

MATH 117: HOMEWORK 2

Due Sunday, October 11th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1*

The textbook proves the following result (see Theorem 3.2, p16):

THEOREM. *Suppose F is an ordered field. Then the following properties hold for all $a, b, c \in F$:*

- (i) *if $a \leq b$, then $-b \leq -a$;*
- (ii) *if $a \leq b$ and $c \leq 0$, then $bc \leq ac$;*
- (iii) *if $0 \leq a$ and $0 \leq b$, then $0 \leq ab$;*
- (iv) *$0 \leq a^2$, where a^2 is an abbreviation for $a \cdot a$;*
- (v) *if $0 < a$, then $0 < 1/a$.*

Using this theorem and the definition of an ordered field, prove the following results for an ordered field F . Explain which properties of a field you use (A1-A4, M1-M4, DL), which properties of an ordered field you use (O1-O5), and which parts of the theorem above you use (i-v).

- (a) For 0 and 1 as in the definition of a field, $0 < 1$.
- (b) For all $a, b \in F$, if $0 < a < b$, then $0 < 1/b < 1/a$.

Question 2*

Prove that if S is a nonempty subset of $\mathbb{R}$ with maximum s_0 , then $\sup(S) = s_0$.

Question 3

Prove by induction that, for $n \in \mathbb{N}$ and $y_1, y_2, \dots, y_n \in \mathbb{R}$,

$$|y_1 + y_2 + \dots + y_n| \leq |y_1| + |y_2| + \dots + |y_n|.$$

Question 4*

Let T be a nonempty subset of $\mathbb{R}$ that is bounded above.

- (a) Prove that $\sup T \in T$ implies $\sup T = \max T$.
- (b) Prove that if $\sup T$ is not an element of T , then the maximum of T does not exist.

Question 5

Let S be a nonempty bounded subset of $\mathbb{R}$.

- (a) Prove that $\inf S \leq \sup S$.
- (b) What can you say about the number of elements in S if $\inf S = \sup S$? Justify your answer.

Question 6*

Suppose A and B are nonempty bounded subsets of $\mathbb{R}$.

- (a) Prove that if $A \subseteq B$, then $\inf B \leq \inf A \leq \sup A \leq \sup B$.
- (b) Prove $\sup(A \cup B) = \max\{\sup A, \sup B\}$. (Note: for this part, do not assume $A \subseteq B$.)

Question 7*

Consider the following proposition:

PROPOSITION. *Every nonempty subset S of $\mathbb{R}$ that is bounded below has an infimum.*

(See also Corollary 4.5 on p23 of the textbook.)

This question will lead you through the proof of the proposition.

- (a) Suppose S is as in the proposition above. Define the set $-S = \{-s : s \in S\}$. Show that $-S$ is bounded above.
- (b) Use the definition of $\mathbb{R}$ to prove that $-S$ has a supremum, $\sup(-S)$.
- (c) Prove that $-\sup(-S)$ is the infimum of S .

Question 8*

Let S and T be nonempty bounded subsets of $\mathbb{R}$, and define $S + T = \{s + t : s \in S \text{ and } t \in T\}$. Prove $\inf(S + T) = \inf S + \inf T$.

Hint: First show that $\inf S + \inf T$ is a lower bound for $S + T$. For the opposite inequality, show that for each $t \in T$, $\inf(S + T) - t$ is a lower bound for S . Hence $\inf S \geq \inf(S + T) - t$ for all $t \in T$. Then show that $\inf(S + T) - \inf S$ is a lower bound for T .

Question 9

Let A and B be nonempty subsets of $\mathbb{R}$ with the following property: $a \leq b$ for all $a \in A$ and $b \in B$.

- (a) Show that A is bounded above and B is bounded below.
- (b) Prove $\sup A \leq \inf B$.
- (c) Given an example of A and B satisfying the above property where $A \cap B$ is nonempty.
- (d) Give an example of A and B satisfying the above property where $\sup A = \inf B$ and $A \cap B$ is the empty set. You do not need to justify your example with a proof.

Question 10*

For each of the sets below, answer the following questions: Is it bounded above? If so, what is its supremum? Is it bounded below? If so, what is its infimum? You do not need to justify your answers.

- (a) $[-\sqrt{2}, \sqrt{2}]$
- (b) $\{-1, 0, e, \pi\}$
- (c) $\{1\}$
- (d) $\cup_{n=1}^{\infty} [2n-1, 2n)$
- (e) $\{1 - \frac{1}{n} : n \in \mathbb{N}\}$
- (f) $\{x \in \mathbb{R} : x^2 < 1\}$
- (g) $\cap_{n=1}^{\infty} (-1 - \frac{1}{n}, 1 + \frac{1}{n})$

Question 11

Follow the same instructions as in Question 10 for the following sets:

- (a) $\{\frac{1}{n^2} : n \in \mathbb{N}\}$
- (b) $\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$
- (c) $\{q \in \mathbb{Q} : q \geq 0\}$
- (d) $\{q \in \mathbb{Q} : q^2 \geq 0\}$
- (e) $\cap_{n=1}^{\infty} (\frac{-1}{n}, \frac{1}{n})$
- (f) $\{x \in \mathbb{R} : x^3 \geq 2\}$
- (g) $\{\sin(n\pi) : n \in \mathbb{N}\}$