Question 1

Homework 4

① First, suppose s_n is a bounded sequence. Then $\exists\ M \geq 0$ s.t. $|s_n| \leq M\ \ \forall\ n \in \mathbb{N}$, so $-M \leq s_n \leq M\ \ \forall\ n \in \mathbb{N}$. This shows $-M$ is a lower bound for S and M is an upper bound for S. Thus S is a bounded set.

Next, suppose S is a bounded set. Then S is bounded above and below, so $\exists\ m, M \in \mathbb{R}$ s.t. $m \leq s_n \leq M\ \ \forall\ n \in \mathbb{N}$. Let $L = \max\{|m|, |M|\}$. Then

$$-L \leq -|m| \leq m \leq s_n \leq M \leq |M| \leq L \quad \forall n \in \mathbb{N},$$

So $|s_n| \leq L\ \ \forall\ n \in \mathbb{N}$. This shows s_n is a bounded sequence.

Q3 (a) Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = s$, for all $\varepsilon > 0$, there exists N_a and N_b so that $n > N_a$ ensures $|a_n - s| < \varepsilon$ and $n > N_b$ ensures $|b_n - s| < \varepsilon$. Recall that $|a_n - s| < \varepsilon \Leftrightarrow s - \varepsilon < a_n < s + \varepsilon$ and $|b_n - s| < \varepsilon \Leftrightarrow s - \varepsilon < b_n < s + \varepsilon$. Define $N = \max\{N_a, N_b\}$. Then for all $\varepsilon > 0$, $n > N$ ensures $s - \varepsilon < a_n$ and $b_n < s + \varepsilon$. Since $a_n \leq s_n \leq b_n$, this shows that for all $\varepsilon > 0$, $n > N$ ensures $s - \varepsilon < s_n < s + \varepsilon$ $\Leftrightarrow |s_n - s| < \varepsilon$, i.e. $\lim_{n\to\infty} s_n = s$.

(b) Apply the squeeze lemma, since

$$-t_n \leq s_n \leq t_n$$

and $\lim_{n\to\infty} t_n = 0$, $\lim_{n\to\infty} -t_n = (-1)(0) = 0$.

(c) If $\lim_{n\to\infty} s_n = 0$, it is not necessarily true that $\lim_{n\to\infty} t_n = 0$. For example, if $s_n = 0 \ \forall n \in \mathbb{N}$ and $t_n = 1 \ \forall n \in \mathbb{N}$, then $|s_n| \leq t_n$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = 0$, but $\lim_{n\to\infty} t_n = 1 \neq 0$.

Question 3

Homework 4

4

(a) Fix $\varepsilon > 0$. Note that, by the reverse triangle inequality,

$$|\,|t| - |t_n|\,| \leq |t - t_n|.$$

Since $\lim_{n\to\infty} t_n = t$, $\exists\ N \in \mathbb{R}$ s.t. $n > N$ ensures $|t - t_n| < \varepsilon$, hence $|\,|t| - |t_n|\,| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows $\lim |t_n| = |t|$.

(b) The converse is not true. Let $t_n = (-1)^n$. Then $|t_n| = 1$ is a convergent sequence, but t_n is not a convergent sequence.

Question 4

Homework 4

4

(a) **Case 1**: $\lim_{n\to\infty} s_n = +\infty$

By definition of divergence to $+\infty$, $\forall M > 0$, $\exists N$ s.t. $n > N$ ensures $s_n > M$. Thus $n > N$ ensures $t_n > M$. Since M was arbitrary, this shows $\lim_{n\to\infty} t_n = +\infty$.

Case 2: $\lim_{n\to\infty} s_n = -\infty$

By theorem from class, $\lim_{n\to\infty} -s_n = +\infty$, so Case 1 ensures $\lim_{n\to\infty} -t_n = +\infty$. By theorem from class, $\lim_{n\to\infty} t_n = -\infty$.

Case 3: $\lim_{n\to\infty} s_n = s$, for $s \in \mathbb{R}$

Fix $\varepsilon > 0$. Since s_n converges to s, there exists N s.t. $n > N$ ensures

$$|s_n - s| < \varepsilon.$$

Then $n > N$ also ensures,
$|t_n - s| = |s_n - s| < \varepsilon$.

