A direct proof of a result of Wassermann

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Abstract

Guided by Wassermann's *Operator Algebras and Conformal Field Theory III*, we will define the basic projective representation of the loop group LSU(n) on Fermionic Fock space. We'll briefly discuss subfactors arising from local loop groups, providing a direct proof (using Sobolev space techniques) that positive energy representations of local loop groups "cannot see points."

1 Introduction

1.1 Positive energy representations

Our main object of study will be *positive energy representations* of loop groups $LG = C^\infty(S^1, G)$ where $G = SU_n$ or $G = S^1$.

Definition 1.1. A *positive energy representation* of LG is a projective unitary representation of $LG \rtimes \mathbb{T}$ on a Hilbert space $\mathcal{H}$ which restricts to an ordinary representation of $\mathbb{T}$, also satisfying

$$\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}(n)$$

where $\dim \mathcal{H}(n) < \infty$ and $\mathbb{T}$ acts on $\mathcal{H}(n)$ by multiplication by $\omega \cdot \xi = \omega^n \xi$.

Since the irreducible representatinos of $\mathbb{T}$ are one-dimensional, they're simply the characters of $\mathbb{T}$ - known to be given by $\omega \mapsto \omega^n$ for $n \in \mathbb{Z}$. The positive energy condition requires that the irreducible summands of the representation only correspond to non-negative n.

Example 1.1. The natural representation of $LG \rtimes \mathbb{T}$ on $L^2(S^1, \mathbb{C}^n)$ is not a positive energy representation. Fix a non-zero vector $v \in \mathbb{C}^n$, and let $e_n(t) = e^{int} \in L^2(S^1, \mathbb{C}^n)$. We then have

$$\omega \cdot \cdot e_{-1} \otimes v = \omega^{-1} e_{-1} \otimes v = \overline{\omega} e_{-1} \otimes v.$$

One potential way to fix this would be to take the natural "representation" of $LG \rtimes \mathbb{T}$ on $p\mathcal{H} \oplus \overline{(1-p)\mathcal{H}}$, where p is the projection onto the Hardy space $H^2(S^1, V)$. The action of $\mathbb{T}$ now has positive energy, but since the action of LG doesn't commute with p, we don't have a $\mathbb{C}$-linear representation.

However, we will build an irreducible positive energy representation π (called the fundamental representation) out of this natural representation, and it will turn out that the irreducible positive energy representations are precisely the irreducible summands of $\pi^{\otimes \ell}$. Subfactors will arise by considering restrictions of positive energy representations to subgropus of LG.

The outline of the talk is as follows.

1. Construct the fundamental representation of LG, and show that it has positive energy according to the actual definition. We will then redefine positive energy representation to mean direct sums of irreducible summands of $\pi^{\otimes \ell}$.
2. State nice results relating to positive energy representations, including relationship with subfactors.
3. Prove that these representations "can't see points."

2 The CAR algebra

Before we can construct the fundamental representation, we need to discuss the Canonical Anticommutation Relations (CAR) algebra of a Hilbert space $\mathcal{H}$. We'll simultaneously follow what the results say for the example $\mathcal{H} = L^2(S^1, \mathbb{C}^n)$.

Let $\Lambda^n \mathcal{H}$ be the subspace of $\bigotimes^n \mathcal{H}$ consisting of antisymmetric elements. We can think of $\Lambda^n \mathcal{H}$ being spanned by symbols $f_1 \wedge \cdots \wedge f_n$ which satisfy the relations $f_{\sigma(1)} \wedge \cdots \wedge f_{\sigma(n)} = (-1)^\sigma f_1 \wedge \cdots \wedge f_n$. Let $\Lambda \mathcal{H} = \bigoplus_{n=0}^{\infty} \Lambda^n \mathcal{H}$ be the exterior algebra of $\mathcal{H}$, where $\Lambda^0(\mathcal{H}) = \mathbb{C}\Omega$. For $f \in \mathcal{H}$, let $a(f)$ be the operator defined by linearly extending $a(f)(f_1 \wedge \cdots \wedge f_n) = f \wedge f_1 \wedge \cdots \wedge f_n$. One can calculate that

$$a(f)^*(f_1 \wedge \cdots \wedge f_{n+1}) = \sum_{k=1}^{n+1} \langle f_k, f \rangle f_1 \wedge \cdots \widehat{f_k} \wedge \cdots \wedge f_{n+1}.$$

