Quiz 5 Solutions

(1) Determine whether the given sequence converges or diverges. If it converges, find the limit.

\[ a_n = \arctan(2n). \]

Since \( \lim_{x \to \infty} \arctan(x) \) exists, we have
\[
\lim_{n \to \infty} \arctan(2n) = \lim_{x \to \infty} \arctan(x) = \pi/2.
\]
In the first step, we used that \( 2n \to \infty \) when \( n \to \infty \).

(2) Determine whether the given series converges or diverges.

\[
\sum_{n=1}^{\infty} \frac{1 + 3^n}{2^n}.
\]

**Solution 1:** We can write
\[
\sum_{n=1}^{\infty} \frac{1 + 3^n}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \frac{3^n}{2^n}.
\]
Both \( \sum_{n=1}^{\infty} \frac{1}{2^n} \) and \( \sum_{n=1}^{\infty} \frac{3^n}{2^n} \) are geometric series with ratios \( r_1 = 1/2 \) and \( r_2 = 3/2 \), respectively. Thus \( \sum_{n=1}^{\infty} \frac{1}{2^n} \) converges and \( \sum_{n=1}^{\infty} \frac{3^n}{2^n} \) diverges. Since \( \sum_{n=1}^{\infty} \frac{1 + 2^n}{3^n} \) is the sum of a divergent series and a convergent series, it diverges.

Note: the sum of two divergent series may diverge, or it may converge. For the first type, think of \( a_n = b_n = 1/n \). For the second, think of \( a_n = 1/n \) and \( b_n = -1/n \).

**Solution 2:** For large \( n \), \( 1 + 3^n \approx 3^n \), so let’s apply the Limit Comparison Test to
\[
a_n = \frac{1 + 3^n}{2^n}, \quad b_n = \frac{3^n}{2^n}.
\]
Using L’Hopital,
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1 + 3^n}{3^n} = \lim_{n \to \infty} \frac{\ln 3 \cdot 3^n}{(\ln 3)3^n} = 1,
\]
so the Limit Comparison Test applies. Since \( \sum_{n=1}^{\infty} b_n \) is a geometric series with ratio \( 3/2 \), it diverges. By the Limit Comparison Test, \( \sum_{n=1}^{\infty} a_n \) also diverges.

**Solution 3:** Since
\[
\frac{1 + 3^n}{2^n} \geq \frac{3^n}{2^n} \geq 0,
\]
and \( \sum_{n=1}^{\infty} \frac{3^n}{2^n} \) diverges, the Comparison Test says that \( \frac{1 + 3^n}{2^n} \) also diverges.

(3) **Determine whether the given series converges or diverges.**
\[
\sum_{n=1}^{\infty} ne^{-n}.
\]

We’ll use the Integral Test. Let \( f(x) = xe^{-x} \). From looking at the formula, \( f \) is continuous everywhere and positive for \( x > 0 \). Since \( f'(x) = (1 - x)e^{-x} \), \( f \) is decreasing for \( x > 1 \). Thus the Integral Test applies. Now evaluate
\[
\int_{5}^{\infty} xe^{-x} \, dx = \lim_{b \to \infty} \left[ -xe^{-x} - e^{-x} \right]_5^b \quad \text{(use integration by parts)}
\]
\[
= 6e^{-5} + \lim_{b \to \infty} -be^{-b} - e^{-b}
\]
\[
= 6e^{-5} \quad \text{(use L’Hopital on the first term)}
\]
Because \( \int_{5}^{\infty} xe^{-x} \, dx \) converges,
\[
\sum_{n=1}^{\infty} ne^{-n}.
\]
also converges by the Integral Test.

Note: I picked 5 to start the integral to show that it doesn’t really matter where you start, as long as you start somewhere after any discontinuities of \( f \).