Since $\varepsilon > 0$ was arbitrary, this shows $\lim_{n \to \infty} t_n = s$.

(b)

Case 1: $\lim_{n\to\infty} s_n = +\infty$.

Fix $m > 0$ arbitrary. There exists $N \in \mathbb{R}$ s.t. $n \in \mathbb{N}$ ensures $s_n > m$.

Thus $n > N$ implies $n + m > N$, so $t_n = s_{m+n} > m$.

Since $m > 0$ was arbitrary, this gives the result. ∎

Case 2: $\lim_{n\to\infty} s_n = -\infty$

By theorem from class, $\lim_{n\to\infty} -s_n = +\infty$, so by Case 1, $\lim_{n\to\infty} -t_n = +\infty$. Thus, by theorem from class again, $\lim_{n\to\infty} t_n = -\infty$. This gives the result.

Case 3: $\lim_{n\to\infty} s_n = s$, for $s \in \mathbb{R}$.

Fix $\varepsilon > 0$. Since s_n converges to s, there exists N s.t. $n > N$ ensures

$$|s_n - s| < \varepsilon.$$

Since $n > N$ implies $n+m > N$, we have

$$|t_n - s| = |s_{n+m} - s| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows $\lim_{n\to\infty} t_n = s = \lim_{n\to\infty} s_n$.

Question 5

Homework 4

Choose a so that $\max\{L, 0\} < a < 1$. Since $|s_{n+1}/s_n| \to L$ and $a - L > 0$, there is an integer $N \geq 1$ such that

$$n > N \quad \Longrightarrow \quad \left| \left| \frac{s_{n+1}}{s_n} \right| - L \right| < a - L.$$

Consequently, $|s_{n+1}/s_n| < a$, and hence

$$|s_{n+1}| < a|s_n| \qquad (n > N).$$

We prove by induction that

$$|s_n| \leq a^{n-N-1}|s_{N+1}| \qquad (n \geq N+1).$$

For $n = N + 1$, this is equality. If the inequality holds for some $n \geq N + 1$, then

$$|s_{n+1}| < a|s_n| \leq a\big(a^{n-N-1}|s_{N+1}|\big) = a^{n-N}|s_{N+1}|,$$

which is the desired inequality for $n + 1$.

Since $0 < a < 1$, we have $a^{n-N-1}|s_{N+1}| \to 0$. Given $\epsilon > 0$, choose $N_1 \geq N + 1$ such that $a^{n-N-1}|s_{N+1}| < \epsilon$ for all $n > N_1$. Then

$$n > N_1 \quad \Longrightarrow \quad |s_n - 0| < \epsilon.$$

Thus $s_n \to 0$.

Question 6

Homework 4

(9) The correct definition is (d).

(a) Consider $s_n = (1, -1, 1, -1, \ldots)$, $s = 0$.
Then for $\varepsilon = 2$ and all $N \in \mathbb{R}$, $n > N$ ensures $|s_n - s| < \varepsilon$.

(b) Consider $s_n = \frac{1}{n}$, $s = 0$. For $\varepsilon = 0$ there is no $N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - 0| < \varepsilon$.

(c) Consider $s_n = \frac{1}{n}$, $s = 0$. For $\varepsilon = \frac{1}{4}$, $|s_n - s| < \varepsilon$ is not true for all $n \in \mathbb{N}$.

Question 7

Homework 4

(10) If $\lim_{n\to\infty} r_n = -\infty$, the result is immediate. Thus, it remains to consider the remaining cases.

Case 1: Suppose $\lim_{n\to\infty} r_n = r \in \mathbb{R}$

Case 1a: If $\lim_{n\to\infty} t_n = +\infty$, we are done.

Case 1b: Suppose $\lim_{n\to\infty} t_n = t \in \mathbb{R}$. Assume for the sake of contradiction that $t < r$. Let $\varepsilon = \frac{r-t}{2} > 0$. Then $\exists N_r, N_t$ s.t. $n > N_r$ ensures $|r_n - r| < \varepsilon$ and $n > N_t$ ensures $|t_n - t| < \varepsilon$. Let $N = \max\{N_r, N_t\}$. Then $n > N$ ensures

$$t_n < t + \varepsilon = t + \frac{r-t}{2} = \frac{t+r}{2} = r - \frac{r-t}{2} = r - \varepsilon < r_n.$$

This contradicts that $r_n \leq t_n \ \forall n \in \mathbb{N}$. Thus $\lim_{n\to\infty} t_n = t \geq r = \lim_{n\to\infty} r_n$.