A direct calculation shows that if $\|f\| = 1$, then $a(f)a(f)^*$ is a projection on $\Lambda^n \mathcal{H}$, whence $\|a(f)\| = \|f\|$ on the algebraic direct sum $\sum \Lambda^n \mathcal{H}$. Hence we can extend $a(f)$ to $\Lambda \mathcal{H}$ as a bounded operator. The map $f \mapsto a(f)$ is complex linear, and satisfies the canonical anticommutation relations

$$\begin{aligned} a(f)a(g) + a(g)a(f) &= 0, \\ a(f)^* a(g) + a(g) a(f)^* &= \langle g, f \rangle. \end{aligned}$$

We'll let $\mathrm{CAR}(\mathcal{H})$ denote the norm closure of $\{a(f)\}$ as operators on $\Lambda \mathcal{H}$.

Theorem 2.1. *CAR($\mathcal{H}$) acts irreducibly on $\Lambda \mathcal{H}$.*

Proof. Observe that $\mathbb{C}\Omega = \bigcap_f \ker(a(f)^*)$, and hence if $x \in (\mathrm{CAR}(\mathcal{H}))'$ then $x\Omega = \lambda\Omega$. Since Ω is cyclic for $\mathrm{CAR}(\mathcal{H})$, we have $x = \lambda$. □

3 Construction of the fundamental representation

Let p be a projection in $B(\mathcal{H})$, and define $\mathcal{H}_p$ to be the real Hilbert space $\mathcal{H}_\mathbb{R}$, with multiplication by i given by $i(2p-1)$. That is, $\mathcal{H}_p = p\mathcal{H} \oplus \overline{(1-p)\mathcal{H}}$. In our example, p will be the projection onto the Hardy space $H^2(S^1, V)$. We have a natural isomorphism $\Lambda \mathcal{H}_p \cong \Lambda p\mathcal{H} \widehat{\otimes} \overline{\Lambda(1-p)\mathcal{H}}$. Using this, we define a map $\pi_p : a(\mathcal{H}) \to B(\Lambda \mathcal{H}_p)$ via $\pi_p(a(f)) = a(pf) \otimes 1 + 1 \otimes a(\overline{(1-p)f})^*$.

Here, we are using a graded tensor product $(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd)$. Alternatively, we can map $a(f)$ to $a(pf) \otimes 1 + D \otimes a(\overline{(1-p)f})^*$, where D is the parity function on $\overline{\Lambda(1-p)\mathcal{H}}$.

One can verify that $\pi_p(a(f))$ satisfy the CAR, allowing us to extend π_p to all of CAR($\mathcal{H}$). We've used the fact that the algebraic representation on $\Lambda\mathcal{H}$ is actually a faithful representation of the universal CAR algebra. This can be proved easily for finite dimensional Hilbert spaces, and extended to separable Hilbert spaces by a hyperfiniteness argument.

Observe that $\mathbb{T}$ acts naturally on $\mathcal{H}_p$ with positive energy, and this extends to $\Lambda\mathcal{H}_p$. To construct a positive energy representation, we need to extend this action to one of $LG \rtimes \mathbb{T}$. We can do this as follows.

