is the sum of a divergent series and a convergent series, it diverges.

Note: the sum of two divergent series may diverge, or it may converge. For the first type, think of $a_n = b_n = 1/n$. For the second, think of $a_n = 1/n$ and $b_n = -1/n$.

Solution 2: For large n, $1 + 3^n \approx 3^n$, so let's apply the Limit Comparison Test to

$$a_n = \frac{1+3^n}{2^n}, \qquad b_n = \frac{3^n}{2^n}.$$

Using L'Hopital,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1+3^n}{3^n} = \lim_{n\to\infty} \frac{(\ln 3)3^n}{(\ln 3)3^n} = 1,$$

so the Limit Comparison Test applies. Since $\sum_{n=1}^{\infty} b_n$ is a geometric series with ratio 3/2, it diverges. By the Limit Comparison Test, $\sum_{n=1}^{\infty} a_n$ also diverges.

Solution 3: Since

$$\frac{1+3^n}{2^n} \geq \frac{3^n}{2^n} \geq 0,$$

and $\sum_{n=1}^{\infty} \frac{3^n}{2^n}$ diverges, the Comparison Test says that $\frac{1+3^n}{2^n}$ also diverges.

(3) *Determine whether the given series converges or diverges.*

$$\sum_{n=1}^{\infty} ne^{-n}.$$

We'll use the Integral Test. Let $f(x) = xe^{-x}$. From looking at the formula, f is continuous everywhere and positive for $x > 0$. Since $f'(x) = (1-x)e^{-x}$, f is decreasing for $x > 1$. Thus the Integral Test applies. Now evaluate

$$\begin{aligned}
\int_5^{\infty} xe^{-x}\, dx &= \lim_{b\to\infty} -xe^{-x} - e^{-x}\Big|_5^b \qquad \text{(use integration by parts)} \\
&= 6e^{-5} + \lim_{b\to\infty} -be^{-b} - e^{-b} \\
&= 6e^{-5} \qquad \text{(use L'Hopital on the first term)}
\end{aligned}$$

Because $\int_5^{\infty} xe^{-x}\, dx$ converges,

$$\sum_{n=1}^{\infty} ne^{-n}.$$

also converges by the Integral Test.

Note: I picked 5 to start the integral to show that it doesn't really matter where you start, as long as you start somewhere after any discontinuities of f.