Case 1c: Suppose $\lim_{n\to\infty} t_n = -\infty$. Then $\exists N_t$ s.t. $\forall n > N_t$, $t_n < r - 1$. There also exists N_r s.t. $\forall n > N_t$, $r - 1 < r_n$. Thus for $N = \max\{N_t, N_r\}$, $n > N$ ensures

$t_n < r-1 < r_n$.

Again, this contradicts that $r_n \leq t_n \ \forall n \in \mathbb{N}$.

Thus $\lim_{n\to\infty} t_n = -\infty$ is impossible.

Case 2: Suppose $\lim_{n\to\infty} r_n = +\infty$. Fix $M > 0$. Then $\exists\ N$ s.t. $\forall\ n \geq N$, $M < r_n \leq t_n$. This shows $\lim_{n\to\infty} t_n = +\infty$.

Question 8

Homework 4

(11) Suppose s_n is increasing. Fix $n \in \mathbb{N}$.
We will prove
$$m \geq n \Rightarrow s_m \geq s_n$$
by induction.

Base case: $m = n$. By definition $s_m = s_n$.

Inductive step: Suppose $m \geq n$ and $s_m \geq s_n$.
Since it is an increasing sequence, $s_{m+1} \geq s_m \geq s_n$.
This shows the inductive step.

Now, suppose $m \geq n \Rightarrow s_m \geq s_n \quad \forall\, n, m \in \mathbb{N}$.
Take $m = n+1$. Then $s_{n+1} \geq s_n \quad \forall n \in \mathbb{N}$.
This shows s_n is increasing.

Question 9

Homework 4

Write $\ell = \lim_{n\to+\infty} s_n$.

If $\ell \in \mathbb{R}$, the constant sequence with value k converges to k. The product limit theorem therefore gives

$$\lim_{n\to+\infty} ks_n = k\ell.$$

If $k = 0$, the sequence ks_n is identically zero, so its limit is 0. This agrees with the stated convention even when $\ell = \pm\infty$.

It remains to consider $k \neq 0$ and $\ell = \pm\infty$.

If $\ell = +\infty$, fix $M > 0$. There exists N such that

$$n > N \quad \Longrightarrow \quad s_n > \frac{M}{|k|}.$$

If $k > 0$, this implies $ks_n > M$, proving $ks_n \to +\infty$. If $k < 0$, multiplying reverses the inequality and gives $ks_n < -M$, proving $ks_n \to -\infty$.

If $\ell = -\infty$, fix $M > 0$. There exists N such that

$$n > N \quad \Longrightarrow \quad s_n < -\frac{M}{|k|}.$$

If $k > 0$, this implies $ks_n < -M$, proving $ks_n \to -\infty$. If $k < 0$, multiplying reverses the inequality and gives $ks_n > M$, proving $ks_n \to +\infty$.

These cases establish the result with all the stated conventions.

Question 1

Practice Midterm 1

(1)

(a) S is bounded below if there exists $m_0 \in \mathbb{R}$ s.t. $m_0 \leq s$ for all $s \in S$.

(b) Suppose $a > 0$ is a lower bound for S. Then $a \leq s \ \forall s \in S$, which is equivalent to $\frac{1}{s} \leq \frac{1}{a} \ \forall s \in S$. Furthermore, this is equivalent to $t \leq \frac{1}{a} \ \forall t \in S'$. Finally, we recognize this is equivalent to $\frac{1}{a}$ being an upper bound for S'.

(c) If $\inf S > 0$, then by part (b), we use the fact that $\inf S$ is a lower bound for S to conclude that $\frac{1}{\inf S}$ is an upper bound for S'.

Suppose M is an upper bound for S'. Since $S' \subseteq (0, +\infty)$ is nonempty, M must be strictly positive. By part (b), $\frac{1}{M}$ is a lower bound for S.

