

Question 1

Homework 3

① Suppose $a = \sup(S)$. By definition, a is an upper bound of S . Fix $\varepsilon > 0$. Then $a - \varepsilon < a$. Thus, $a - \varepsilon$ must not be an upper bound of S . Hence, there exists $s \in S$ s.t. $a - \varepsilon < s$.

Suppose a is an upper bound of S and, for all $\varepsilon > 0$, $\exists s \in S$ s.t. $s > a - \varepsilon$.

Let m be an upper bound of S .

Assume, for the sake of contradiction, that $m < a$. Then $\varepsilon := a - m > 0$.

Thus, $\exists s \in S$ s.t. $s > a - \varepsilon = m$.

This contradicts that m was an upper bound. Therefore $m \geq a$.

This shows a is the least upper bound or supremum of S .

Question 2

Homework 3

②

① We estimate as follows:

$$(x+\varepsilon)^2 = x^2 + 2\varepsilon x + \varepsilon^2 \stackrel{x \leq 2}{\leq} x^2 + 4\varepsilon + \varepsilon^2 \stackrel{\varepsilon \leq 1 \Rightarrow \varepsilon^2 \leq \varepsilon}{\leq} x^2 + 5\varepsilon$$

$$(y-\varepsilon)^2 = y^2 - 2\varepsilon y + \varepsilon^2 \stackrel{y \leq 2}{\geq} y^2 - 4\varepsilon + \varepsilon^2 \stackrel{\varepsilon^2 \geq 0}{\geq} y^2 - 4\varepsilon.$$

② Since $x^2 < 2$, $\tilde{\varepsilon}_1 := 2 - x^2 > 0$. Let $\varepsilon_1 = \frac{\tilde{\varepsilon}_1}{10}$.
Then $x^2 + 5\varepsilon_1 = x^2 + \frac{\tilde{\varepsilon}_1}{2} < x^2 + \tilde{\varepsilon}_1 = 2$.

Since $y^2 > 2$, $\tilde{\varepsilon}_2 = y^2 - 2 > 0$. Let $\varepsilon_2 = \frac{\tilde{\varepsilon}_2}{8}$.
Then $y^2 - 4\varepsilon_2 = y^2 - \frac{\tilde{\varepsilon}_2}{2} > y^2 - \tilde{\varepsilon}_2 = 2$.

③ We estimate as follows:

$$(x+\varepsilon_1)^2 \stackrel{\text{①}}{\leq} x^2 + 5\varepsilon_1 \stackrel{\text{②}}{<} 2$$

$$(y-\varepsilon_2)^2 \stackrel{\text{①}}{\geq} y^2 - 4\varepsilon_2 \stackrel{\text{②}}{>} 2.$$

Question 3

Homework 3

③

(a) As shown in class, $0^2 = 0$. Thus $0 \in S$, so $S \neq \emptyset$.

Define $m = \max\{a, 1\}$. Suppose $c \in S$. If $c \leq 1$, then $c \leq 1 \leq m$. If $c \geq 1$, then by (05), $c^2 \geq c$, so $c \leq c^2 \leq a \leq m$. This shows S is bounded above by m . By definition, S is bounded below by 0 .

(b) By definition of $\mathbb{R}$, for any nonempty subset of $\mathbb{R}$ that is bounded above, the supremum exists.

Question 4

Homework 3

④

① In ③①, we showed $m = \max\{2, 1\} = 2$ is an upper bound for S . Thus $b \leq 2$. Since $1 \geq 0$ and $1^2 \leq 2$, $1 \in S$, so $b \geq 1$.

② If $b^2 < 2$, by ②① $\exists \varepsilon, \varepsilon \in (0, 1)$ s.t. $(b + \varepsilon)^2 < 2$. Thus $b + \varepsilon \in S$. This contradicts the fact that b is an upper bound of S .

Question 4 — continued

Homework 3

© If $b^2 > 2$, by ②©, $\exists \varepsilon_2 \in (0, 1)$ s.t.
 $(b - \varepsilon_2)^2 > 2$. Thus, if $c > b - \varepsilon_2$,

$$c^2 \geq \overset{c \geq 0}{c} \overset{b - \varepsilon_2 \geq 0}{(b - \varepsilon_2)} \geq (b - \varepsilon_2)^2 > 2,$$

so $c \notin S$. By Q1, this contradicts that b is the supremum.

Question 5

Homework 3

- (5) We first show $\lim a^n = 0$ if $|a| < 1$.
(a) Note that, if $a = 0$, then for all $\varepsilon > 0$ and any $N \in \mathbb{R}$, $n > N$ ensures $|a^n - 0| = 0 < \varepsilon$. Thus, $\lim a^n = 0$.

