Question 1

Homework 2

(1)(a) By (M3), $1^2 = 1$, so by part (iv) of the theorem, $0 \neq 1^2 = 1$. By (M3), $0 \neq 1$, so $0 < 1$.

(b) By part (v) of the Theorem, $0 < \frac{1}{b}$ and $0 < \frac{1}{a}$, so it suffices to show $\frac{1}{b} < \frac{1}{a}$.

Note that $a < b \Rightarrow a \leq b$

$\overset{(OS)}{\Rightarrow} a \cdot \left(\frac{1}{a} \cdot \frac{1}{b}\right) \leq b \cdot \left(\frac{1}{a} \cdot \frac{1}{b}\right) \overset{(M2)}{\Rightarrow} a \cdot \left(\frac{1}{a} \cdot \frac{1}{b}\right) \leq b\left(\frac{1}{b} \cdot \frac{1}{a}\right)$

$\overset{(M1)}{\Rightarrow} \left(a \cdot \frac{1}{a}\right) \cdot \frac{1}{b} \leq \left(b \cdot \frac{1}{b}\right) \cdot \frac{1}{a} \overset{(M4)}{\Rightarrow} 1 \cdot \frac{1}{b} \leq 1 \cdot \frac{1}{a}$

$\overset{(M2)}{\Rightarrow} \frac{1}{b} \cdot 1 \leq \frac{1}{a} \cdot 1 \overset{(M3)}{\Rightarrow} \frac{1}{b} \leq \frac{1}{a}$.

It remains to show $\frac{1}{b} \neq \frac{1}{a}$. Suppose for the sake of contradiction that $\frac{1}{b} = \frac{1}{a}$.

Then, $\frac{1}{b} = \frac{1}{a} \Rightarrow \frac{1}{b}(ab) = \frac{1}{a}(a \cdot b)$

$\overset{(M1 \text{ and } M2)}{\Rightarrow} \left(\frac{1}{b} \cdot b\right) \cdot a = \left(\frac{1}{a} \cdot a\right) \cdot b \overset{(M4)}{\Rightarrow} 1 \cdot a = 1 \cdot b \overset{(M2, M3)}{\Rightarrow} a = b$,

which contradicts that $a < b$. Therefore $\frac{1}{b} \neq \frac{1}{a}$

Question 2

Homework 2

(2) If $s_0 = \max(S)$, then $s_0 \in S$ and $s_0 \geq s \ \forall s \in S$. Consequently, s_0 is an upper bound of S. Suppose s_1 is another upper bound of S. Since $s_0 \in S$, $s_1 \geq s_0$. Thus, s_0 is the least upper bound of S, so $s_0 = \sup(S)$.

Question 3

Homework 2

③ mild notational difference: replace a_i with y_i.

Base case: When $n=1$, $|a_1| \leq |a_1|$.

Inductive step: Assume $|a_1+a_2+\ldots+a_n| \leq |a_1|+\ldots+|a_n|$

By the triangle inequality (as stated in part (a)) with $x=a_1+a_2+\ldots+a_n$ and $y=a_{n+1}$

$|a_1+a_2+\ldots+a_n+a_{n+1}| \leq |a_1+a_2+\ldots+a_n| + |a_{n+1}|$.

By the inductive hypothesis, the right hand side is bounded above by $|a_1|+|a_2|+\ldots+|a_n|+|a_{n+1}|$, which completes the proof.

Question 4

Homework 2

mild notational difference: replace S with T

(4)(a) By definition, $\sup(S) \geq s$ for all $s \in S$. Thus, if $\sup(S) \in S$, it is the largest element of S and $\max(S) = \sup(S)$.

(b) Suppose S has a maximum M_0. By Q2, $M_0 = \sup(S)$. This contradicts the fact that $\sup(S) \notin S$. Therefore, S must not have a maximum.

Question 5

Homework 2

(5)(a) Let s be an element of S. Since $\inf(S)$ is a lower bound for S, $\inf(S) \leq s$. Since $\sup(S)$ is an upper bound for S, $s \leq \sup(S)$. Therefore, $\inf(S) \leq \sup(S)$.

(b) We will show $S = \{\inf(S)\}$, so there is one element in the set. Since $\inf(S) = \sup(S)$, $\inf(S)$ is both an upper and lower bound for S. In particular, for any $s \in S$, $\inf(S) \leq s$ and $\inf(S) \geq s$. Thus $\inf(S) = s$ for all $s \in S$. This shows $S = \{\inf(S)\}$.

Question 6

Homework 2

(7) (a) Since inf(T) is a lower bound for the set T, if $S \subseteq T$, then inf(T) is also a lower bound for the set S. Since inf(S) is the greatest lower bound of S, $\inf(T) \leq \inf(S)$. The fact that $\sup(S) \leq \sup(T)$ follows from an analogous argument. The fact that $\inf(S) \leq \sup(S)$ follows from Q5@.

/ mild notational change: A, B becomes S, T

(b) Since sup(S) is an upper bound for S and sup(T) is an upper bound for T, $\max\{\sup(S), \sup(T)\}$ is an upper bound for SUT. Thus, since sup(SUT) is the least upper bound for SUT, $\sup(SUT) \leq \max\{\sup(S), \sup(T)\}$.