Thus, by definition of the greatest lower bound, $\frac{1}{m} \leq \inf S$. Thus, $\frac{1}{\inf S} \leq m$. This shows that $\frac{1}{\inf S}$ is the least upper bound of S'. Thus $\frac{1}{\inf S} = \sup S'$.

(d) By definition of what $\sup S' = +\infty$ means, it suffices to show S' is not bounded above.

Suppose, for the sake of contradiction, that M were an upper bound for S'. Again, since $S' \subseteq (0, +\infty)$ is nonempty, we have $M > 0$. By part (b), $\frac{1}{M} > 0$. is a lower bound for S. This contradicts that $\inf S = 0$ is the greatest lower bound.

Thus, S' is not bounded above.

Question 2

Practice Midterm 1

(a) Quotient theorem

Suppose $x_n \to x$ and $y_n \to y$, where $x, y \in \mathbb{R}$ and $y \neq 0$. Then $y_n \neq 0$ for all sufficiently large n, and

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}.$$

(b) The limit of $cn^m + d$

Let $M > 0$ and choose

$$N \geq \max\left\{1, \frac{M + |d|}{|c|}\right\}.$$

Since m is a positive integer, $n > N$ implies $n^m \geq n$.

If $c > 0$, then for $n > N$,

$$cn^m + d \geq cn - |d| > M.$$

By the definition of divergence to $+\infty$, this proves $cn^m + d \to +\infty$.

If $c < 0$, then for $n > N$,

$$cn^m + d \leq cn + |d| = -|c|n + |d| < -M.$$

By the definition of divergence to $-\infty$, this proves $cn^m + d \to -\infty$.

(c) The denominator is eventually nonzero

By part (b), if $c > 0$, then $cn^m + d > 1$ for all sufficiently large n. If $c < 0$, then $cn^m + d < -1$ for all sufficiently large n. In either case, $cn^m + d \neq 0$ for all sufficiently large n.

(d) The cases $k < m$ and $k = m$

For all sufficiently large n, divide the numerator and denominator by n^m:

$$\frac{an^k + b}{cn^m + d} = \frac{an^{k-m} + b/n^m}{c + d/n^m}.$$

Since $m \geq 1$, we have $b/n^m \to 0$ and $d/n^m \to 0$, so the denominator on the right converges to $c \neq 0$.

If $k < m$, then $n^{k-m} = 1/n^{m-k} \to 0$, so the numerator converges to 0. By the quotient theorem,

$$\lim_{n\to\infty} \frac{an^k + b}{cn^m + d} = \frac{0}{c} = 0.$$

If $k = m$, then $n^{k-m} = 1$, so the numerator converges to a. By the quotient theorem,

$$\lim_{n\to\infty} \frac{an^k + b}{cn^m + d} = \frac{a}{c}.$$

Question 3

Practice Midterm 1

3

a

Pf: For all $n \in \mathbb{N}$, $\sup(A) - \frac{1}{n} < \sup(A)$. Since $\sup(A)$ is the <u>least</u> upper bound of A, $\sup(A) - \frac{1}{n}$ cannot be an upper bound for A. That is, $\exists\ a_n \in A$ s.t. $\sup(A) - \frac{1}{n} \leq a_n$. Since $a_n \in A$, for all $n \in \mathbb{N}$, we have

$$\sup A - \frac{1}{n} \leq a_n \leq \sup A.$$

(b) Since $\sup A - \frac{1}{n} \le a_n \le \sup A$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \sup A - \frac{1}{n} = \lim_{n\to\infty} \sup A = \sup A$ by squeeze lemma we obtain $\lim a_n = \sup A$.

(c) Fix $n \in \mathbb{N}$. Since A is not bounded above, n is not an upper bound for A, so there exists $a_n \in A$ s.t. $a_n > n$. In this way, we know there exists a sequence a_n with $a_n \in A$ and $a_n > n$ for all $n \in \mathbb{N}$.

(d) Fix $m > 0$. Let $N = m$. Then for all $n > N$, $a_n > n > N = m$. Thus a_n diverges to $+\infty$. This shows $\lim_{n\to\infty} a_n = \sup A$

Question 4

Practice Midterm 1

(4)

(1) (i) no, $\sup(C) = +\infty$

(ii) yes, $\inf(C) = 0$

(2) (d)

(3) (d)