

MATH 117: HOMEWORK 4

Due Sunday, October 25, 2026, at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1

Consider a sequences s_n . Prove that s_n is a bounded sequence if and only if $S = \{s_n : n \in \mathbb{N}\}$ is a bounded set of real numbers.

Question 2*

An important lemma in the analysis of sequences is known as the *squeeze lemma*.

LEMMA 1 (Squeeze Lemma). *Consider three sequences a_n, b_n , and s_n . If $a_n \leq s_n \leq b_n$ for all $n \in \mathbb{N}$ and there exists $s \in \mathbb{R}$ so that*

$$\lim_{n \rightarrow +\infty} a_n = \lim_{n \rightarrow +\infty} b_n = s,$$

then $\lim_{n \rightarrow +\infty} s_n = s$.

In this question, you will prove the squeeze lemma and consider an important consequence of this lemma.

- (a) Prove the squeeze lemma.
- (b) Suppose s_n and t_n are sequences such that $|s_n| \leq t_n$ for all $n \in \mathbb{N}$. Prove that, if $\lim_{n \rightarrow +\infty} t_n = 0$, then $\lim_{n \rightarrow +\infty} s_n = 0$.
- (c) Is the converse to part (b) true? If so, prove it. If not, give a counterexample and justify your counterexample.

Question 3

Recall that the *reverse triangle inequality* ensures that, for all $a, b \in \mathbb{R}$, $||a| - |b|| \leq |a - b|$.

Suppose t_n is a convergent sequence, with $\lim_{n \rightarrow +\infty} t_n = t$.

- (a) Use the definition of convergence to prove that

$$\lim_{n \rightarrow +\infty} |t_n| = |t|.$$

- (b) As a consequence of part (a), you have proved the following statement:

If t_n is a convergent sequence, then $|t_n|$ is a convergent sequence.

Is the converse true? If so, prove it. If not, give a counterexample and justify your counterexample.

Question 4*

Suppose that s_n is a sequence whose limit exists, with domain $n \in \{1, 2, 3, \dots\} = \mathbb{N}$.

(a) Fix $m \in \mathbb{N}$ and let t_n be a sequence with domain $n \in \{m, m+1, m+2, \dots\}$ defined by

$$t_n = s_n \text{ for all } n \in \{m, m+1, m+2, \dots\}.$$

Prove that $\lim t_n = \lim s_n$.

(b) Fix $m \in \mathbb{N}$ and let t_n be a sequence with domain $n \in \mathbb{N}$ defined by

$$t_n = s_{n+m} \text{ for all } n \in \mathbb{N}.$$

Prove that $\lim t_n = \lim s_n$.

The moral of part (a) is that the domain of a sequence doesn't affect its limit—all that matters are the values of the sequence and the order in which the values appear. **The moral of part (b)** is that, if you chop off the first m elements of a sequence, it doesn't change the limit.

Question 5

Consider a sequence s_n satisfying $s_n \neq 0$ for all n and for which $\left| \frac{s_{n+1}}{s_n} \right|$ converges to L . If $L < 1$, show that $\lim s_n = 0$.

Hint: Explain why you can select a so that $L < a < 1$ and prove that there exists N so that $n > N$ ensures $|s_{n+1}| < a|s_n|$. Then use induction to show $|s_n| \leq a^{n-N-1}|s_{N+1}|$ for all $n > N$.

Question 6*

Which of the following is a correct version of the definition of convergence?

A sequence s_n converges to a limit s if...

- (a) there exists $\epsilon > 0$ so that, for all $N \in \mathbb{R}$, $n > N$ ensures $|s_n - s| < \epsilon$.
- (b) for all $\epsilon \geq 0$, there exists $N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - s| < \epsilon$.
- (c) for all $\epsilon > 0$, $|s_n - s| < \epsilon$ for all $n \in \mathbb{N}$.
- (d) given $\epsilon > 0$, there is some $N \in \mathbb{R}$ so that $|s_n - s| < \epsilon$ for all $n > N$.

For each of the statements above that are *not* the correct definition of convergence, give an example of either

- a sequence that satisfies the statement but does not converge or
- a sequence that converges but does not satisfy the statement.

In each case, the existence of such an example will illustrate why the statement is *not* equivalent to the correct definition of convergence.

Question 7

Prove the following useful lemma:

LEMMA 2. *If r_n and t_n are sequences whose limits exist and $r_n \leq t_n \forall n \in \mathbb{N}$, then*

$$\lim_{n \rightarrow +\infty} r_n \leq \lim_{n \rightarrow +\infty} t_n.$$

Do **not** assume that the sequences r_n and t_n converge, only that the limits exist.

