Operator algebras and geometric conformal field theory

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October 12, 2013

CFT: Variations on a theme

- Physical notion is not mathematically well-defined
- Different axiomatizations have different features
- There is a large gap between “what is known” and “what should be true”

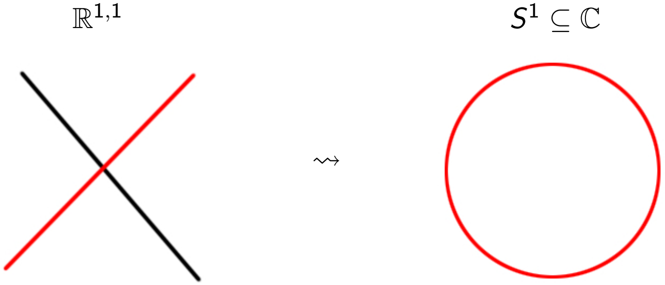
Our topic:

Relationships between different notions of mathematical conformal field theory

Common elements

Common elements: Space-time

- Most interesting with $(1+1)$-dim space-time (e.g. $\mathbb{R}^{1,1}$).
- Chiral CFT is "$\frac{1}{2}+\frac{1}{2}$" dimensional. Space-time is the compactified light-ray S^1.



Common elements: Symmetry and states

- Symmetry group is the conformal group of space-time.
- Chiral CFT: Diff(S^1) or the Virasoro algebra *VIR*

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.$$

- Hilbert space of states with a unitary representation of Diff(S^1) or *VIR*.
- Vacuum state Ω.

Version 1: Conformal nets

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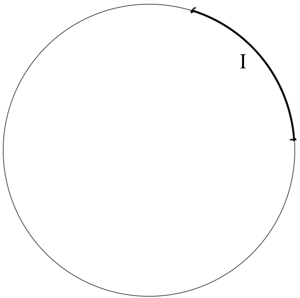

- $\mathcal{A}(I)$ - von Neumann algebras of local observables on $\mathcal{H}$
- Locality: $[A(I), A(J)] = 0$ if $I \cap J = \emptyset$.
- Conformal symmetry: $\pi(g)\mathcal{A}(I)\pi(g)^* = \mathcal{A}(g \cdot I)$ for $g \in \mathrm{Diff}^+(S^1)$

Version 1: Conformal nets (continued)

- Main interest - representations $\lambda_I : \mathcal{A}(I) \to \mathcal{B}(H_\lambda)$.
- Jones-Wassermann subfactors $\lambda_I(\mathcal{A}(I)) \subseteq \lambda_{I^c}(\mathcal{A}(I^c))'$.
- "Tensor" product of representations: composition of sectors.
- We'll talk about rational models (finite index, finite depth).

Version 1: Conformal nets (continued)

Example (Loop groups)

$\mathcal{A}(I) = \pi_0(L_I G)''$

- $G = SU(N), \quad LG = C^\infty(S^1, G)$.
- $L_I G =$ loops supported in I
- π_0 the vacuum representation (at level ℓ)
- Representations of $LG \rightsquigarrow$ representations of the net

Version 1: Conformal nets (continued)

Theorem (Wassermann)

The $SU(N)_\ell$ nets are rational. For $N = 2$, the J-W subfactor is A_n with index $4\cos^2(\frac{\pi}{\ell+2})$.

Proof.

Long, difficult paper. Key idea: identify physical notion of "fusion" of representations with Connes-Sauvageot relative tensor product. Explicit construction of intertwiners $\mathcal{H}_\lambda \to \mathcal{H}_\mu \boxtimes \mathcal{H}_\nu$. □

Version 2: Vertex operator algebras

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- Algebraic notion: pre-Hilbert space $\mathcal{H}^0$
- State-field correspondence: $\mathcal{H}^0 \to \mathrm{End}(\mathcal{H}^0)[[z^{\pm 1}]]$.

$$a \mapsto Y(a,z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \quad a_n \in \mathrm{End}(\mathcal{H}^0)$$

- Quantum fields $\longleftrightarrow$ an observable at every point of spacetime

Version 2: Vertex operator algebras (continued)

- The fields are *formal distributions*. For $f \in L^2(S^1)$,

$$Y(a, f) = \int Y(a, z) f(z) dz = \sum_{n \in \mathbb{Z}} \hat{f}(n) a_n.$$

- Von Neumann's idea: replace distributions with

$$\mathcal{A}(I) = \{Y(a, f) : \operatorname{supp}(f) \subseteq I\}''$$

Conformal nets vs VOAs

- Representations of conformal nets $\longleftrightarrow$ modules over VOAs
- Product of representations $\longleftrightarrow$ tensor product of modules
- Fusion rules $N_{\lambda\mu}^{\nu} = \dim \mathrm{Hom}(\mathcal{H}_\nu, \mathcal{H}_\lambda \boxtimes \mathcal{H}_\mu)$

$$\mathcal{H}_\lambda \boxtimes \mathcal{H}_\mu \cong \bigoplus N_{\lambda\mu}^{\nu} \mathcal{H}_\nu.$$

Problem

Given a conformal net and a VOA that encode the same data, identify their fusion rules.