Theorem 3.1. *If $u \in U(\mathcal{H})$ satisfies $\|[u,p]\|_2 < \infty$, then there exists a unique element of $PU(\Lambda\mathcal{H}_p)$, which we call $\pi(u)$, such that $\pi_p(a(uf)) = \pi(u)\pi_p(a(f))\pi(u)^*$.*

Proof. The steps go something like this

1. Show that $\langle \pi_p(a(f_1)^* \cdots a(f_n)^* a(g_1) \cdots a(g_m))\Omega_p, \Omega_p\rangle$ gives the quasi-free state of covariance p.
2. Show that ϕ_q is pure for any projection q.
3. Let $\alpha(a(f)) = a(uf)$. Show that if $\phi_p \circ \alpha$ is equivalent to ϕ_p, then α is implemented on the GNS space of ϕ_p. Recall that to prove equivalent of states, we just need to prove the norm of the difference is less than 2.
4. Show that $\phi_p \circ \alpha = \phi_q$, where $q = u^*pu$, and that $\|\phi_p - \phi_q\| \leq C\|p-q\|_2$.
5. Split $\mathcal{H}$ into a finite-dimensional subspace, and one on which $\|p-q\|_2$ is small to get the equivalence.

$\square$

If $U_{\text{res}} = \{u \in U(\mathcal{H}) : \|[u,p]\|_2 < \infty\}$, then $\pi : U_{\text{res}} \to PU(\Lambda\mathcal{H}_p)$ is a projective representation. In fact, if U_{res} is given the strong operator topology, combined with the premetric $d(u,v) = \|[u-v,p]\|_2$, then π is continuous. That is, if $u_n \to u$ in U_{res}, then there exists $V, V_n \in U(\Lambda\mathcal{H}_p)$ such that $\pi(u_n) = [V_n]$, $\pi(u) = [V]$, and $V_n \to V$ strongly.

We would like to use this theorem to obtain a positive energy representation of LG. We begin by carefully calculating $\|[M_f,p]\|_2$ for $f \in LG$ and $p\mathcal{H} = H^2(S^1, \mathbb{C}^n)$. From now on, we will use $G = S^1$ for simplicity.

Proposition 3.1. *If $f = \sum_{n=-\infty}^{\infty} c_n e_n$, then $\|[M_f,p]\|_2^2 = \sum_{n=-\infty}^{\infty} n|c_n|^2$.*

Proof. If $n \geq 0$, we have

$$\|(pM_f - M_f p)e_n\|^2 = \|(1-p)M_f e_n\|^2 = \sum_{k=n+1}^{\infty} |c_{-k}|^2.$$

Similarly, if $n < 0$,

$$\|(pM_f - M_f p)e_n\|^2 = \|pM_f e_n\|^2 = \sum_{k=-n}^{\infty} |c_k|^2$$

Observe that $|c_n|$ shows up in exactly $|n|$ times, which gives the desired result. $\square$

The preceding proposition shows that $\|[M_f, p]\|_2 \leq \|f'\|_2$, so in particular $LG \subseteq U_{\text{res}}$.

4 Local loop groups (Jones-Wassermann inclusions)

We'll now look at how this gives us subfactors. Let I be the open, upper half of the semi-circle, and let $L_I G$ be the subgroup of LG consisting of loops such that $L\ |_{I^c} \equiv 1$. If π_i is a positive energy representation of LSU_n, one can show that $\pi_i(LG)''$ is a III_1 factor, and that $\pi_i(L_I G)'' \subseteq \pi_i(L_{I^c} G)'$. Observe that all of the difficulties arise from the fact that π_i is a *projective* representation.

5 The irreducibility result

Let A be a finite subset of S^1, and let $L^A G$ be the subset of LG consisting of loops such that $f(a) = 1$ and $f^{(n)}(a) = 0$ for all $a \in A$ and $n \geq 1$. Wassermann states and proves that for any positive energy representation π_i of LG, we have $\pi_i(L^A G)'' = \pi_i(LG)''$. As a corollary, Schur's Lemma tells us that irreducible positive energy representations of LG stay irreducible when restricted to $L^A G$. Wassermann proves this result using some heavy machinery. Our goal is to obtain it by some direct analysis, using some basic theory of Sobolev spaces.