Question 8*

Prove that a sequence s_n is increasing if and only if for all $n, m \in \mathbb{N}$, $n \leq m \implies s_n \leq s_m$.

Question 9

Suppose the limit of s_n exists and $k \in \mathbb{R}$. Then $\lim_{n \rightarrow +\infty} ks_n = k \lim_{n \rightarrow +\infty} s_n$, where we take the convention that

$$k \cdot (+\infty) = \begin{cases} +\infty & \text{if } k > 0 \\ 0 & \text{if } k = 0 \\ -\infty & \text{if } k < 0, \end{cases} \quad \text{and} \quad k \cdot (-\infty) = \begin{cases} -\infty & \text{if } k > 0 \\ 0 & \text{if } k = 0 \\ +\infty & \text{if } k < 0. \end{cases}$$

MATH 117: PRACTICE MIDTERM 1

All questions are extra practice. The midterm will be four questions long, where the last of the four questions is a “lightning round”. The first two questions will be given on Monday, October 26, and the second two questions will be given on Wednesday, October 28. The format will be very similar to this practice exam, but the questions will be different. Don’t forget that one of the extra practice problems from HW1, HW2, HW3, or HW4 will be on the midterm :).

Question 1

Let $(0, +\infty)$ denote the set $\{x \in \mathbb{R} : x > 0\}$.

- (a) State the definition of what it means for a nonempty set $S \subseteq \mathbb{R}$ to be bounded below.
- (b) Consider a nonempty set $S \subseteq (0, +\infty)$. Define $S' = \{1/s : s \in S\}$. Prove that $a > 0$ is a lower bound for S if and only if $1/a > 0$ is an upper bound for S' .
- (c) Suppose $\inf S > 0$. Prove that $\sup S' = 1/\inf S$.
- (d) Suppose $\inf S = 0$. Prove that $\sup S' = +\infty$.

Question 2

Let k, m be positive integers, and let $a, b, c, d \in \mathbb{R}$ with $a \neq 0$ and $c \neq 0$.

- (a) State the theorem that the limit of a quotient is the quotient of the limits.
- (b) Prove that

$$\lim_{n \rightarrow +\infty} (cn^m + d) = \begin{cases} +\infty, & c > 0, \\ -\infty, & c < 0. \end{cases}$$

- (c) Using part (b), explain why $cn^m + d \neq 0$ for all sufficiently large n .
- (d) Prove that

$$\lim_{n \rightarrow +\infty} \frac{an^k + b}{cn^m + d} = \begin{cases} 0, & k < m, \\ a/c, & k = m. \end{cases}$$

Question 3

Consider a nonempty set $A \subseteq \mathbb{R}$.

- (a) Suppose A is bounded above. Prove that there exists a sequence a_n , satisfying

$$\{a_n : n \in \mathbb{N}\} \subseteq A$$

and

$$\sup A - \frac{1}{n} \leq a_n \leq \sup A \text{ for all } n \in \mathbb{N}.$$

(Hint: Use the fact that $\sup A - \frac{1}{n}$ *cannot* be an upper bound.)

- (b) Prove that the sequence you found in the previous part satisfies $\lim_{n \rightarrow \infty} a_n = \sup A$.

- (c) Now suppose A is not bounded above. Prove that there exists a sequence a_n satisfying

$$\{a_n : n \in \mathbb{N}\} \subseteq A$$

and

$$a_n \geq n \text{ for all } n \in \mathbb{N}.$$

- (d) Prove that the sequence you found in the previous part satisfies $\lim_{n \rightarrow +\infty} a_n = \sup A$

In summary, you have proved the following important result: for any nonempty set $A \subseteq \mathbb{R}$, we may always find a sequence of elements a_n in A so that $\lim_{n \rightarrow +\infty} a_n = \sup A$.

Question 4

Lightning Round!

You do not need to show your work or justify your answers.

- (1) Consider the set $C = \left\{ n^2 + \frac{(-1)^n}{n^2} : n \in \mathbb{N} \right\}$.

(i) Does the set have a maximum? What is $\sup(C)$?

(ii) Does the set have a minimum? What is $\inf(C)$?

- (2) Circle the correct answer: Given a nonempty set $S \subseteq \mathbb{R}$, if M_0 is the supremum of S and M is a /an _____, then $M_0 \leq M$.

- (a) infimum of S
- (b) minimum of S
- (c) additive identity
- (d) upper bound of S
- (e) multiplicative identity

- (3) Circle the correct answer: Given $S \subseteq \mathbb{R}$ nonempty, what criteria must S satisfy in order to have a supremum?

- (a) $\max(S)$ D.N.E.
- (b) S has finitely many elements
- (c) S is unbounded
- (d) S is bounded above
- (e) S is bounded below