By Q7@, since $S \subseteq SUT$ and $T \subseteq SUT$, we have $\sup(S) \leq \sup(SUT)$ and $\sup(T) \leq \sup(SUT)$. Thus $\max\{\sup(S), \sup(T)\} \leq \sup(SUT)$.

Combining these two inequalities, we conclude $\max\{\sup(S), \sup(T)\} = \sup(SUT)$.

(7) (a) Since S is bounded below, $\exists\, m_0 \in \mathbb{R}$ s.t. $s \geq m_0$ $\forall s \in S$. This implies $-m_0 \geq -s$ $\forall s \in S$, so $-S$ is bounded above.

(b) Since S is nonempty, so is $-S$. Since $-S$ is bounded above, by definition of the real numbers, it has a supremum, $\sup(-S)$.

(c) Since $\sup(-S)$ is an upper bound for $-S$, $-s \leq \sup(-S)$ for all $s \in S$, hence $s \geq -\sup(-S)$ for all $s \in S$. Therefore $-\sup(-S)$ is a lower bound for S, and it suffices to show it is the greatest lower bound.

Suppose for the sake of contradiction that m_0 is a lower bound for S with $m_0 > -\sup(-S)$. As argued in part (a), $-m_0$ is an upper bound for $-S$. Furthermore, $m_0 > -\sup(-S)$ implies $-m_0 < \sup(-S)$. This is a contradiction, since $\sup(-S)$ is the least upper bound for $-S$.

Therefore $-\sup(-S)$ must be the greatest lower bound of S.

Question 8

Homework 2

(8)

Step 1: Show that for all $t \in T$, $\inf(S+T) - t$ is a lower bound for S.

By defn of $S+T$ and the infimum, $\inf(S+T)$ is a lower bound for $S+T$, so $s+t \geq \inf(S+T) \iff s \geq \inf(S+T) - t$ for all $s \in S, t \in T$. Thus, for all $t \in T$, $\inf(S+T) - t$ is a lower bound for S.

Step 2: Show that $\inf(S+T) - \inf(S)$ is a lower bound for T.

By Step 1, for all $t \in T$, $\inf(S+T) - t$ is a lower bound for S. By defn, $\inf(S)$ is the greatest lower bound of S. Thus, $\inf(S) \geq \inf(S+T) - t$
$\iff t \geq \inf(S+T) - \inf(S)$ for all $t \in T$.

Since $\inf(S+T) - \inf(S)$ is a lower bound for T and $\inf(T)$ is the greatest lower bound,

$$\inf(T) \geq \inf(S+T) - \inf(S).$$

$$\Updownarrow$$

$$\inf(S) + \inf(T) \geq \inf(S+T). \quad (*)$$

It remains to prove the opposite inequality. Since $\inf(S)$ and $\inf(T)$ are lower bounds for S and T, for all $s \in S$ and $t \in T$,

$\inf(S) \leq s$ and $\inf(T) \leq t \Rightarrow \inf(S) + \inf(T) \leq s+t$.

Thus, $\inf(S) + \inf(T)$ is a lower bound for $S+T$. Since $\inf(S+T)$ is the greatest lower bound,

$$\inf(S) + \inf(T) \leq \inf(S+T). \quad (**)$$

Thus, combining inequalities $(*)$ and $(**)$, we obtain

$$\inf(S) + \inf(T) = \inf(S+T). \quad \square$$

Question 9

Homework 2

slight notational change: $A=S$, $B=T$

(9) (a) Since $s \leq t$ for all $s \in S$ and $t \in T$, any $t \in T$ is an upper bound for S and any $s \in S$ is a lower bound for T. Hence, S is bounded above and T is bounded below.

(b) As shown in part (a), any $t \in T$ is an upper bound for S. Since $\sup(S)$ is the least upper bound, $\sup(S) \leq t$ for all $t \in T$. Thus, $\sup(S)$ is a lower bound for T, and since $\inf(T)$ is the greatest lower bound, $\sup(S) \leq \inf(T)$.

(c) $S = [0,1]$, $T = [1,2]$

(d) $S = [0,1)$, $T = (1,2]$

Question 10

Homework 2

(10) Throughout, we use S to denote the set under consideration.

(a) $\sup(S) = \sqrt{2}$, $\inf(S) = -\sqrt{2}$

(b) $\sup(S) = \pi$, $\inf(S) = -1$

(c) $\sup(S) = \inf(S) = 1$

(d) S is not bounded above, $\inf(S) = 1$

(e) $\sup(S) = 1$, $\inf(S) = 0$

(f) $\sup(S) = 1$, $\inf(S) = -1$

(g) $S = [-1, 1]$, so $\sup(S) = 1$ and $\inf(S) = -1$

Question 11

Homework 2

(11) (a) $\sup(S) = 1$, $\inf(S) = 0$

(b) S is not bounded above, $\inf(S) = 0$

(c) S is not bounded above, $\inf(S) = 0$

(d) S is neither bounded above or below

(e) $S = \{0\}$, so $\sup(S) = \inf(S) = 0$

(f) S is not bounded above, $\inf(S) = 2^{1/3}$

(g) $\sup(S) = \inf(S) = 0$