We begin with two key reductions. The frist lemma reduces our question to the analysis of the topology of LG inherited as a subgroup of $U_{\text{res}}(L^2(S^1))_p$.

Recall that the topology of U_{res} is strong operator convergence along with convergence in $d(u, v) = \|[u - v, p]\|_2$. We first observe that L^2 convergence for elements of U_{res} implies SOT convergence. Since U_{res} is bounded in norm, it is sufficient to show that L^2 convergence implies pointwise convergence on a dense subset of $L^2(S^1)$. If $g \in L^\infty(S^1)$, then we have

$$\int_{S^1} |f_n - f|^2 |g|^2 \to 0$$

if $f_n \to f$ in L^2.

Thus the topology on U_{res} is controlled by

$$\|f\|_{H^{1/2}}^2 := \sum_{n=-\infty}^{\infty} (1 + |n|)|c_n(f)|^2.$$

We define $H^{1/2}(S^1) := \{f \in L^2(S^1) : \|f\|_{H^{1/2}} < \infty\}$, the Sobolev space of half-differentiable functions on the circle.

Lemma 5.1. *If π_i is an irreducible positive energy representation of LG, and $X \subseteq LG$ is a subset of LG closed under multiplication by unimodular constants, then $\pi_i(X)'' = \pi_i(\overline{X})''$, where the closure of X is in the topology of U_{res}.*

Proof. We saw that $\pi : U_{\text{res}} \to PU(\Lambda\mathcal{H}_p)$ was continuous. Observe that if u_n are unitary operators on a Hilbert space, and $u_n \to u$ strongly, then $u_n \otimes \cdots \otimes u_n \to u \otimes \cdots \otimes u$ strongly (we clear have strong convergence on linear combinations of simple tensors, and since our operators are uniformly bounded in norm this extends to the closure). Hence $\pi^{\otimes \ell}$ is also a continuous representation, as is $\pi^{\otimes \ell} p$ for any projection p. In particular, this means that if π_i is a positive energy representation on $\mathcal{H}$ and $g_n \to g$ in U_{res}, then we have a sequence of unitaries $v_n \to v$ in $U(\mathcal{H})$ such that $\pi_i(u_n) = [v_n]$ and $\pi_i(u) = [v]$. The result follows. $\square$

It remains to prove that $L^A G$ is dense in LG in the topology of $H^{\frac{1}{2}}$. Clearly it is sufficient to do this for $A = \{0\}$. The next reduction allows us to consider the Lie algebra $C^\infty(S^1, \mathbb{R})$ instead of $C^\infty(S^1, S^1)$.

Lemma 5.2. *Suppose that loops in $C^\infty(S^1, \mathbb{R})$ such that $f^{(n)}(0) = 0$ for $n \geq 0$ are $H^{\frac{1}{2}}$ dense in $C^\infty(S^1, \mathbb{R})$. Then loops in $C^\infty(S^1, S^1)$ such that $g^{(n)}(0) = \delta_{n0}$ for $n \geq 0$ are $H^{\frac{1}{2}}$ dense in $C^\infty(S^1, S^1)$.*

Proof. Before beginning the proof, we wish to establish that U_{res} is a topological group in its natural topology. Since U_{res} is norm bounded and consists of normal operators, both adjoint and multiplication are strongly continuous. Clearly the adjoint is continuous with respect to $\|[u, p]\|_2$, and the simple estimate

$$\|[uv, p]\|_2 \leq \|u(vp - pv)\|_2 + \|(up - pu)v\|_2 = \|[v, p]\|_2 + \|[u, p]\|_2$$

gives that multiplication is jointly continuous.