Now, suppose that $a \neq 0$.
Fix $\varepsilon > 0$. Note that $|a^n - 0| < \varepsilon$
 $\Leftrightarrow |a^n| < \varepsilon \Leftrightarrow |a|^n < \varepsilon \Leftrightarrow n \log(|a|) < \log \varepsilon$
 $\downarrow$ since $|a| < 1$ ensures $\log(|a|) < 0$ $\uparrow$ since $|a^n| = |a|^n$
 $\Leftrightarrow n > \frac{\log \varepsilon}{\log(|a|)}$. Let $N = \frac{\log \varepsilon}{\log(|a|)}$.
Then $n > N$ ensures $|a^n - 0| < \varepsilon$, so
 $\lim a^n = 0$.

We now show $\lim a^n = 1$ if $a = 1$.
Note that if $a = 1$, then $a^n = 1$ for all n .
Fix $\varepsilon > 0$ and choose $N = 1$. Then $n > N$
ensures $|a^n - 1| = 0 < \varepsilon$.

- (b) We conclude by showing a^n
does not converge if $a \leq -1$.

Question 5 — continued

Homework 3

Suppose for the sake of contradiction that a^n converges to some $a \in \mathbb{R}$.
Let $\varepsilon = 1$. Then there exists N s.t. $n > N$ ensures $|a^n - a| < 1 \Leftrightarrow a - 1 < a^n < a + 1$.
For n even, $a^n = |a|^n > 1$ so $1 < a + 1 \Rightarrow 0 < a$.
For n odd, $a^n = -|a|^n < -1$, so $a - 1 < -1 \Rightarrow a < 0$.
This is a contradiction, since no $a \in \mathbb{R}$ can satisfy both $a > 0$ and $a < 0$.

Question 6

Homework 3

6

- (a) We will show that the minimum of S exists if $a \in \mathbb{Q}$ and does not exist if $a \in \mathbb{R} \setminus \mathbb{Q}$.

Suppose $a \in \mathbb{Q}$. By definition of S , $a \in S$. Furthermore, for any $s \in S$, $a \leq s$. Thus $\min(S) = a$.

Suppose $a \in \mathbb{R} \setminus \mathbb{Q}$. Assume, for the sake of contradiction, that the minimum of S exists, and $\min(S) = s_0$. Since $s_0 \in S$, we have $s_0 \geq a$. However, since $a \notin \mathbb{Q}$, we must have $s_0 > a$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ so that $s_0 > r > a$. By definition of S , we must have $r \in S$. This contradicts that s_0 was the minimum of S . Thus, the minimum of S must not exist.

- (b) We will show that $\inf(S) = a$. By definition of S , $a \leq s$ for all $s \in S$, so a is a lower bound for S . Suppose $m_0 \in \mathbb{R}$ is another lower bound of S . Assume for the sake of contradiction that $m_0 > a$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ so that $m_0 > r > a$. Thus, $r \in S$, which contradicts the definition of m_0 as a lower bound of S . This shows $m_0 \leq a$, so a is the greatest lower bound.

Question 7

Homework 3

⑦ Let $S = (a, b]$.

- $\max(S) = b$, since by defn, b is the largest element in S .
- $\sup(S) = b$. b is an upper bound for S and since $b \in S$, no number smaller than b can be an upper bound. Thus b is the least upper bound.
- The minimum of S does not exist. Suppose, for the sake of contradiction that $\min(S) = m_0$. Since $m_0 \in S$, $m_0 > a$. However $\frac{m_0 + a}{2} \in (a, m_0)$, so $\frac{m_0 + a}{2} \in S$ and $\frac{m_0 + a}{2} < m_0$. This contradicts that m_0 was the smallest element in S .
- $\inf(S) = a$. a is a lower bound for S . Suppose $m_0 > a$ was another lower bound. Since $b \in S$, we have $m_0 \in (a, b]$. Furthermore, since $\frac{m_0 + a}{2} \in (a, m_0)$, we have $\frac{m_0 + a}{2} \in S$ and $\frac{m_0 + a}{2} < m_0$.

This contradicts that m_0 was a lower bound of S .

Question 8

Homework 3

⑧ By the Archimedean Property, if $x > 0$ and $y > 0$, then there exists $n \in \mathbb{N}$ so that $nx > y$.

Taking $x=1$ and $y=a$ gives that there exists $n_1 \in \mathbb{N}$ so that $n_1 > a$.

Taking $x=a$ and $y=1$ gives that there exists $n_2 \in \mathbb{N}$ so that $n_2 a > 1 \Leftrightarrow a > \frac{1}{n_2}$.

Let $n = \max\{n_1, n_2\}$. Then $n \in \mathbb{N}$ and $\frac{1}{n} \leq \frac{1}{n_2} < a < n_1 < n$, which gives the result.

