Unitary equivalence to a complex symmetric matrix

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Definitions

Definition

A *complex symmetric matrix* (or CSM) is an $n \times n$ matrix $T = T^t$. This should not be confused with real symmetric (i.e. self-adjoint) matrices.

- Complex symmetric matrices have been studied for a long time. For instance, the Grunksy inequalities in complex analysis can be written using CSM's.
- We think of these matrices acting on the Hilbert space $(\mathbb{C}^n, \|.\|_2)$. To study the linear operators induced by CSM's, we also need to consider matrices that are unitarily equivalent to CSM's.

Definition

A matrix T is called UECSM if it is unitarily equivalent to a complex symmetric matrix. That is, if there is a unitary matrix U such that $S = U^* T U$.

Recall that unitary matrices take orthonormal sets to orthonormal sets. They correspond to rigid motions of the space. Equivalent definitions are:

1. $\|Ux\| = \|x\|$ for all x
2. $\langle x, y \rangle = \langle Ux, Uy \rangle$ for all x, y
3. $U^* = U^{-1}$

Why unitary equivalence?

- Unitary equivalence is the natural notion of equivalence in $B(\mathcal{H})$. Similarity alone does not preserve normality, etc.
- It is not a very interesting question if we use similarity.

Theorem

Every matrix is similar to a complex symmetric matrix.

Proof: We prove this classical theorem with an example.

It is not hard to check that:

Example

$\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$ is UECSM if and only if $|a| = |b|$.

- T is UECSM if and only if $T + \lambda I$ is, so every Jordan block is UECSM.
- Jordan form tells us **nothing** about whether or not a matrix is UECSM.

Classes of UECSMs

Certain classes of matrices are known to be UECSM.

- Jordan forms
- Normal matrices ($TT^* = T^*T$)
- Algebraic of degree ≤ 2. ($T^2 + \alpha T + \beta I = 0$)
- Hankel and Toeplitz matrices
- Partial isometries $\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$, where $A^*A + B^*B = I$, are UECSM if and only if A is UECSM.

- If T is UECSM, then $T + \lambda I$, λT and T^* are UECSM
- The set of UECSM's is closed under any norm
- If $T = U|T|$ is UECSM, then $\hat{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ is as well (the Aluthge transform). In general, the Aluthge transform of a CSM is not CSM.
- Bad news: The UECSM matrices are not closed under addition or multiplication.
- None of the proofs that certain kinds of matrices are UECSM can be used to get a hold of general matrices.

The Question

Question

Given a matrix, is it possible to tell if it is UECSM?

Example

Exactly one of the following matrices is UECSM:

$$T_1 = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 6 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 3 \end{bmatrix}.$$

- We want a test that we can easily apply using standard software (`Mathematica`, `Maple`, `Matlab`).
- We cannot use eigenvalues or Jordan form, so we need new techniques.

The plan

1. Try to avoid considering *every* unitary matrix. We'll accomplish this by using conjugation operators.
2. Try to avoid considering *every* matrix. We'll use the Cartesian decomposition $T = A + iB$ to reduce the problem to self-adjoint matrices.
3. Where necessary, make simplifying assumptions that hold on all but a set of measure 0.

Conjugations

Definition

The map $C : \mathbb{C}^n \to \mathbb{C}^n$ is called a *conjugation* if:

(i) $C(\alpha x + \beta y) = \overline{\alpha} Cx + \overline{\beta} Cy$ (conjugate linear)

(ii) $C^2 = I$ (involutive)

(iii) $\langle x, y \rangle = \langle Cy, Cx \rangle$ (isometric)

Example

The simplest example is coordinate-wise complex conjugation on $\mathbb{C}^n$:

$$K \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \overline{x_1} \\ \vdots \\ \overline{x_n} \end{bmatrix}$$

In fact, *all* conjugation operators arise in this way.

Lemma

For every conjugation C, there is an orthonormal basis $\{e_k\}$ such that $Ce_k = e_k$.

Such a basis is called *C-real*. C is nothing but complex conjugation with respect to this basis. That is,

$$\begin{aligned} Cx &= C\left(\sum_{k=1}^{n} \langle x, e_k \rangle e_k\right) \\ &= \sum_{k=1}^{n} \overline{\langle x, e_k \rangle} e_k \end{aligned}$$

The standard conjugation K has the standard basis as a K-real basis

C-symmetry

Conjugations encode basis information, but because they are involutions they can be easier to deal with than unitary matrices.

Definition

If C is a conjugation and T is a linear operator satisfying $T = CT^*C$, then T is called *C-symmetric*.

Lemma

A matrix is UECSM if and only if it's operator is C-symmetric for some C.

Proof: It is easy to check that T is complex symmetric with respect to any C-real basis.

Cartesian decomposition

- We can work with conjugations instead of unitaries. Now we want to simplify the matrices that we have to work with.
- We will accomplish this via a matrix's Cartesian decomposition.

Proposition (Cartesian decomposition)

Given any matrix T, there exist unique self-adjoint matrices A and B such that $T = A + iB$.