Now fix $F \in C^\infty(S^1, S^1)$. Let ℓ be the winding number of F, and let G be an element of $C^\infty(S^1, S^1)$ whose winding number is $-\ell$ and whose support is bounded away from 0 (say, contained in $(\frac{1}{2}, 1)$). Now FG has winding number 0. If we can show that FG can be approximated by elements $H_k \in C^\infty(S^1, S^1)$ such that $H_k^{(n)}(0) = \delta_{n0}$, then F can be approximated by $H_k \overline{G}$ by the continuity of multiplication. Thus we will assume without loss of generality that the winding number of F is 0.

Using analytic continuation of $-i \log(z)$, we may construct a map $f : S^1 \to \mathbb{R}$ such that $e^{if(t)} = F(t)$ that is smooth in $(-1, 1)$. Since F has winding number 0, we in fact have $f \in C^\infty(S^1, \mathbb{R})$. By our hypothesis, we can choose a sequence $f_n \in C^\infty(S^1, \mathbb{R})$ such that $f_n \to f$ in $H^{\frac{1}{2}}$ and f_n has a zero of infinite order at $t = 0$. Then the sequence e^{if_n} is in $C^\infty(S^1, S^1)$, and takes on the value 1 with all derivatives vanishing at $t = 0$. We now show $e^{if_n} \to e^{if} = F$ in $H^{\frac{1}{2}}$.

Let $h_n = f_n - f$, so that $h_n \to 0$ in $H^{\frac{1}{2}}$. We now invoke an alternate characterization of the $H^{\frac{1}{2}}$ norm (for more, see *Loop Groups* by Pressley and Segal). We have

$$\|\gamma(t)\|^2_{H^{\frac{1}{2}}} = \|\gamma(t)\|^2_{L^2} + \int_{-1}^{1} \int_{-1}^{1} |\gamma(t) - \gamma(s)|^2 \cot^2\left(\tfrac{1}{2}(t - s)\right).$$

Applying the mean value theorem, we get

$$\begin{aligned}
\|\exp(ih_n)\|^2_{H^{\frac{1}{2}}} &= \int_{-1}^{1}\int_{-1}^{1} |\exp(ih_n(t)) - \exp(ih_n(s))|^2 \cot^2\left(\tfrac{1}{2}(t-s)\right) \\
&\leq \int_{-1}^{1}\int_{-1}^{1} |h_n(t)) - h_n(s)|^2 \cot^2\left(\tfrac{1}{2}(t-s)\right) \\
&= \|h_n\|_{H^{\frac{1}{2}}}.
\end{aligned}$$

A similar (but simpler) argument shows that $e^{ih_n} \to 1$ in L^2. Thus since $h_n \to 0$ in $H^{\frac{1}{2}}$, we have $e^{ih_n} \to 1$ in $H^{\frac{1}{2}}$. Thus $e^{if_n} = e^{ih_n}e^{if} \to e^{if} = F$ by the continuity of multiplication. □

We now employ Sobolev space techniques to verify the hypothesis of Lemma 5.2.

5.1 Intro to Sobolev spaces

Our proof that L^AG is $H^{1/2}$-dense in LG will rely on intermediate steps that involve elements of $H^{1/2}$ that are neither smooth nor take values in the circle. We will also be using the Sobolev space $H^1(S^1)$, whose norm is given by

$$\|f\|^2_{H^1} := \sum_{n=-\infty}^{\infty} (1+n^2)|c_n(f)|^2.$$

In practice, it is easier to work with H^1, as it can be characterized as the space of weakly differentiable functions whose (weak) derivatives are L^2 functions. However, L^AG is *not* H^1-dense in LG. First, some basic results.