Question 9

Homework 3

⑨ Suppose for the sake of contradiction that $a > b$. Then if we define $y = a - b$, $y > 0$, and by Q9, there exists $n \in \mathbb{N}$ so that $\frac{1}{n} < y = a - b$. This implies that there exists $n \in \mathbb{N}$ so that $b + \frac{1}{n} < a$, which is a contradiction. Therefore, we must have $a \leq b$.

Question 10

Homework 3

⑩

(a) A sequence s_n converges to a limit $s \in \mathbb{R}$ if, $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ s.t. $n > N$ ensures $|s_n - s| < \varepsilon$.

(b) A sequence s_n does not converge to a limit $s \in \mathbb{R}$ if $\exists \varepsilon > 0$ s.t. for all $N \in \mathbb{R}$, $\exists n > N$ s.t. $|s_n - s| \geq \varepsilon$.

Question 10 — continued

Homework 3

① Fix $\varepsilon > 0$. Let $N = \frac{4}{\varepsilon}$. Then $n > N$ ensures

$$\frac{4}{n} < \varepsilon \iff \frac{4n}{n^2} < \varepsilon \iff \frac{n+3n}{n^2} < \varepsilon \implies \frac{n+3}{n^2} < \varepsilon \implies$$

$$\frac{|n-3|}{n^2} < \varepsilon \implies \left| \frac{n-3}{n^2} \right| < \varepsilon \implies \left| \frac{n-3}{n^2+9} \right| < \varepsilon \iff \left| \frac{n-3}{n^2+9} - 0 \right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this gives the result.

② Assume, for the sake of contradiction, that s_n converges to some $s \in \mathbb{R}$. Then, for $\varepsilon = 1$, there exists $N \in \mathbb{R}$ so that $n > N$ ensures

$$\begin{aligned} |s_n - s| < 1 &\iff s - 1 < s_n < s + 1 \\ &\iff s - 1 < (n+1)^2 - 2 < s + 1 \\ &\iff s + 1 < (n+1)^2 < s + 3 \end{aligned}$$

By the lemma following the Archimedean Property, $\exists m \in \mathbb{N}$ so that $m > s + 3$.

Question 10 — continued

Homework 3

Again, by the Archimedean Property, $\exists k \in \mathbb{N}$ so that $k > \max(m, N+1)$.

Then $k \geq m > s+3$ and $k > N$.


The latter ensures

Question 10 — continued

Homework 3

$$(k+1)^2 < s+3 \Rightarrow k < k^2 + 2k + 1 < s+3$$

This contradicts $\star$. Thus s_n must not converge to any $s \in \mathbb{R}$.

Question 11

Homework 3

(a) We will show a_n converges to $a = 7/3$. Fix $\epsilon > 0$. Note that

$$\begin{aligned} |a_n - a| &= \left| \frac{7n - 19}{3n + 7} - \frac{7}{3} \right| = \left| \frac{21n - 57}{3(3n + 7)} - \frac{7(3n + 7)}{3(3n + 7)} \right| \\ &= \left| \frac{-106}{3(3n + 7)} \right| = \frac{106}{3(3n + 7)}. \end{aligned}$$

Thus, if $N = \frac{106}{\epsilon}$, for all $n > N$, we have

$$|a_n - a| = \frac{106}{3(3n + 7)} < \frac{106}{n} < \epsilon.$$

Since ϵ was arbitrary, this shows $\lim_{n \rightarrow +\infty} a_n = a$.

(b) We will show b_n does not converge. Note that the elements in the sequence b_n are

$$\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \dots \right),$$

repeating in this way. Assume, for the sake of contradiction, that b_n converges to some $b \in \mathbb{R}$. Then for $\epsilon = \frac{1}{4} > 0$, there exists N so that $n > N$ ensures

$$|b_n - b| < \epsilon \iff b - \epsilon < b_n < b + \epsilon \iff b - \frac{1}{4} < b_n < b + \frac{1}{4}.$$

Since there are infinitely many $n > N$ for which $b_n = -\frac{\sqrt{3}}{2}$, we see that

$$b - \frac{1}{4} < -\frac{\sqrt{3}}{2} < -\frac{1}{2} \implies b < -\frac{1}{4}.$$

Likewise, since there are infinitely many $n > N$ for which $b_n = \frac{\sqrt{3}}{2}$, we see that

$$\frac{1}{2} < \frac{\sqrt{3}}{2} < b + \frac{1}{4} \implies \frac{1}{4} < b.$$

It is impossible to have both $b < -\frac{1}{4}$ and $\frac{1}{4} < b$. Thus, we have found a contradiction. This shows b_n does not converge.