- We can find the Cartesian decomposition using standard math software. In fact,

$$A = \frac{1}{2}(T + T^*), \quad B = \frac{1}{2i}(T - T^*)$$

- The Spectral Theorem says that any self-adjoint matrix is UECSM. For T to be UECSM, we need A and B to work together “nicely.”

The following lemma says what it means for A and B to work together "nicely":

Lemma

For a given C, T is C-symmetric if and only if A and B are C-symmetric.

($\Longrightarrow$) holds because $A = \frac{1}{2}(T + T^*)$ and $B = \frac{1}{2i}(T - T^*)$.
($\Longleftarrow$) is a simple calculation.

New Question

Given self-adjoint matrices A and B, can we tell if they are ever simultaneously C-symmetric?

C-symmetry of self-adjoint matrices

We need to understand the C-symmetry of self-adjoint matrices better.

Lemma

If $A = A^*$ and A is C-symmetric, then there is a C-real basis of eigenvectors of A

Proof: Recall that self-adjoint matrices have mutually orthogonal eigenspaces. C preserves the eigenspaces of A, and so we can just take a C-real basis of each eigenspace.

If A and B are *both* C-symmetric, how do these bases interact?

Lemma

A and B are simultaneously C-symmetric if and only if they have orthonormal bases of eigenvectors $\{e_k\}$ and $\{f_j\}$ such that $\langle e_k, f_j \rangle \in \mathbb{R}$.

Proof.

($\Longrightarrow$) Suppose A and B both simultaneously C-symmetric. Then they have C-real bases of eigenvectors $\{e_k\}$ and $\{f_j\}$, respectively.

$$\begin{aligned} \langle e_k, f_j \rangle &= \langle Ce_k, Cf_j \rangle \\ &= \langle f_j, e_k \rangle \\ &= \overline{\langle e_k, f_j \rangle} \end{aligned}$$

□

Proof (cont.)

($\Longleftarrow$) Suppose A and B have bases of eigenvectors $\{e_k\}$ and $\{f_j\}$ such that $\langle e_k, f_j \rangle \in \mathbb{R}$. Define

$$Cx = \sum_{k=1}^{n} \overline{\langle x, e_k \rangle} e_k.$$

It is easy to check that $Ce_k = e_k$, $Cf_j = f_j$ and A and B are C-symmetric. □

- Notice that whether or not $T = A + iB$ is UECSM depends only on the eigenvectors of A and B.
- We can do anything we want to the eigenvalues of A and B.

Exhibiting the unitary equivalence

- The previous lemma is entirely constructive.
- If $T = A + iB$ and the bases of eigenvectors $\{e_k\}$ and $\{f_j\}$ satisfy $\langle e_k, f_j \rangle \in \mathbb{R}$, then define

$$U = \left(\; e_1 \; \middle| \; \ldots \; \middle| \; e_n \; \right).$$

Proposition

T is C-symmetric for $C = UU^t K$, where K is the standard conjugation, and $U^* T U$ is a CSM.

The general algorithm

- Our goal is to be able to compute "nice" eigenvectors for A and B using standard software packages, if possible, or determine that it cannot be done.
- `Mathematica` will give you a set of eigenvectors for A and B, but not all of them.
- Repeated eigenvalues give a lot of flexibility, so first consider if both A and B have simple spectra.
- Given one set of eigenvectors $\{e_k\}$ for A, any other set of eigenvectors is of the form $\{\omega_k e_k\}$ where $|\omega_k| = 1$. Similarly, any other bases for B can be written $\{\zeta_j f_j\}$.

- So `Mathematica` gives us $\{e_k\}$ and $\{f_j\}$, eigenvectors of A and B, respectively.
- We have $2n$ degrees of freedom in finding other bases of eigenvectors, one unimodular constant for each e_k and f_j.
- We have to satisfy n^2 conditions: $\langle e_k, f_j \rangle \in \mathbb{R}$. That is, we need the following matrix to have real entries:

$$\begin{pmatrix} \langle e_1, f_1 \rangle & \cdots & \langle e_1, f_n \rangle \\ \vdots & \ddots & \vdots \\ \langle e_n, f_1 \rangle & \cdots & \langle e_n, f_n \rangle \end{pmatrix}$$

- Multiplying e_k by ω_k scales every entry in the kth row by ω_k. Multiplying f_j by ζ_j scales every entry in the jth column by $\overline{\zeta_j}$.

- To get a general answer, we are forced to temporarily make another non-degeneracy assumption: assume $\langle e_k, f_j \rangle \neq 0$ if $k = 1$ or $j = 1$.

Definition

If $T = A + iB$, and $\{e_k\}$ and $\{f_j\}$ are ONB of eigenvectors of A and B, we call the setup non-degenerate if A and B have simple spectra and $\langle e_k, f_j \rangle \neq 0$ for all $k = 1$ and $j = 1$.

Under the assumption of non-degeneracy we get:

Theorem

$T = A + iB$ is UECSM if and only if

$$\frac{\langle e_k, f_j \rangle \langle e_1, f_1 \rangle}{\langle e_1, f_j \rangle \langle e_k, f_1 \rangle} \in \mathbb{R}$$

for $k, j > 1$.