Proposition 5.1. *$C^\infty(S^1)$ is $H^{1/2}$-dense in $H^{1/2}(S^1)$.*

Proof. If $f \in H^s$, then its Fourier series converges to it in H^s. □

Proposition 5.2. *If $f : S^1 \to \mathbb{C}$ is piecewise C^1 and continuous, then it is an element of H^1.*

Proof. Let $f'(t)$ be the a.e. defined derivative of f, which we will show is its derivative in the sense of H^1. For simplicity, we will assume that f's only point of discontinuity is at 0. If $g \in C^\infty(S^1)$, then

$$\begin{aligned}
\int_{-1}^{1} f(t)g'(t)dt &= \int_{-1}^{0} f(t)g'(t)dt + \int_{0}^{1} f(t)g'(t)dt \\
&= -\int_{-1}^{0} f'(t)g(t)dt + f(0^-)g(0^-) - f(-1^+)g(-1^+) - \\
&\quad -\int_{0}^{1} f'(t)g(t)dt + f(1^-)g(1^-) - f(0^+)g(0^+) \\
&= -\int_{-1}^{1} f'(t)g(t)dt.
\end{aligned}$$

□

5.2 Proof of the main result

Proposition 5.3. *The set* $\{f \in C^\infty(S^1, \mathbb{R}) : f(0) = 0\}$ *is* $H^{1/2}$*-dense in* LG.

Proof. We first show that the linear functional δ_0 on $C^\infty(S^1)$ is unbounded with respect to the $H^{1/2}$ norm. Suppose to a contradiction that there is a constant C such that

$$\left| \sum_{n=-\infty}^{\infty} c_n \right| \leq C \left(\sum_{n=-\infty}^{\infty} (1 + |n|)|c_n|^2 \right)^{\frac{1}{2}}$$

for every $c_n \in \mathcal{F}(H^{1/2}(S^1))$. This is equivalent to saying that for every $(d_n) \in \ell^2(\mathbb{Z})$ we have

$$\begin{aligned} \left| d_0 + \sum_{n \neq 0} \frac{d_n}{\sqrt{|n|}} \right| &\leq C \left(|d_0|^2 + \sum_{n \neq 0} (\frac{1}{|n|} + 1)|d_n|^2 \right)^{\frac{1}{2}} \\ &\leq 2C \|d_n\|_{\ell^2} \end{aligned}$$

This would say that the vector $v = (v_n) \in \ell^2(\mathbb{Z})$, where $v_0 = 1$ and $v_n = |n|^{-\frac{1}{2}}$, which is a contradiction.

Hence $\ker \delta_0$ (thinking of δ_0 defined on smooth functions) must be dense in $H^{1/2}$, as otherwise we could choose by Hahn-Banach a continuous linear functional on $H^{1/2}$ that vanished on $\overline{\ker \delta_0}$, which would then restrict to a bounded linear functional on C^∞. □

We will now approximate loops that vanish at 0 with piecewise smooth loops that vanish in a neighborhood of 0.

Proposition 5.4. *If* $f \in C^\infty(S^1, \mathbb{R})$ *with* $f(0) = 0$ *is as above, then there is a sequence of piecewise smooth functions* $f_n \in H^1(S^1, \mathbb{R})$ *such that* $f_n \equiv 0$ *in some neighborhood of 0,* $f_n \to f$ *in* H^1.

Proof. Define

$$f_\delta(t) = \begin{cases} f(t) & |t| > 2\delta \\ 0 & |t| < \delta \\ f(2(t+\delta)) & -2\delta < t < \delta \\ f(2(t-\delta)) & \delta < t < 2\delta \end{cases}$$

This function is piecewise-smooth and continuous, and is thus an element of H^1, with weak derivative given by the piecewise derivatives. Clearly

$$\|f_\delta - f\|_{L^2} + \|f'_\delta - f'\|_{L^2} \leq 4\delta \left(2\|f\|_\infty + 3\|f'\|_\infty\right).$$

This shows that $f_\delta \to f$ in the H^1 topology as $\delta \to 0$. Also,

$$\|f_\delta - f\|_\infty \leq \sup_{t,s \in [-2\delta, 2\delta]} |f(t) - f(s)|$$

which goes to zero by the uniform continuity of f. □

We now wish to replace the piecewise smooth functions from the previous proposition with smooth functions that still vanish in some neighborhood of 0. Simply truncating the Fourier series will not preserve this property, so we will use mollifiers.