(Without computing them)

Version 3: Segal CFT

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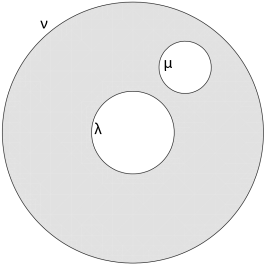
$\Sigma =$ 

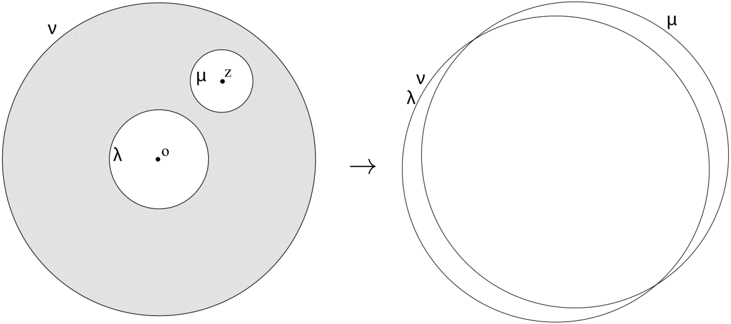
$\rightsquigarrow$

$$\rightsquigarrow E(\Sigma) \subseteq \text{trace class}(\mathcal{H}_\lambda \otimes_{\mathbb{C}} \mathcal{H}_\mu \to \mathcal{H}_\nu).$$

Gluing of surfaces $\longleftrightarrow$ composition of maps

Segal CFT vs VOAs

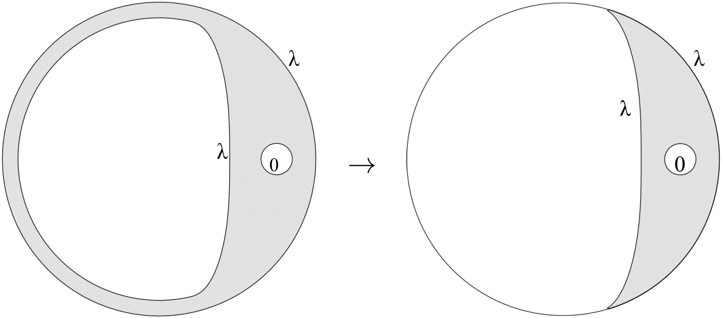
Philosophy: Segal CFT $\to$ intertwiners $Y(\,\cdot\,, z)\,\cdot$



As radii $\to 1$.

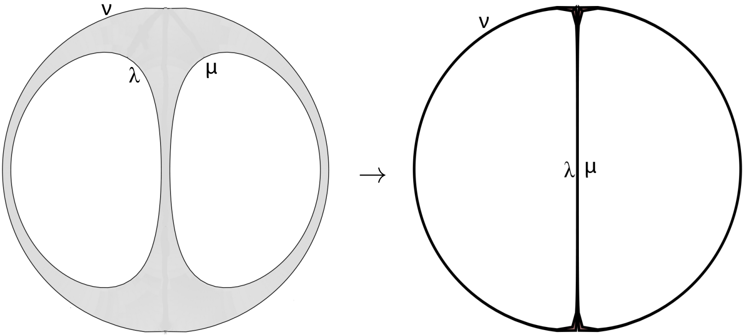
Segal CFT vs conformal nets

Philosophy: Segal CFT $\to \mathcal{A}(I)$ and its representations



Segal CFT vs conformal nets (continued)

Philosophy: Segal CFT $\to$ fusion of conformal nets (Wassermann's conjecture)



Idea

A VOA and a conformal net are "the same" if there is an interpolating Segal CFT

Problem

Shortage of examples of Segal CFTs

(Following the original definition, with nice analytic properties)

Theorem (T '12, '13)

- *There exist Segal CFTs for the free fermion (all genus) and $SU(N)_\ell$ (genus zero)*
- *The "VOA" and "conformal net" limits converge (to the correct VOA and conformal net)*
- *The Segal CFT encodes the fusion rules for the VOA*

Work in progress

Work in progress:

- Obtain conformal net fusion rules from the Segal CFT
- Construct $SU(N)_\ell$ Segal CFT in higher genus
- More constructions of Segal CFTs
- New constructions of irreducible, finite index, finite depth subfactors

Thank you!