`Mathematica` can check this condition for an arbitrary matrix that satisfies our non-degeneracy conditions.

Proof sketch

Consider the matrix

$$\begin{pmatrix} \langle e_1, f_1 \rangle & \cdots & \langle e_1, f_n \rangle \\ \vdots & \ddots & \vdots \\ \langle e_n, f_1 \rangle & \cdots & \langle e_n, f_n \rangle \end{pmatrix}$$

We can scale rows and columns, and want every entry to be real. The previous theorem says that all we can do is make the first row and column real. T is UECSM if and only if we made all of the entries real in the process.

Applications

- We now have a numerical test that can answer whether or not $T = A + iB$ is UECSM, assuming that A and B have simple spectra and no eigenvector of A is orthogonal to an eigenvector of B.
- These assumptions hold for almost every matrix. This is useful for probabilistic searches.
- Example: Are all 4×4 partial isometries UECSM? Affirmative answer was known if rank is 0, 1, 3 or 4.
- Testing 100,000 randomly generated rank-2 4×4 partial isometries yielded 100,000 UECSM's.
- Garcia and Wogen later proved it.

Theorem

All 2×2 matrices are UECSM.

Proof.

Suppose $T = A + iB$ is 2×2. Take arbitrary bases of eigenvectors, $\{e_k\}$ and $\{f_j\}$, of A and B respectively. Form the matrix $U = (\langle e_k, f_j \rangle)$ and observe that U is unitary. If any of the entries are 0, the T is normal and the theorem is trivially true. Otherwise, the rows of U are orthogonal, which tells us

$$\langle e_1, f_1 \rangle \overline{\langle e_1, f_2 \rangle} + \langle e_2, f_1 \rangle \overline{\langle e_2, f_2 \rangle} = 0.$$

A simple computation shows that the hypothesis of the previous theorem holds. □

Dealing with degenerate matrices

Right now, we have a test that works as follows:

- Given T, calculate self-adjoint A and B such that $T = A + iB$.
- Calculate orthonormal bases of eigenvectors of A and B
- If neither A nor B has a repeated eigenvalue, and no pair of eigenvectors is orthogonal, we have a necessary and sufficient numerical test.
- Even if one of the degeneracy conditions holds, the test is sufficient.

If T is 3×3 (the simplest non-trivial case), we can also handle the degenerate cases.

Lemma

If $T = A + iB$ is 3×3, and either of the degeneracy conditions holds, then T is UECSM.

The proof of this lemma is not enlightening by itself, but it is nice to have a complete characterization all the way up through 3×3. Moreover, for 3×3 matrices, every step in the algorithm can be performed exactly. The most complicated operation is finding roots of a cubic equation.

Back to our puzzle!

We can finally return to our original example.

Example

Exactly one of the following matrices is UECSM:

$$T_1 = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 6 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 3 \end{bmatrix}.$$

Neither matrix is degenerate. Using the steps above, we get that

$$U^* T_1 U = \begin{pmatrix} \frac{56}{37} - i\sqrt{\frac{37}{2}} & -\frac{55}{37} & \frac{35\sqrt{55}}{74} \\ -\frac{55}{37} & \frac{56}{37} + i\sqrt{\frac{37}{2}} & \frac{35\sqrt{55}}{74} \\ \frac{35\sqrt{55}}{74} & \frac{35\sqrt{55}}{74} & \frac{147}{37} \end{pmatrix}$$

Where

$$U = \begin{pmatrix} \frac{7}{2\sqrt{37}} & \frac{7\left(-19+6i\sqrt{74}\right)}{110\sqrt{37}} & -\sqrt{\frac{5}{814}\left(-19+6i\sqrt{74}\right)} \\ -\frac{i}{\sqrt{2}} & -\frac{19i}{55\sqrt{2}} - \frac{6\sqrt{37}}{55} & 0 \\ \frac{5}{2\sqrt{37}} & \frac{-19+6i\sqrt{74}}{22\sqrt{37}} & 7\sqrt{\frac{-19+6i\sqrt{74}}{4070}} \end{pmatrix}$$

Sadly, T_2 just won't work... we can calculate A and B, find bases of eigenvectors $\{e_k, f_j\}$, and calculate the inner product matrix

$$(\langle e_k, f_j \rangle) = \begin{pmatrix} 1.8 & 1.8 & 1.5 \\ 5.2 & 2.9 - 4.3i & -2.5 + 1.3i \\ 2.3 & -1.9 + 1.3i & -0.1 - 0.4i \end{pmatrix}$$

The first row and column are real, but the rest of the entries are strictly complex. Therefore T_2 is not UECSM.

- There are categories of matrices known to be UECSM (normal, certain partial isometries, etc.)
- Neither of these matrices fit any prior results. We don't know any deep reason why one is UECSM, and the other is not.

Future work

- Unfortunately, this test does not tell us anything about the topological properties of the class of UECSM matrices.
- Even restricted to self-adjoint matrices, taking eigenvectors is discontinuous.

Question

Does the set of UECSM matrices have empty interior? (I would imagine "yes.")