5.3 Properties of mollifiers

Let $\eta(x) = Ce^{-1/(1-x^2)}$ if $|x| < 1$ and $\eta(x) = 0$ otherwise. Choose the constant C so that $\|\eta\|_{L^1} = 1$. Let $\eta^\epsilon(x) = \epsilon^{-1}\eta(\frac{x}{\epsilon})$. We will use the following basic facts from PDE.

Proposition 5.5. *If $f \in L^2(S^1)$, then $\eta^\epsilon * f \in C^\infty(S^1)$ and $\eta^\epsilon * f \to f$ in L^2. If $f \in H^1(S^1)$, then $(\eta^\epsilon * f)' = \eta^\epsilon * f'$.*

Proof.

$$h^{-1}\left((\eta^\epsilon * f)(x+h) - (\eta^\epsilon * f)(x)\right) = \int_{-1}^{1} \frac{1}{h}\left[\eta\left(\frac{x+h-y}{\epsilon}\right) - \eta\left(\frac{x-y}{\epsilon}\right)\right] f(y)dy$$

which converges to $(\eta^\epsilon)' * f$ by the dominated convergence theorem. Hence $\eta^\epsilon * f$ is smooth. If $f \in H^1$, integration by parts shows that $(\eta^\epsilon * f)' = \eta^\epsilon * f'$. □

Proposition 5.6. *If $f \in C(S^1) \cap H^1(S^1)$, then $\eta^\epsilon * f \to f$ uniformly and in H^1.*

Proof. Let $f^\epsilon = \eta^\epsilon * f$. First we prove the uniform convergence. We have

$$\begin{aligned} |f^\epsilon(x) - f(x)| &\leq \frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon} \eta\left(\frac{x-y}{\epsilon}\right)|f(y) - f(x)|dy \\ &\leq 2\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} |f(y) - f(x)|dy \end{aligned}$$

which goes to zero uniformly in x by the uniform continuity of f.

By the preceding proposition, $(\eta^\epsilon * f)' = \eta^\epsilon * f'$. We will show that this converges to f' in L^2. First we show that if $g \in L^2(S^1)$, then $\|g^\epsilon\|_2 \leq \|g\|_2$. Observe

$$\begin{aligned} |g^\epsilon(x)| &= \left|\int_{-1}^{1} \eta^\epsilon(x-y)g(y)dy\right| \\ &\leq \left(\int_{-1}^{1} \eta^\epsilon(x-y)dy\right)^{\frac{1}{2}} \left(\int_{-1}^{1} \eta^\epsilon(x-y)|g(y)|^2dy\right)^{\frac{1}{2}} \\ &= \left(\int_{-1}^{1} \eta^\epsilon(x-y)|g(y)|^2dys\right)^{\frac{1}{2}}. \end{aligned}$$

Taking two norm and interchanging the integrals gives

$$\|g^\epsilon\|_2^2 \leq \int_{-1}^{1} |g(y)|^2 \left(\int_{-1}^{1} \eta^\epsilon(x-y)dx\right) dy = \|g\|_2^2$$

Now let h be a continuous function such that $\|h - f'\|_2$ is very small. We then have

$$\begin{aligned} \|f' - \eta^\epsilon * f'\|_2 &\leq \|f' - h\|_2 + \|h - h * \eta^\epsilon\|_2 + \|(h * \eta^\epsilon - f' * \eta^\epsilon\|_2 \\ &\leq 2\|f' - h\|_2 + \|h - h * \eta^\epsilon\|_2. \end{aligned}$$

By the first half of the proposition, $h^\epsilon \to h$ uniformly, which completes the proof. □

Combining the results of Section 5, we get the desired theorem.

Theorem 5.1. *If π_i is an irreducible positive energy representation of LS^1, then $(\pi_i(L^AS^1))'' = (\pi_i(LS^1))''$.*